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79 Numeration exercises (bases 5, 7, 12, 10, 12, 60) in translating from each nondecimal base to base 10 and conversely.

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INTRODUCTION

I. A PERSPECTIVE

Loose talk about reform in the teaching of elementary mathematics has spread confusion and shattered the confidence of many teachers. Uneasiness and fear have developed. This mood has damaging effects in the classroom.

Teachers have been told they are teaching horse-and-buggy mathematics in a space age. New and esoteric language has begun to flood the literature and to be introduced into discussions. Some innovators have pointed the finger of scorn at traditional subject matter, pronouncing it "bad mathematics."

Relax! The mathematics our pupils are learning in traditional classrooms is good, sound, useful mathematics; it is good enough to play a central role in any moon shot. Traditional content in elementary school arithmetic is sound; it is at the core of man's advance, including his recent bursts into space.

Our real concern is with finding solutions for two interrelated problems. How can we help pupils (a) to learn *more* mathematics, and (b) to develop a *deep appreciation* for this increasingly important part of our culture? All proposals and discussions should be evaluated in terms of their contributions to the solution of these fundamental problems.

The traditional content taught to pupils in elementary schools is completely valid. Our elementary schools must teach pupils how to carry out the fundamental operations with whole numbers and fractions and how to use these skills in solving practical problems. But the truth of the matter is that pupils simply aren't mastering the traditional content. Many have too weak a grasp on facts such as that $7+2=9$ and that $8\times9=72$. (Most of these pupils will drop mathematics as soon as they can, losing contact with a tool that is essential in coping with the problems of the times.) The lack of mastery cannot be remedied just by giving more drill. Pupils will be able to master more of the traditional content if we give them freedom to *invent* ways of finding answers, if we take steps to make sure that they *understand* what they are doing, and if they have ample opportunity to *use* what they learn.

The job of teaching more mathematics will become easier to the extent that pupils develop an appreciation for the subject. This appreciation can develop only when mathematics is learned with understanding and through active participation—it will not take place when mathematics is learned by rote. Mathematics is a world of endless fascination. It is a world brimming with surprises, patterns, and enticing vistas. To see a pattern emerge from a conglomeration of facts, to organize seemingly disconnected ideas—these are some of the joys of learning mathematics. It is this sense of excitement in mathematics which pupils ought to be experiencing in elementary schools.

Giant steps are currently being taken to alter the traditional approach to teaching mathematics in order to get more mathematics into the elementary and secondary curricula. To this end, brilliant pioneering work is being done at the Universities of Illinois, Maryland, Minnesota, Stanford, Syracuse, and elsewhere. Generous financial support has been given by the National Science

Foundation, the Carnegie Corporation of New York, and the U. S. Office of Education for research projects, workshops, and institutes.

The authors of MATH WORKSHOP have participated in this pioneering work. Our program has evolved during a period of over ten years. Its ideas and techniques have been field-tested with thousands of teachers and tens of thousands of pupils.

II. CONTENT AND TEACHING APPROACH

The authors of MATH WORKSHOP believe that a sound mathematics program can be made ineffective by methods of teaching which rely mainly on rote learning. We also believe that even inspired teaching of isolated fragments of mathematics will fail to produce pupils who understand and can use what they have learned. The MATH WORKSHOP puts its *major subject matter emphasis* on basic principles and unifying ideas which tie together the facts and manipulative procedures of arithmetic. This emphasis simplifies the subject, makes it understandable, insures recall, and increases interest. Our *major pedagogical emphasis* is that of teaching for discovery. We present content to pupils so that they are challenged to think about relationships, to look for patterns and clues, to draw logical conclusions for themselves, and to invent their own ways of finding answers.

A. The *What* of Mathematics

The Spiral Curriculum

The *24th Yearbook* of the National Council of Teachers of Mathematics talks about "the basic mathematical understandings which should be continuously developed and extended throughout the entire mathematics curriculum, grades K-12 and beyond." This concept of the spiral curriculum has been used in designing MATH WORKSHOP. At each level of the MATH WORKSHOP the main ideas of mathematics have been introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication.

However, pupils are not expected to master such ideas as, say, multiplication of fractions or addition of positive and negative numbers the first time they meet them. They will encounter such ideas again and again, building always more layers of understanding.

The spiral curriculum rejects the "building block" approach to arithmetic which has been used in the past. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas with repeated reconsideration at later times. It calls for continual nurturing of important concepts year after year as understanding gradually develops.

The hope for absolute mastery must be limited to the basic facts of arithmetic. Here, complete mastery can and must be achieved. But where big ideas are involved, mastery does not come in this way. It develops slowly and is based completely on understanding.

The Unifying Ideas or Strands

MATH WORKSHOP emphasizes nine unifying ideas in mathematics:

1. *Structure*
2. *Sets*
3. *Number and counting*
4. *Numeration*
5. *Addition and subtraction*
6. *Multiplication and division*
7. *Functions and relations*
8. *Geometry*
9. *Measurement*

The coverage of these ideas on the pupil pages and on the guide pages is summarized in the chart on pages 38-39. Each teacher's guide page lists the unifying ideas considered on the corresponding pupil page and contains many suggestions for extending these ideas beyond what is called for on the pupil page.

Let us consider each of these ideas.

Structure. Suppose that you came upon some sentences from a Martian arithmetic book:

1) | ☆ □ = □ ☆ |
2) ⊓ ☆ ◪ = ◪ ☆ ⊓
3) (⊠ ☆ ⌜) ☆ □ = ⊠ ☆ (⌜ ☆ □)
4) (□ ☆ ◪) ☆ | = □ ☆ (◪ ☆ |)
5) □ ☆ ⊠ = □
6) ◪ ☆ ⊠ = ◪

Somehow you have found out that the symbols:

|, □, ⊓, ◪, ⊠, ⌜

represent different cardinal numbers. You guess that the symbol ☆ is an operation sign, and you wonder which operation it stands for. Sentences 1 and 2 convince you that ☆ stands for neither subtraction nor division, for these sentences suggest that ☆ stands for an operation for which order does not affect the results. Sentences 3 and 4 also support this conviction because they suggest that grouping does not affect the results. However, there is nothing about sentences 1-4 which would help you choose between addition and multiplication as the operation denoted by ☆. Perhaps the other sentences will help.

Sentences 5 and 6 turn out to be of little help. In fact, both of them suggest that ☆ could denote addition, multiplication, subtraction, or division. Fortunately, sentences 1-4 rule out both subtraction and division. If ☆ stands for addition, then ⊠ denotes the number 0; if ☆ stands for multiplication, then ⊠ denotes the number 1.

It turns out that, based on what can be learned from sentences 1-6, the operation sign ☆ could stand for addition or it could stand for multiplication.

We can't decide from sentences 1-6 whether it is addition or multiplication that is denoted by ☆ because these operations have certain properties in common and these properties are reflected in sentences 1-6. The *additive structure* of the cardinal numbers is similar to the *multiplicative structure.* Each operation is commutative (sentences 1 and 2) and each is associative (sentences 3 and 4). Also, each operation has an identity element (sentences 5 and 6).

Whenever you provide exercises for pupils which bring any of these properties to the surface, you are paying attention to the structural properties of the number system. For example, exercises such as the ones which ask pupils to fill the frames to make the following into true sentences:

$$8 \times 7 = \bigcirc \times 8$$
$$7 + (3 + 5) = (7 + \triangle) + 5$$

are exercises which help pupils develop an awareness of commutativity and associativity. Another such exercise is the one which asks pupils to complete the addition *Cross Number Puzzle:*

4	3	
5	1	

Notice that the cell in the lower right-hand corner gets filled with the same number whether the pupil adds the numbers in the rows and then adds the sums:

4	3	7
5	1	6
		13

or adds the numbers in the columns and then adds the sums:

4	3	
5	1	
9	4	13

This invariance of the total highlights the rearrangement properties of addition. (See pages 18-23 in the Introduction.)

Of course, the advice that one should give pupils exercises which bring structural properties to the surface does not mean that pupils need to be told the names for these properties. A pupil who has discovered these properties

by working with displays of physical objects has a sound understanding of the properties. He thinks of them almost as laws of nature. Any attempt to name these properties or to formalize them would be regarded by him as an attempt to glorify the obvious. A young pupil who tells you that

$$7+3 \text{ is the same number as } 3+7$$

because "addition is commutative" has surely been coerced into using these words. He knows that $7+3=3+7$ because any display of objects which represents $7+3$ can be rearranged (without changing the total number of objects) into a display which represents $3+7$.

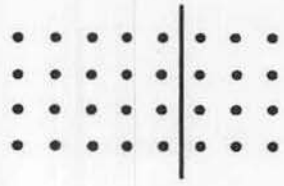
A pupil can be aware of and can use the structural properties without knowing their names. Exercises can be found which encourage pupils to make use of the properties. For example, a number line exercise which asks pupils to find the landing point when the starting point is 3 and the jump is 16 to the right could be solved by starting at 16 and jumping 3 to the right. The fact that this alternative procedure gives the correct answer strengthens a pupil's awareness of the commutative property of addition and helps to convince him of its usefulness.

If a pupil finds the sum of 16 and 3 by finding the sum of 6 and 3 and then finding the sum of 10 and 9, he has made use of the associative property of addition in a helpful way. (Of course, he has also made use of the fact that the numbers are expressed in decimal notation.) He does not become convinced of the correctness of his technique by being told that it is based on the fact that addition is an associative operation. Rather he becomes convinced by observing that this display of counters:

can be rearranged into this display:

In addition to the structural properties of commutativity, associativity, and the existence of an identity element for addition, the operation of adding a number has the property that it can be "undone" by subtracting that number. Similarly, the operation of multiplying by a (nonzero) number can be "undone" by dividing by that number.

Pupils also learn that certain multiplication facts can be discovered by using other multiplication facts and addition. For example, pupils can find the product of 8 by 4 by adding the products of 5 by 4 and 3 by 4. That this procedure is valid is based on the "physical" fact that an 8 by 4 array of counters can be partitioned into a 5 by 4 array and a 3 by 4 array:



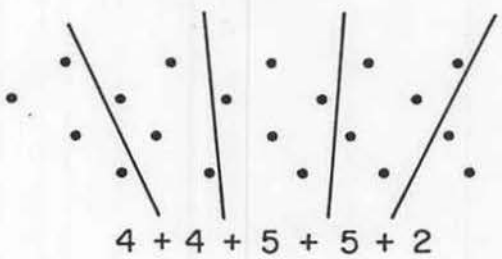
This experience helps pupils become aware of the fact that multiplication distributes over addition—a property of multiplication and addition. The term "distributive property" is not used on the pupil pages and need not be used in class. The important thing is that pupils use this technique in deriving certain products from others.

Sets. The principal contribution of sets to elementary arithmetic is that they can be used to clarify the notion of cardinal number. For a long time people thought that the word "two" in the expression "two apples" played the same kind of role as the word "green" in the expression "green apples." This was the cause of much confusion about the nature of cardinal numbers. An individual apple can have the property of "green-ness" but it can't have the property of "two-ness." Only sets can have the property of number. A set consisting of two green apples is a set whose cardinal number is two. Also, any set whose members can be matched with those of a set of two apples is a set whose cardinal number is two. From here it is an easy step to say that the cardinal number two is the class of all such sets.

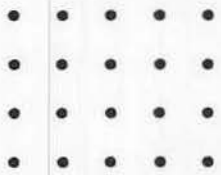
Now, a set is an abstract entity just as a number is abstract. You can see two apples but you cannot see a set of two apples any more than you can see the number two or justice or liberty. For example, it is complete nonsense to ask the *set* of boys in the classroom to stand or to ask pupils to look at the *set* of counters on the table. There will come a time in the pupil's mathematical education when he will need to use the word "set" and we do not think it is a good idea for him to develop in the primary grades the notion that a set is a tangible object. Therefore, we have rarely used the word "set" in the pupil's text, and have used it sparingly in the teacher's guide pages. Instead we have used nontechnical terms such as "display," "group," and "array."

Number and Counting. In the primary grades the basic mathematical idea is that of counting. The operations of addition and multiplication are presented as shortcuts to counting. Given a display of objects:

and asked to tell how many objects are in the display, a pupil might proceed to count them one by one. This is the most primitive approach. A more sophisticated approach is to partition the display, obtain the count for each of the smaller displays, and add the counts:



An even more sophisticated approach is to rearrange the objects in the display into a rectangular array:



If one is familiar with the count for a 4 by 5 array, he knows immediately that the original display contained 20 objects.

Numeration. The idea of numeration grows out of the need of reporting a count, and the mode of reporting is culturally determined. For example, if the objects in the display shown above represent glasses of milk, we might report "1 gallon and 1 quart" in answer to the question: How many? If the objects represent eggs, we might report "1 dozen and 8 more" or "4 less than 2 dozen."

Each of these reports is a description of the result of some partitioning procedure. When decimal partitioning is used, the English language provides us with the words for giving brief reports. For example, the report "3 ten-groups and 7 one-groups" or "3 tens and 7 ones" can be abbreviated to "thirty-seven" and can be recorded in positional notation as 37.

Addition and Subtraction. The operation of addition is developed as a shortcut to counting. Given enough opportunities to find the sum of 5 and 4 by counting, for example, a pupil soon commits to memory the fact that $5+4$ is 9. He makes use of this fact as well as of positional notation and associativity when finding the sum of 25 and 4: $25+4=(20+5)+4=20+(5+4)=20+9=29$. The addition facts from "$0+0=0$" through "$9+9=18$" should be memorized during the primary grades.

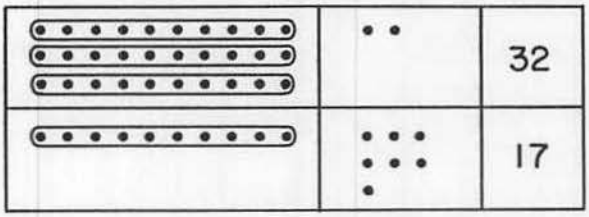
We use the word subtraction to denote the process used in undoing the result of adding. So, for example, to compute $8-3$ amounts to finding the number whose sum with 3 is 8. From this point of view, it is natural to learn subtraction facts at the same time one is learning addition facts. Thus, we have *fact teams*, groups of four sentences, such as:

$$5+3=8$$
$$3+5=8$$
$$8-3=5$$
$$8-5=3$$

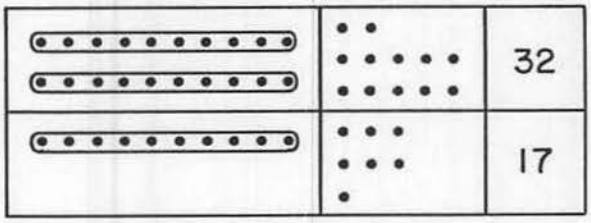
which are derived from one addition fact with the help of commutativity and the fact that subtracting a number is the *inverse* of adding that number.

Addition and subtraction algorithms are developed as end products of activities in which pupils manipulate physical objects and keep records of the manipulations. An algorithm becomes a record-keeping device for imagined manipulations. Thus, from the viewpoint of the pupil, an algorithm works because of laws which govern the manipulation of physical objects. For exam-

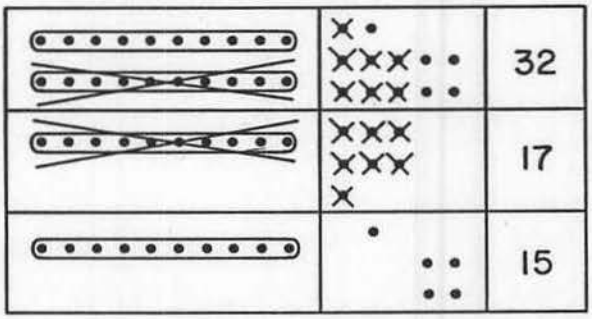
ple, consider an algorithm used in subtracting 17 from 32. In terms of physical objects, one can find the difference by counting what remains when 17 objects are removed from a display of 32 objects. We start by giving ourselves decimally partitioned displays of 32 objects and 17 objects:



In order to carry out the take-away step, we need to break into one of the ten-groups. So, we rearrange the display for 32 by exchanging a ten-group for 10 one-groups:



and then take away 1 ten-group and 7 one-groups:

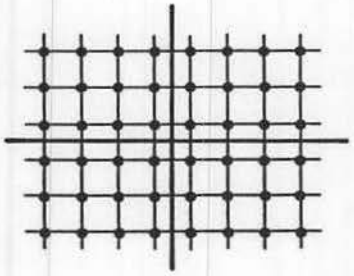


These three stages are recorded in the steps of the algorithm:

$$\begin{array}{r} 32 \\ -17 \\ \hline \end{array} \qquad \begin{array}{r} 2\not{3}12 \\ -17 \\ \hline \end{array} \qquad \begin{array}{r} 2\not{3}12 \\ -17 \\ \hline 15 \end{array}$$

Multiplication and Division. As in the case of addition, multiplication is developed as a shortcut to counting. In this case the counting involves objects arranged in rectangular arrays or otherwise grouped to encourage counting by groups. For example, counting the spaces in an egg carton and telling the end point of a sequence of 5 number line jumps of 2 to the right starting at 0

are multiplicative situations. By continual practice in the primary years a pupil should commit to memory the multiplication facts from "$0\times0=0$" to "$9\times9=81$," and each of these facts should be discovered by the pupil. Some of the facts should be discovered by direct counting, but the discovery of others should involve the partitioning of arrays and adding the counts of the parts. For example, a pupil might learn that 3×4 is 12 by counting the spaces in a certain type of egg carton. He should use this fact to learn that 6×8 is 48 by drawing 6 horizontal lines and 8 vertical lines and then partitioning the resulting display of crossing points:



12
12
12
12
48

Division is the process used in undoing the result of multiplying. So, for example, to compute $24\div8$ amounts to finding the number whose product with 8 is 24. From this point of view, it is natural to learn division facts at the same time one is learning multiplication facts. Thus, as in the case of addition and subtraction, we have fact teams for multiplication and division such as:

$$8\times3=24$$
$$3\times8=24$$
$$24\div3=8$$
$$24\div8=3$$

which are derived from one multiplication fact with the help of commutativity and the fact that dividing by a (nonzero) number is the *inverse* of multiplying by that number.

Functions and Relations. An example of a function is the operation addition-of-2. This function assigns a unique number to each number. For example, it assigns 5 to 3, 19 to 17, and 202 to 200. A common way to exhibit a function is to show a table which lists some of the assignments. For example, to exhibit the function addition-of-2 we might show the table:

3	4	9	12	18	30
5	6	11	14	20	32

The numbers listed in the top row are the arguments of the function and those listed in the bottom row are the values of the function. The use of two colors helps to distinguish arguments from values. This same table shows another function—subtraction-of-2—which is the inverse of addition-of-2.

All functions are examples of mathematical objects known as relations. Numerical relations which are not functions are those which assign more than one number to given numbers. For example, the greater-than relation assigns 4, 5, 6, etc., to the number 3.

Geometry. Under this general heading one finds a variety of topics—classification of common figures, comparisons of area and perimeter, dissections of regions, and similarity transformations (the doubling and halving of line segments).

Measurement. Under this heading we find work with denominate numbers and the conversion of units. There is a substantial amount of this work in LEVEL C.

B. How to Teach Mathematics

It's easy to ask a question so easy that almost anyone can answer it. And it's easy to dig up trick questions—there are books full of them.

But challenging questions that encourage independent research and lead to discovery must be neither too hard nor too easy, and must not rely on clever tricks. Rather, challenging questions are less than obvious but will yield answers if the pupil applies thoughtfully what he already knows.

Such questions are hard to come by. They are worth their weight in gold. Often they lurk in very simple situations.

Practically every page and activity in MATH WORKSHOP poses such less than obvious, challenging questions. MATH WORKSHOP presents these questions through a wide assortment of techniques, many of which were invented by the authors of the program. Challenging questions are posed through the game of *What's My Rule?*, through *Cross Number Puzzles*, and through problem solving. Let us examine examples of these vehicles we use for asking challenging questions.

What's My Rule?

In first grade, Miss Atkins tells the children that she has a secret rule (a function) and she will help them read her mind.

"Call out a number and I will answer with a number."

"Five," a child calls out.

"Four," Miss Atkins answers. "Someone call out another number."

"One," a child calls out.

"Zero," the teacher replies.

"Seven," another child calls out.

"Six," the teacher answers. "Have you guessed my rule?"

"Yes," Bill is quick to reply.

"Don't give my secret away. Just call out a number and tell me what you think I will reply."

"If I say 9, you are going to say 8."

"That's right. I *think* that you know my secret." Thus the children have learned a way to demonstrate their discovery.

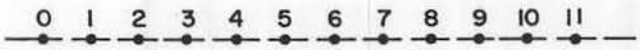
Many children are not as quick as Bill. So Miss Atkins may put on the board a record of the development thus far, first listing the number called out and then the reply:

5→4
1→0
7→6
9→8

The last item was Bill's contribution. Others might be:

6→5
2→1
4→3

When the game is new or the rule difficult, a number line on the board will help:



"Let's act out the game we have played. When someone called out 5, I answered 4. Point to 5 on the number line. Keep your finger on that point. I answered by saying 4. Point to 4 on the line."

Each step is similarly acted out.

"Now, if you call out 8, put your finger at that point on the number line."

8→____

"What do you think I will reply?"

"Seven, of course!"

In some such way the idea can be made clear even to the slowest children. Care was taken not to talk about the teacher's rule itself—only about the results of applying it. Thus, every child must make the discovery for himself—some with minimum clues, others only after several different kinds of clues. There is a bit of artistry in giving clues only when needed and finding ways to reserve at least a little for even the slowest child to find out for himself.

Usually some child will note a curious fact about this rule.

"Suppose that I call out zero. What will you answer?" a child may ask.

"Point to zero on the number line," the teacher suggests. "You can see that I would have to ask you to pick some other number." (Unless the teacher would like to talk about positive and negative numbers and zero.)

A new game begins with another secret rule. Some child may want to take over the teacher's role and invent a rule of his own.

Rules can become more complicated.

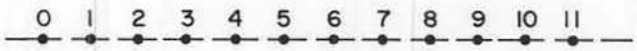
"I have a new rule. Call out a pair of numbers and I will reply by saying a number." Here is a record of a simple rule:

2, 3→5
4, 0→4
3, 1→4
1, 2→3
1, 1→2
3, 1→___

A more difficult rule is behind the following record:

2, 4→3
0, 2→1
4, 0→2
5, 9→7

These last two rules may puzzle the pupils. Perhaps a number line can help shed light on the rule.



"If someone called out 2 and 4, I would reply with 3. Point to the numbers called out (2 and 4). I would reply with 3.

"If someone called out 4 and 0, I would reply with 2. Point to 4 and 0 on the number line. I would call out 2.

"Point to 5 and 9—and remember that I would answer 7. What would I answer if someone called out 1 and 7?"

1, 7→

"Four," Bill might answer, and quickly ask: "But what would you do if I called out 5 and 6 or if I called out 3 and 0?"

"I would have to ask you to pick another pair of numbers." (Unless the teacher is ready to discuss fractions.)

An unlimited number of rules can be devised on any level of sophistication. Further, there are many formats to be used in presenting rules, such as:

a. 1, 4, 7, 10, 13, , , . . .

b.

	3	1	4	10	0	9	6
	7	3	9	21	1		

c.

	7	4	3	1
	9	9	0	9
	11	6	5	1
		8	6	
	6			2

Mathematics is essentially a study of patterns. It has its own fascinating patterns and it is often employed to help reveal patterns in the physical world. When a scientist or any observer is searching for a better understanding of what he notices, he usually tries to set down a description of events. Whenever possible he will try to write down numbers that in some way help him discover relationships. He might record the following observations:

	1st	2nd	3rd	4th	5th
minutes	1	3	4	5	10
miles	1	9	16		

In his first three observations he notes time and distance. A pattern is strongly suggested. He may check his guess by making further checks for 5 minutes and 10 minutes. If his results are 25 miles and 100 miles, respectively, he will feel that he can formulate a rule or law that will suggest a pattern in certain real events ($1 \times 1 = 1, 3 \times 3 = 9, 4 \times 4 = 16, \ldots$).

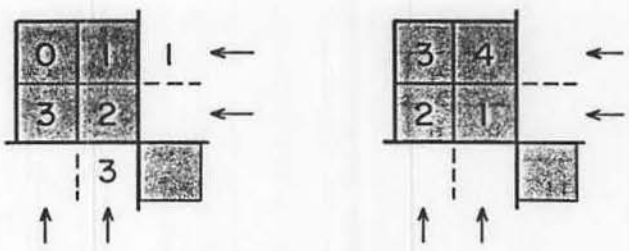
In mathematics we study patterns because they suggest basic principles of the subject itself and because mathematics helps record and explore patterns of the world we live in. The discovery of *one* basic principle can remove the necessity for learning many separate facts. Learning to discover patterns is a big step toward self-education. The recognition of the mathematical relationship among quantities in a scientific experiment allows further costly experimentation to be replaced by inexpensive computation.

Thus, this exercise or game of *What's My Rule?* has three important dimensions: (1) It can be used to motivate drill in the basic combinations of arithmetic. (2) It poses problems that pupils can think through to a solution. (3) It uses a method fundamental to the experimental sciences.

Cross Number Puzzles

Too few of us realize that even six- and seven-year-olds can carry out research problems in arithmetic. Do you doubt it? Let's leave speculation and get to work.

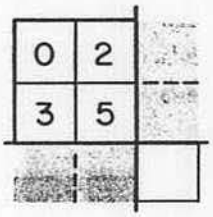
In the spaces indicated by arrows, write the sum of the two numbers listed in each row and in each column:

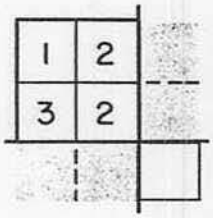


Do you have a desire to fill in the lower right-hand corner of each diagram above? Notice the numbers listed above the corner and the numbers listed to

the left of the corner: 1 and 5, 3 and 3 in the first example; 7 and 3, 5 and 5 in the second example.

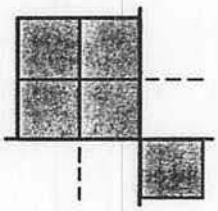
Most people would feel an urge to write 6 in the lower right-hand corner of the first example, and 10 in the corresponding corner of the second example. Try two more:

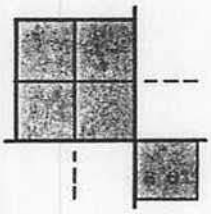




Again, a similar result. This is the kind of surprise that is the spice of arithmetic!

Can you select four numbers to list in the four blue-shaded cells so that some surprise happens in the lower right-hand corner? Try your hand at two examples:

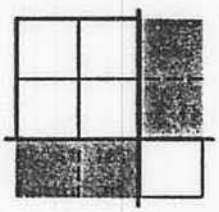


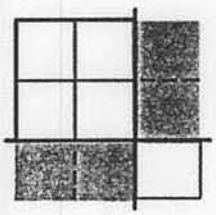


Did you succeed?

This problem is relatively simple. Even first-graders can find examples that work. They are putting their arithmetic to use, practicing the basic facts of addition.

Now try another problem. Find four numbers to list in the four green-shaded cells at the upper left—numbers that will not work in the same way. Try two examples:





Do you find that this problem is easier? Or is it more difficult?

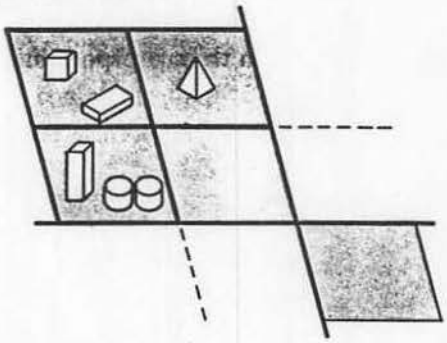
When pupils tackle this problem, they sometimes claim their numbers won't work. But when the arithmetic involved is checked, mistakes turn up —and the examples work after all.

Some pupils jump to the conclusion that it's impossible to find exceptions. Most will extend the search, trying the largest numbers they can handle. But the search will not yield an exception.

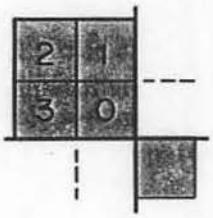
This certainly will be a surprising result to most children (and adults). Yet all pupils who have at least a few facts of addition at their command can discover this principle.

An understanding of the principle can be illustrated in an activity designed for pupils who are just learning to count and to record their results.

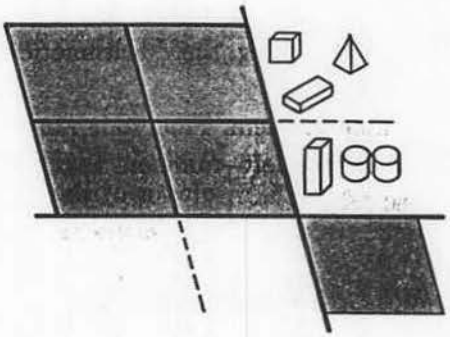
Suppose that we create a small model on a table with masking tape—or on the floor of the classroom. We place objects in three of the four boxes:



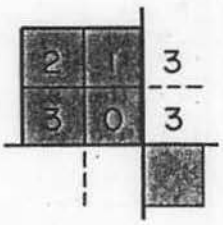
On the chalkboard a record is made for later use:



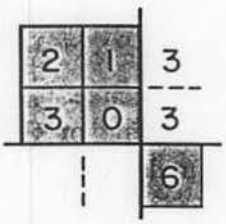
We now sweep to the right the objects in the top row of boxes and count them. There are 3 (2 plus 1). Next we sweep to the right the objects in the second row and count them. There are 3 (3 plus 0).



Include these results in the record on the board:

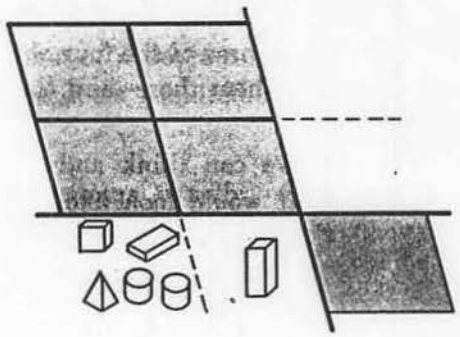


Now sweep the objects into the box in the lower right-hand corner and count them. There are 6 (3 plus 3). Include this in the record:

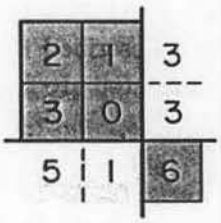


Can we put the objects back into the original four boxes so that each has the same number of objects as it had in the beginning? We have an adequate record on the board. It does not tell us which objects were in each box, but it does tell us the number of objects—and we make it plain that we are interested only in the number of objects.

After the objects have been replaced so that they conform with the original record, we sweep the objects in the boxes on the left down to the bottom and count them. There are 5 (2 plus 3). Similarly, we sweep the other object down and count it. There is 1 (1 plus 0).



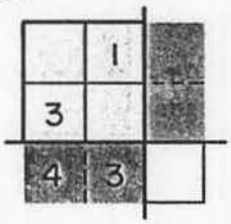
Include these results in the record on the board:



We sweep all the objects into the bottom right-hand corner and count them. There are 6 (5 plus 1).

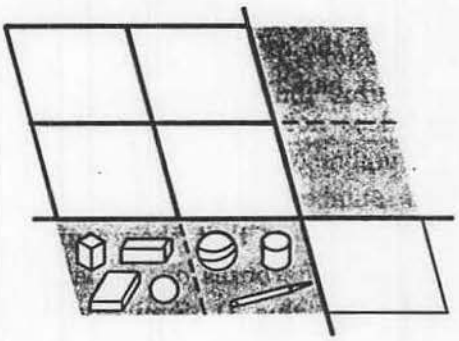
We find that we do not need to record this result, since we already noted that 6 objects were in that corner when we swept them there by a different route. Are you surprised? Certainly not. In fact, if any other result had occurred, you would be understandably disturbed.

Reconstructing the puzzle. Now suppose we start this activity a second time by putting a partially completed record on the board:



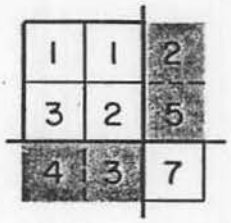
Can we find out the number of objects that were originally placed in each of the four main squares?

If we place four objects in the space beneath the two boxes on the left, and three objects in the space beneath the other two boxes, the result might look like this:

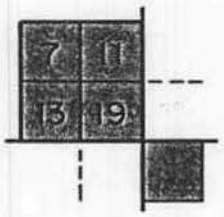


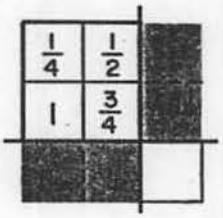
Since the record tells us that three of the four objects on the left were in one of the boxes, place three of them there—and there is one left for the other box on the left.

In a similar fashion, we can think and work our way through to a completion of the record. It would look like this when completed:

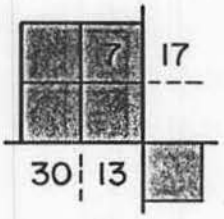


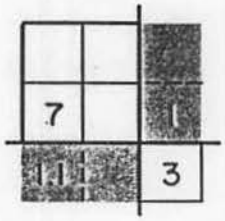
Although the simple activity just described is for beginners, the underlying principle is the same when the initiated are faced with such *Cross Number Puzzles* as:





Similar but more sophisticated reasoning is required for reconstructing the full record from such fragments as:





The *Cross Number Puzzle* requires considerable practice of the basic facts of arithmetic; it produces surprises; it helps to build a conviction that the outcome in addition is not influenced by the order or the grouping of the addends.

The *Cross Number Puzzle* can be used with great effectiveness to get pupils to discover algorithms for addition and subtraction.

Problem Solving and Applications

Story Problems. Over the years, "story problems" or "word problems" have been a source of vexation to many teachers of arithmetic, to many children—and to many parents. This activity, which properly seeks to relate the abstractions of mathematics to the world of events, often ends without a sense of accomplishment for anyone.

Story or word problems are usually reserved for the end of a development, as a sort of test or exercise of the pupil's ability to apply an acquired knowledge of arithmetic.

Examples vs. problems. A critical analysis of the questions usually asked in traditional arithmetic programs reveals that they constitute primarily a test of a child's reading ability, and that if the child can read, the questions are trivial and constitute *examples* rather than *problems.*

Example: Mr. Jones bought 8 fence posts. He paid $10 apiece. What was his total expense? (If you can read, this constitutes an example of the fact that $8 \times 10 = 80$.)

Problem: Mr. Jones bought 8 fence posts. He set them out in a straight line, 10 feet apart. What is the distance from the center of one end post to the center of the other end post? (If you can read, this is a problem that requires careful consideration of the situation presented; perhaps a sketch to show that there are 7 spaces between posts.)

The first situation is an example; the second is more of a problem. If a pupil clears the reading hurdle then the example taxes little more than his memory, while the problem depends on a less than obvious question that makes greater demands on the mind.

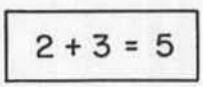
Examples and problems play different roles in helping pupils understand mathematics—and both are important.

When we notice a pattern repeating itself in the limited world around us—the world we can manipulate—we often use an expression in the language of arithmetic that points to that pattern, such as the sentence:

$$2 + 3 = 5$$

At this point we ought to direct pupils' attention to an application of this idea to the wide world of fact and fancy that they live in.

"Suppose that this is a headline."



"Tell a story that might fit such a title or headline," the teacher urges.

"There are five children in our family, two boys and three girls."

"Fine. Who has another story that would fit?" the teacher asks.

"I have three cookies out of the five I started with. I ate two of them."

"It's the second of December and my birthday is on the fifth. My birthday is in three days."

"On my hand I have a thumb and a little finger. There are three fingers in between."

"Out of my allowance of five cents I have two cents left."

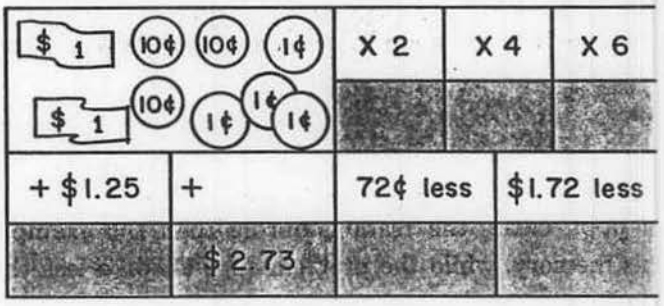
"Three hours from now will be five o'clock." (Check with a clock to determine whether or not this is a true sentence.)

Each of these situations has within it the conditions we point to in the headline. Each of the story-tellers reveals that he is beginning to grasp the relationship between arithmetic and the real world.

Whenever we abstract a bit of mathematics out of the world at hand and write it down in the language of arithmetic, good classroom technique should include the next logical step—to find examples from the world that reflect this idea.

The authors of MATH WORKSHOP believe strongly that basic mathematics should be derived from the world around us. For this reason we continually refer to practical situations on both the pupil pages and in the guide pages.

In LEVEL C we have introduced two types of problem-solving exercises that will go far in developing problem-solving skills. One type goes under the heading of *What Are My Rules?* Here is a typical exercise:



The pupil looks at the display of money, determines the amount shown, and then operates on this amount according to the operations given. For example, the "× 2" tells him to double the amount. He is free to compute the double in any manner he chooses. He might use the distributive principle and double the amount of each "coin" and then add. Or, he might multiply 234 by 2 using a multiplication algorithm. Or, he might work the problem using play money or other representational materials.

When he comes to the "× 4" part, he might start from scratch, or he might use the associative principle for multiplication and merely double his previous answer, thus: $234 \times 4 = 234 \times (2 \times 2) = (234 \times 2) \times 2$.

When he reaches the "× 6" part, he can again start from scratch, or he can triple his first answer, or he can use the distributive principle and add his first and second answers.

The problems in the bottom row involve addition or subtraction. Each of the first two requires exchanging. So does the third. The fourth can be solved by building on the result of the third.

This type of problem-solving activity involves very few words, it is semi-concrete, it encourages mental computation, and it gives pupils opportunity to become aware of the usefulness of the basic properties of the number system.

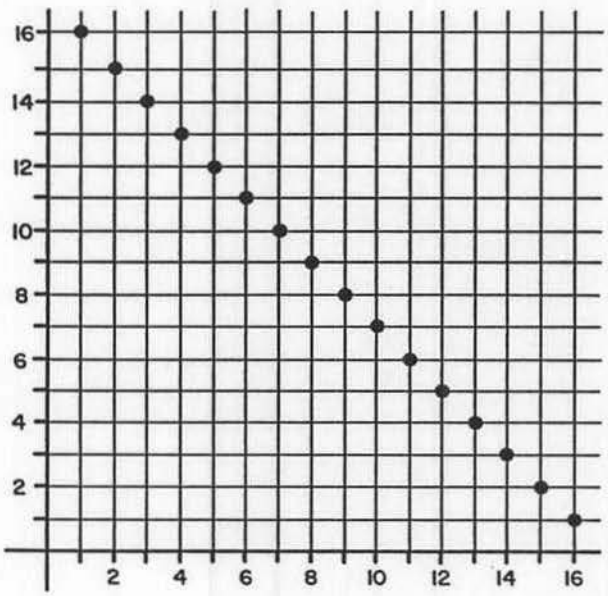
A second type of problem-solving activity introduced in LEVEL C (especially in *Headline for the Day* work) is the *What Can You Say?* situation. Here is an example of this type of exercise:

> Fred had 17 marbles. He put some of them in his left-hand pocket and the rest in his right-hand pocket. What can you say?

In this case, a pupil might say: If Fred put 4 marbles in his left-hand pocket then he must have put 13 marbles in his right-hand pocket. Other pupils propose various conditions. The situation calls for systematic tablemaking:

left-hand pocket	1	2	3	4	5	etc.
right-hand pocket	16	15	14	13	12	

or graphing:



Each guide page contains a section entitled *Headline for the Day*. This might be copied on the chalkboard at the opening of the school day, thus giving pupils a chance to ponder it. For example, you might write the head-

line "2 + 3 = 5" given above, and then let pupils tell various stories. Or, you might put a story on the chalkboard and ask for headlines. For example, if you wrote the story:

Bill has 2 toy cars. His father gave him 3 more.

You should expect headlines such as:

$$2 + 3 = 5$$
$$5 - 3 = 2$$
$$3 > 2$$
$$5 = 3 + 2$$
$$5 - 2 = 3$$
$$5 > 2$$
$$2 + 3 = \square$$
$$\square - 3 = 2$$
$$2 < 5$$

Please allow the pupils plenty of freedom in composing headlines.

We have included in the guide pages stories for "headline practice" which you may wish to use on a drill or put on the chalkboard. At least part of the discussion in each lesson should remind pupils that mankind has developed the language of mathematics to help report events that happen in the world of things.

Teaching for Discovery

Each page in MATH WORKSHOP and the activities suggested in the guide pages are designed so that there are several levels on which pupils can participate—including opportunities for success for even the slowest learners.

The activities we present are truly open ended, with challenge for even the most advanced and able pupils in the class. While some questions will have been settled, more will have been left open. Some pupils will wish to explore these open questions under their own steam. Such pupils ought to learn to work independently, sparing the teacher's time for those who need more of it. Actually, what the more able students need most is encouragement and opportunity to develop a taste for independent study.

The highest level of learning has been reached when a pupil begins to pose questions of his own and to work toward their solution.

There is in MATH WORKSHOP provision for introducing or practicing the basic facts of arithmetic. Usually such facts must be recalled and organized so that a pattern emerges. As patterns appear, they provide answers to less than obvious questions.

Discovering patterns is one of the most rewarding experiences in mathematics. It is called the "Aha!" or the "Gee-Whiz!" experience—a gasp, the light dawns! "I see it!" These are events teachers bend their best efforts to help bring about. One of the most disheartening facts teachers must face

is that some pupils simply "don't see it." Yet this is also a challenge—it calls for exercising skill in teaching. Slow learners need more clues—more obvious clues; but the finest teacher will reserve a wee bit for each pupil to "see" for himself. If the teacher's best efforts fail, he may have to give away part of the secret and try again next time. This is an art that does not come easily, but the eventual results are deeply rewarding.

When the results of work in arithmetic exhibit an unusual regularity, the inquiring mind looks for an explanation. One problem may be solved but others are suggested. This is an exciting characteristic of mathematics.

When pupils discover a relationship, they are excited and eager to stay with that discovery. They will readily put this new idea to work, showing more examples of *it*. They come alive, often with an "Oo-o-o! I've got it!" or with a burst of industry using a newly-discovered relationship to reel off answers with speed and confidence.

What is the *it* they have glimpsed? Well, if we distract their attention with a request for them to *tell about it* or *put it in words*, we have selected a method that could not be better calculated to turn pupils against discovery.

That a child can move ahead in putting *it* to work successfully is certainly all the evidence we need in order to know that he *has it*. Our only justification for veering off in the direction of a word-pulling exercise is a feeling that *it* is a generalization that ought to be verbalized in order to be usable. But research does not support this as a valid reason.

Transfer to future experiences is not necessarily increased by attempting to *verbalize a generalization*. Subvocal or nonverbal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, is our main goal.

When pupils experience an insight—when they leap ahead because they have discovered a pattern—we ought to be very careful about letting words into the act too soon. The pupils probably do not have the language needed to express their discovery in words. If they try, we must, of course, help them. But too often such help and explanation degenerate into hair-splitting, an activity far removed from the act of discovery.

Gertrude Hendrix of the University of Illinois Committee on School Mathematics, who has done much research on discovery in the teaching of mathematics, warns that imprecise verbalization may "mutilate" a discovery.

On the other hand, when some pupils (and teachers) make discoveries, they wish others to know about it. But herein lies a danger. Whenever a person reveals his secret to one who has not been as fortunate, the latter is deprived of the delight of discovering it for himself.

So we invent ways in which pupils can share their discovery without revealing the secret too widely. We have them whisper their new idea in the teacher's ear; we encourage them to make up additional examples which will provide further clues for those who haven't found a pattern that underlies several examples. We name the new idea after its discoverer—Bill's Rule or Mary's Law or John's Pattern. But we keep urging children to keep the secret—for a while at least.

Once in a while we find pupils who have not invented shortcuts for solving certain kinds of problems. For example, suppose that pupils use counting by one in counting problems which involve fairly large totals. Hopefully, such pupils will become discouraged and begin to look for easier ways of determining the total, such as counting by five. If the easier ways are invented, the pupils should be highly praised and urged to demonstrate these techniques to the entire class. If these examples are not sufficient to shake all pupils loose from counting by one, then you must step in and urge the conservative ones to drop the slower technique. Please temper the discovery method with common sense.

III. REPRESENTATIONAL MATERIALS

When mathematics loses contact with the world of things and with the events out of which it came, the subject becomes mysterious to most pupils. If they do not understand that in mathematics we are discussing the way things happen as we manipulate objects, pupils must resort to rote memory.

MATH WORKSHOP activities stress the fact that arithmetic provides a means of recording or reporting what happens in the physical world.

Many kinds of manipulative or representational materials are suggested and discussed in this Teacher's Guide. These include various types of counting materials, geometric shapes, Cuisenaire® rods,* number lines, etc. (See the reference in section G on page 33 of the Introduction.)

Perhaps the most accessible source of representational materials is the pupil pages themselves. We believe that all of elementary mathematics can be abstracted out of experiences with the physical world. The very word WORKSHOP in the title of our program indicates our belief in the value of learning mathematics by working with physical objects. In some cases, the pupil pages are the workshop—pupils work with the two-dimensional pictorial representations of physical objects and apparatus (*Cross Number Puzzles*, number lines, colored strips) and use the forms provided to keep records. Frequently the teacher will decide that the pupils need to work with three-dimensional equipment. In those cases, the teacher may provide equipment like that pictured on the pupil pages or like that suggested in the guide pages.

IV. MASTERY OF BASIC FACTS AND ALGORITHMS

Mathematics provides a language to help report events that happen in the world of things. Counting is at its core. Because this process becomes tedious when many things must be counted, shortcuts to counting have been invented and improved. Thus, elementary school mathematics is essentially concerned with counting and with shortcuts to counting.

At first glance, MATH WORKSHOP activities may appear to be highly motivated exercises to provide practice in the basic facts and combinations

*Cuisenaire—Registered trademark of Cuisenaire Company of America, Inc.

and processes of arithmetic. In the vernacular, they wear the garb of "sugar-coated drill." Actually, they are valid educational devices, because they help establish that automatic recall of "fundamentals" without which pupils are mathematical stutterers.

Every teacher knows that many pupils are mathematical stutterers because they can't easily recall the basic facts. They hesitate and flounder; the discussion bogs down; they lose the argument; they miss the feeling of movement toward a solution; they cannot enjoy mathematics.

This area of trouble is so universally known that it is surprising so many are only casually concerned! There is much hue and cry to bring new and unusual content into the elementary school curriculum—numeration in bases other than ten, sets, properties of numbers, and other concepts—yet there seems to be silence about such a crucial matter as helping pupils master the fundamental combinations without which mastery their progress in understanding mathematical ideas is very painful.

While we have emphasized the scope of our program in terms of nine unifying ideas, *the basic facts of arithmetic and algorithms are taught and retaught throughout the program.*

To provide you with help toward the goal of mastery, we have included on practically every guide page (in a section entitled Drill) a sample work sheet which can be easily converted into a Ditto®* sheet. We have felt that it would be too expensive and perhaps a little silly to use any of the pupil pages for strictly drill purposes. (You may find it useful to send a Ditto drill sheet home with a pupil rather than have him take home his text.)

V. THREE WAYS TO USE PUPIL PAGES

There are at least three generally successful ways in which individual pages or groups of pages can be used in the classroom:

1) Use the page to conclude a development.
2) Use the page as the basis for discussion.
3) Let pupils work independently.

1) In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching—with liberal use of manipulative material, the chalkboard, and class discussion. The page is then used late in the development to tie the idea down, or simply to record the progress that has been made toward deeper understanding.

There are limitless ways to initiate lively activities around important and useful concepts. The MATH WORKSHOP provides many such activities for you to choose from.

After handling manipulative materials, playing number games, and other such preliminary work, pupils are ready to move through the page in the final few minutes of the arithmetic period.

*The name Ditto is a registered trademark of Ditto Incorporated, Chicago, Illinois.

2) In contrast, some pages might be gone through very carefully with an entire class. Such a page or sequence of pages offers many opportunities to dig deeply into: What's going on here?
3) A third approach might be called *Do-it-all-by-yourself*. Begin by saying, "Open your book to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let's make today a 'quiet day.'"

Often, the first question the page poses to pupils is: What am I supposed to do? There are almost always sufficient clues provided so that with a bit of study pupils can discover the intent of the exercises. Traditional materials have provided too little incentive for pupils to figure out for themselves the nature of the business at hand.

Now you feign reluctance to help but are, of course, watching carefully for those who haven't sufficient self-confidence to get started on their own. If several children are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity under way for everyone. Let time run out on the session if you wish, suggesting that those who haven't finished may wish to take the work home with them to finish.

VI. TEACHING AIDS IN PROVIDING FOR INDIVIDUAL DIFFERENCES

A. In the Teacher's Guide

The Introduction presents a point of view and a rationale for our approach to arithmetic. You will find this useful in interpreting many of the things we have included in the pupil pages. You will also find it useful in explaining the program to parents.

Each pupil page is reproduced in the Teacher's Guide in full size and in full color with answers printed in red.

Each pupil page is accompanied by guide pages which are divided into several sections. Let's consider some of these sections:

Tasks. This section contains a very brief description of what the pupil will be doing as he works on the pupil page. We suggest that you study the pupil page in a pupil's book and work on it before you read our description and the answers we give.

Purpose. Here we describe what we think the pupil will learn as he works on the pupil page or as he carries out some of the activities described in The Lesson. (Naturally, our description contains a conservative estimate of what might happen. Mathematics is full of surprises and so are pupils.) If you are aware of our purpose, you will find it easier to help pupils achieve it.

Unifying Ideas. In this section we list those of the nine unifying ideas mentioned on page 8 to which the exercises on the pupil page and the recommendations in the guide can be expected to contribute. Please regard the list

as a minimum statement, and be prepared for pupils who may wish to go beyond what we have described for the page.

Materials for Teacher. This is a description of the equipment you would need if you followed all of the suggestions made in The Lesson. Some teachers will be able to look at this description and then plan their own lessons without reading further.

Materials for Pupils. This section describes the equipment needed by individual pupils in carrying out the tasks or in pursuing the activities suggested in The Lesson.

The Lesson. In this section we describe one or more of the lessons we might teach after studying the pupil page.

Practically all of our lessons would involve representational materials. We might decide to use a workshop and laboratory approach, using the equipment directly suggested by the pupil page (see section III on page 28). In that case, the pupil page would be used as review material.

We might use equipment quite unlike that suggested by the pupil page, but in keeping with the purpose and unifying ideas. In some cases, the pupil page might be worked on at the beginning of the lesson, so that the laboratory would then provide an application of what was learned on the pupil page might be worked on at the beginning of the lesson, so that pupil page the application of what was learned in the laboratory.

We would assign some pupil pages to be worked on with very little explanation from us (see section V on page 29). For other pages, we would ask many questions. Hopefully, we would find some means of projecting such pages on the chalkboard so that we could talk about them easily (see section B, below).

No teacher could possibly find the time to use all our suggestions in any school year. Choose the ones which best suit your style of teaching, and vary your choices from year to year.

B. "Math Workshop Supplementary Activities"

A package of supplementary activity worksheets is available at every level, A through F. Each package includes 32 copies of 64 different practice pages. These can be used independently by pupils for seatwork and for homework. Able pupils should be given supplementary activity material at advanced levels. Each set of supplementary activities includes a Teacher's Manual with brief suggestions for each page and an answer key.

C. "Work Forms"

Many of the interesting and versatile page-forms developed in MATH WORKSHOP texts have been extracted and made available as Ditto and Multilith masters called *Work Forms.* In a few minutes, a teacher can convert one of these forms with pen or pencil into a rich supplementary activity for the

whole class or for a group within the class. Providing, as they do, significant independent learning activities for creative seatwork and meaningful homework, and taking into consideration, as they do, the wide range of individual abilities within any class, the *Work Forms* should prove very useful to busy teachers.

The Supplementary Work Forms Manual itself suggests many ways in which these forms can be used to help develop important ideas (and, at the same time, provide more drill in the fundamental facts). Teachers can create similar activities by adding whatever numbers they want in a matter of minutes.

Countless other uses for the *Work Forms* can be found in the pupil's texts and the teacher's guides to the other levels of the MATH WORKSHOP. The ideas in the particular adaptations found there are easily applied to any level of development.

The *Work Forms* provide the teacher with a flexibility and versatility never before available in an organized and integrated program. While they are not essential to the use of the MATH WORKSHOP texts, they are an invaluable supplement, an effective means of meeting individual needs, and a help to every teacher in expressing his own individual approach to helping pupils understand arithmetic.

D. "Mastery Tests of the 390 Basic Facts of Arithmetic"

These tests are contained in a booklet including a battery of six tests, each of which presents all the basic facts of addition, subtraction, multiplication, and division. An answer sheet is provided so that pupils can check their own or their neighbor's work. In addition, an individual record sheet enables each pupil to chart his own progress in terms of accuracy and time required to recall all the basic facts.

E. "Discovery in Elementary School Mathematics"

This is a professional development textbook which suggests to teachers many ideas for teaching mathematics at every grade level. This book is divided into two major sections—Multilevel Research Activities and Multilevel Worksheet Activities.

Multilevel Research Activities present 27 exploratory experiences which can be introduced on the chalkboard and with simple, teacher-made manipulative materials at any grade level. Each activity starts simply and spirals upward with extensions and elaborations which will appeal to the more mature and sophisticated mind. Each activity focuses on mathematical structure in a learning context of discovery.

Multilevel Worksheet Activities present 24 units, each of which explores some of the unifying ideas of mathematics. Each unit begins with a brief introductory statement regarding mathematical structure and discovery. Each unit includes six activity worksheets, scaled in difficulty so that the

teacher can present the ideas discussed in the unit to pupils at any level. These pages may be copied or modified by teachers for classroom use.

F. "Helping Children Discover Arithmetic"

This is a brief guide to help parents understand the ideas behind MATH WORKSHOP. Specifically, it helps parents see:

1. How we can get children to think creatively in arithmetic by posing challenging questions.
2. How we help children master the basic facts of arithmetic.
3. How parents can help children learn arithmetic.
4. How multiplication, one of the central ideas of arithmetic, is taught from the first through sixth grades in a spiral curriculum.

G. "Games and Enrichment Activities"

We have prepared a manual on the use of representational materials, such as Cuisenaire rods and Geo-boards. These materials can be used as the basis for games and other enrichment activities—many of which are mentioned or introduced in the guide pages. They contain many unusual applications of the concepts considered in the pupil pages.

H. Tests in the Pupil Book

Scattered throughout MATH WORKSHOP, LEVEL C are pages which we suggest you use as tests. These are pages 22, 39, 52, 65, 88, and 99. The guide pages suggest point distributions for these tests.

As with any evaluation techniques for young pupils, care must be taken not to put more faith in tests than they deserve. Your own judgment of a pupil based on your face-to-face work with him is more valid and probably more reliable than scores on tests.

This brings up the general problem of marking or grouping pupils as they work on all of the pages of the text. The pages should be considered opportunities for investigation and exploration on the part of the pupil rather than as pages to mark for a grade. There are many pages on which pupils will not be expected to get 100% on the first try. Nothing would be more damaging to their regard for mathematical inquiry than to be pummeled with red marks and low grades. By all means, keep a light touch rather than a heavy hand when exploring these pages.

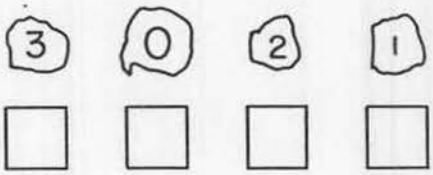
VII. SOME FINAL THOUGHTS

A. Number or Numeral—What Distinctions Shall We Make?

Suppose that you were conducting the following activity with your class and the class was being observed by a visitor who knows nothing about numbers:

"Let's cut out numbers and paste them on a piece of paper in some sensible order. In what order shall we paste them?

"Suppose that these are my cut-out numbers (put a sketch on the board):



How should I paste them down?"

A discussion may ensue: Let's put the smallest number on the left. Which one is that? Zero, of course. Then the one, the two, and the three. The pupils then work alone or in groups, cutting out the various shapes and pasting them on a sheet of paper with the smallest number on the left.

The visitor would be completely mystified by the pupils' paste-ups. Surely, the pupils must be mistaken, because the smallest cut-out is the 2, not the 0.

The pupils are not confused. When you tell these pupils to draw a tree, they know that you expect them to draw a *picture* of a tree. When you tell them to cut out numbers, they know that you expect them to cut out *symbols* for numbers. They know this because they have learned from activities involving displays of objects and one-to-one correspondence that numbers are abstract entities. They know that they can see 3 stars, they can eat 3 candy bars, and they can lift 3 books, but they can't see, feel, hear, smell, or taste the number 3. They can write a name for this number—the simplest name looks a little like a turned-around capital E. They can say this name.

If pupils have a rich background of experience in handling physical objects and comparing displays of such objects, they will not confuse numbers with the symbols (called *numerals*) which are the names of the numbers. The reformers of secondary school mathematics education in the 1950's found it necessary to make students aware of the distinction between numbers and numerals because many of these students came out of elementary school with impoverished backgrounds in arithmetic. Such students were taught a brand of arithmetic which consisted largely of symbol-pushing. Big mathematical ideas for such students were rules about turning fractions upside down and moving decimal points. Whatever understanding they may have developed about the abstract existence of numbers through manipulation of physical objects in the first grade diminished severely in the later grades under an avalanche of symbol-pushing activities.

The authors of MATH WORKSHOP have decided to abide by common usage in which the word "number" has two meanings—the abstraction and the symbol. To help pupils build the kind of understanding which will enable them to pick the correct meaning from context, the authors urge a rich and varied series of activities which involve physical objects. (Of course, we also abide by common usage when we speak of Roman numerals and Arabic numerals.)

B. Punctuation and Order of Operations

In LEVEL C the pupils will sometimes be faced with expressions such as:

$$8 - 2 + 3$$

which appear to be ambiguous. Does this expression stand for 9 or does it stand for 3? That is, should one subtract 2 from 8 and then add 3 to the result, or should one add 3 to 2 and subtract the result from 8?

One way to remove the ambiguity is to punctuate the expression with a pair of parentheses. Thus,

$$(8 - 2) + 3 = 6 + 3 = 9$$

and

$$8 - (2 + 3) = 8 - 5 = 3.$$

A second way to remove the ambiguity is to establish a convention or agreement. The convention we follow in MATH WORKSHOP is that in the case of an expression involving addition or subtraction operations and no parentheses, one should proceed from left to right in carrying out the indicated operations. Thus,

$$8 - 2 + 3 = 6 + 3 = 9$$

and

$$9 + 4 - 1 = 13 - 1 = 12$$

and

$$7 - 2 - 1 - 3 = 5 - 1 - 3 = 4 - 3 = 1.$$

In the case of an expression involving multiplication operations separated by addition or subtraction operations and no parentheses, the convention we follow says that one should perform the multiplications first, and then the additions or subtractions from left to right. Thus,

$$2 \times 3 - 1 = 6 - 1 = 5,$$

$$4 + 3 \times 2 = 4 + 6 = 10,$$

$$3 \times 5 + 7 \times 5 = 15 + 35 = 50,$$

and

$$\begin{aligned} 4 \times 2 + 5 \times 3 - 6 \times 2 &= 8 + 15 - 12 \\ &= 23 - 12 \\ &= 11. \end{aligned}$$

C. Achievement and Intelligence

Makers of I.Q. tests search for measures of the pupil's ability to think. They try to rule out questions that merely assess achievement in subject-matter areas. Therefore, they would not ask pupils to complete the following familiar examples from arithmetic. These questions would more likely appear in an achievement test.

$$\begin{array}{r} 3 \\ +7 \\ \hline \end{array} \qquad \begin{array}{r} 41 \\ -16 \\ \hline \end{array} \qquad \begin{array}{r} 346 \\ \times\ 27 \\ \hline \end{array} \qquad 5\overline{)135} \qquad 8 \times 3 =$$

$$\frac{1}{2} + \frac{2}{3} = \qquad 1.19 = 17\% \times \_\_$$

For measurement of an I.Q. the questions from mathematics would be more like those that follow:

I. Extend each of the following sequences:

a. 2, 5, 8, 11, 14, ___, ___, ___ .....

b. 1, 2, 4, 8, 16, ___, ___, ___ .....

c. 3, 4, 6, 9, 13, ___, ___, ___ .....

d. 1, 1, 2, 3, 5, ___, ___, ___ .....

II. Fill in the blanks in the following examples:

a.

6	3	1	4	2	
(10)	(7)	(5)	( )	( )	(11)

b.

2	4	3	1	5	
↓	↓	↓	↓	↓	↓
4	16	9	___	___	0

c.

1, 3 → 8	3, 2 → 10
5, 0 → 10	6, 3 → ___
2, 5 → 14	, 4 → 12

III. Circle the number that seems most out of place in each row:

a. 6, 10, 2, 7, 8, 4

b. 25, 70, 43, 16, 31, 52

c. 13, 15, 19, 3, 11, 7

Such problems requiring a familiarity with arithmetic are held to be more appropriate as measures of the ability to think. They are designed to test intelligence rather than achievement.

Mathematics is a fertile source for I.Q. test questions because mathematics is fundamentally a study of patterns and relationships. Many of the questions selected are similar to the examples given above. Behind that choice is the test-maker's estimate that pupils have not already explored and investigated such questions.

What an indictment of the elementary school curriculum in mathematics!

If arithmetic is limited to memory of number combinations and rules for manipulation, then it is a dead, dull, dreary task against which an active, eager, creative mind must surely rebel. However, if we broaden our thinking and look everywhere for patterns and relationships, the subject stimulates the mind to inquire and search out and discover!

Every page in MATH WORKSHOP probes beneath the basic facts and processes. We are always exploring the question: Why does arithmetic work? Each activity invites the pupil to ask: What's going on here?—and urges him to find out.

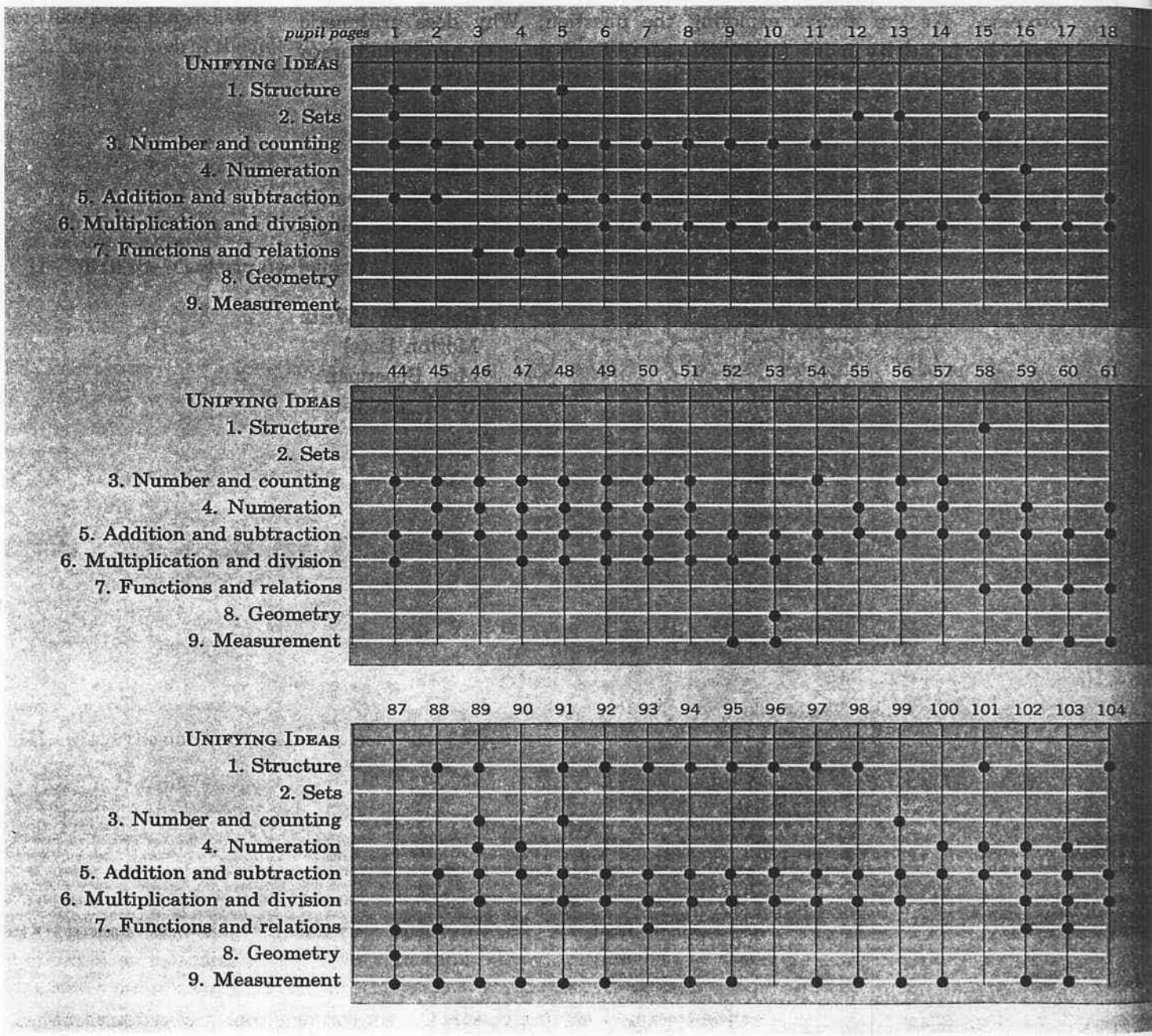
D. A Final Word

The MATH WORKSHOP has not invented a new mathematics. Nor have its authors spun novel theories of education. We have tried to incorporate the most universally accepted ideas of both educators and mathematicians.

Robert W. Wirtz
Morton Botel
Max Beberman
W. W. Sawyer

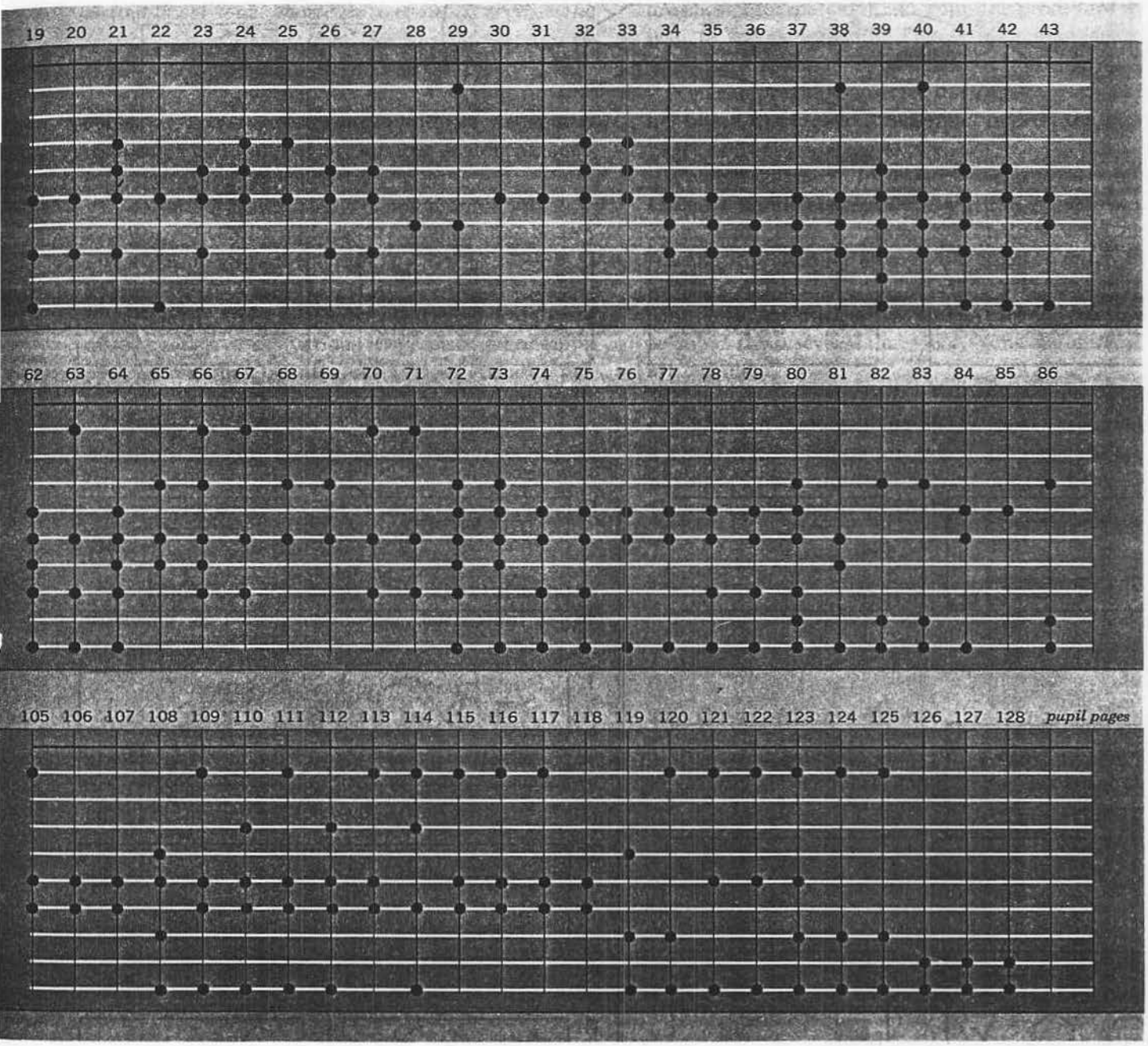
SCOPE AND SEQUENCE CHART

Level C Revised


pupil pages 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
Unifying Ideas
1. Structure
2. Sets
3. Number and counting
4. Numeration
5. Addition and subtraction
6. Multiplication and division
7. Functions and relations
8. Geometry
9. Measurement
44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61
Unifying Ideas
1. Structure
2. Sets
3. Number and counting
4. Numeration
5. Addition and subtraction
6. Multiplication and division
7. Functions and relations
8. Geometry
9. Measurement
87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104
Unifying Ideas
1. Structure
2. Sets
3. Number and counting
4. Numeration
5. Addition and subtraction
6. Multiplication and division
7. Functions and relations
8. Geometry
9. Measurement

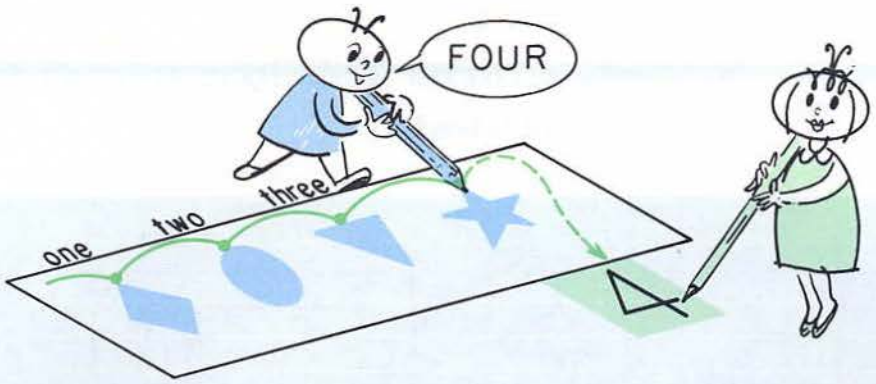
Consider each unifying idea . . . on how many pages will you find it?

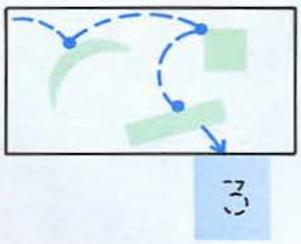
1. Structure	47 pages
2. Sets	4 pages
3. Number and Counting	43 pages
4. Numeration	44 pages
5. Addition and Subtraction	101 pages
6. Multiplication and Division	65 pages
7. Functions and Relations	46 pages
8. Geometry	10 pages
9. Measurement	58 pages



Consider each page . . . how many unifying ideas will you find on it?

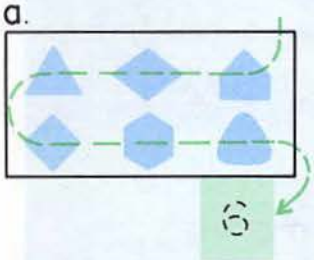
6 pages contain just one unifying idea,
25 pages contain two unifying ideas,
51 pages contain three unifying ideas,
34 pages contain four unifying ideas,
8 pages contain five unifying ideas, and
4 pages contain six unifying ideas.



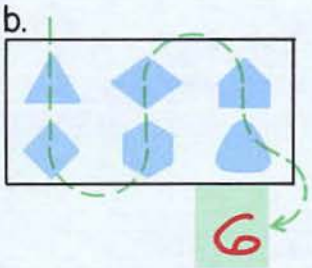


Follow the dotted lines as you count.

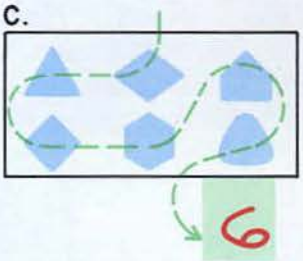
a.



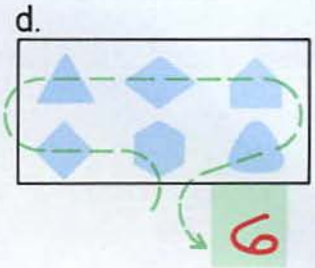
b.



c.



d.



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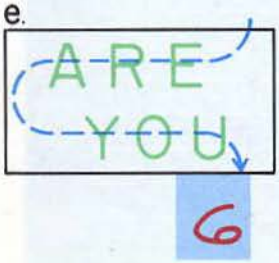

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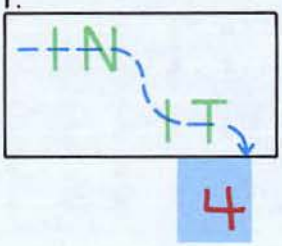

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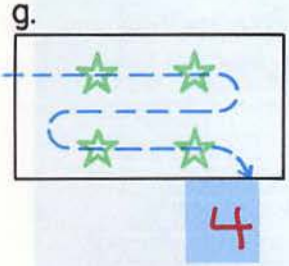

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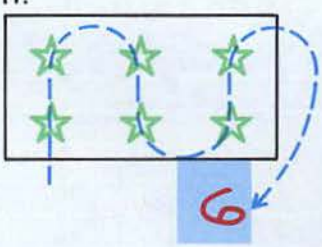

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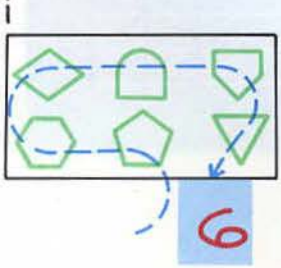

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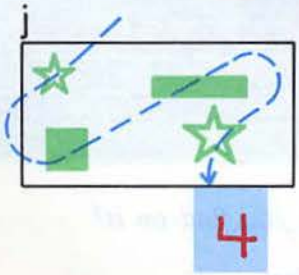
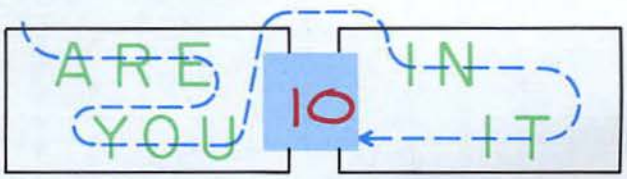
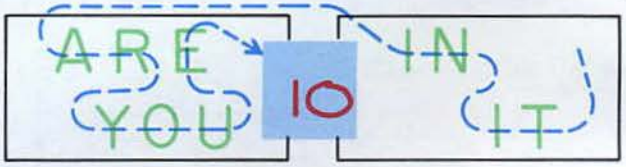


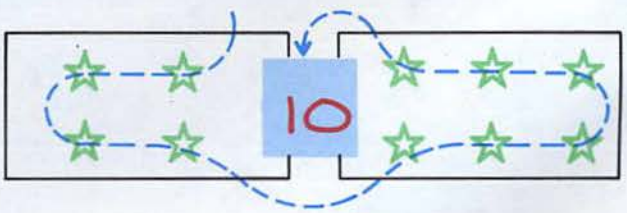
k.



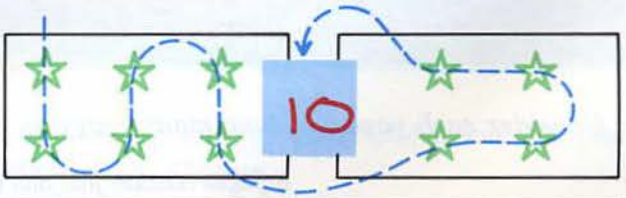
l.



m.



n.



o. 6 + 4 = 10

q. 2 + 3 = 5

s. 4 + 6 = 10

p. 4 + 6 = 10

r. 3 + 2 = 5

t. 6 + 4 = 10

1 Counting designs - simple addition - objects and equations.

Tasks: *Pupils count (in sequences suggested by the dotted lines) or use shortcuts to counting to fill in the blanks in* a-n. *Pupils then use this information to complete the number sentences in* o-t.

Purpose: To review counting, addition (particularly with sums of 10 and of 5), number sentences, commutativity, and associativity.

Unifying Ideas: Structure; sets; number and counting; addition and subtraction.

Materials for Teacher: Demonstration-size tagboard or felt models of the various items pictured on pupil page 1 for flannel board, corkboard, or chalkrail displays; three empty boxes; counters; Ditto masters.

Materials for Pupils: Cuisenaire rods or counters; Ditto sheets.

The Lesson:

Counting the objects in a display amounts to matching, or pairing one-to-one, each word in the sequence

one, two, three, four, five, six, . . .

with an object in the display. The last word used stands for the number of objects in the display. The first word in the sequence can be matched with any object in the display, the second word with any of the remaining objects, etc. No matter how the objects are chosen for matching, the counting procedure will lead to the same result. This is the major idea on pupil page 1.

You can prepare pupils for this page by using flannel board displays like those pictured on the pupil page.

Put ten counters of various shapes and sizes on a table or flannel board and have the pupils count them, each time changing the arrangement. Ask whether the number of counters changed when the arrangement was changed. Have pupils arrange ten counters at their desks in a random way and ask them to count them several times, each time changing the arrangement.

Have pupils place a few of their counters on a sheet of paper and the remainder off the sheet (preferably a Ditto sheet marked for record keeping). Keep a record of how many such separations can be made. (They should find 11 separations.)

On	Off	Number Sentences
2	8	$2 + 8 = 10$
10	0	$10 + 0 = 10$

The Game of Three Boxes

Prepare three boxes each with a different shaped lid—a triangle, a square, and a circle. Distribute ten

more

Headline for the Day: Write one story to fit all the headlines O, P, S, T at the bottom of page 1.

objects among the boxes and let pupils count the objects in two of the boxes and guess the number of objects in the third. Let them keep a record:

$$10 = \triangle + \square + \bigcirc$$
$$10 = 3 + 2 + 5$$
$$10 = 4 + 3 + 3$$
$$10 = 8 + 1 + 1$$

This activity illustrates that a number (in this case, 10) can be partitioned in a variety of ways. When pupils understand that partitioning doesn't affect the total, they have an understanding of the mathematical notions of commutativity and associativity of addition.

Cuisenaire Rods

Let pupils make and compare trains of rods to illustrate such consequences of commutativity and associativity as:

$$2 + 3 = 3 + 2$$
$$4 + 10 = 10 + 4$$
$$(3 + 1) + 7 = 3 + (1 + 7)$$
$$(3 + 1) + 7 = (3 + 7) + 1$$
$$(4 + 2) + (3 + 5) = (3 + 5) + (4 + 2)$$
$$(4 + 2) + (3 + 5) = (4 + 3) + (2 + 5)$$

See pages 31-79 of the booklet MATH WORKSHOP *Games and Enrichment Activities* published by Encyclopaedia Britannica Press.

Making Up Stories from Number Sentences

Put on the chalkboard a mathematical sentence or "headline" such as: $9 + \_\_\_ = 14$. Let a pupil complete the headline: $9 + 5 = 14$. Then ask: Who can make up a story to fit this headline? The response may be slow at first if the pupils have not had previous experience with this gamelike situation. They may begin with examples which are on the dull side, such as: There were 9 boys and 5 girls, or 14 altogether. Or, if a pupil prefers a question form, he might change the story to: There were 9 boys and some girls. There were 14 in all. How many were girls?

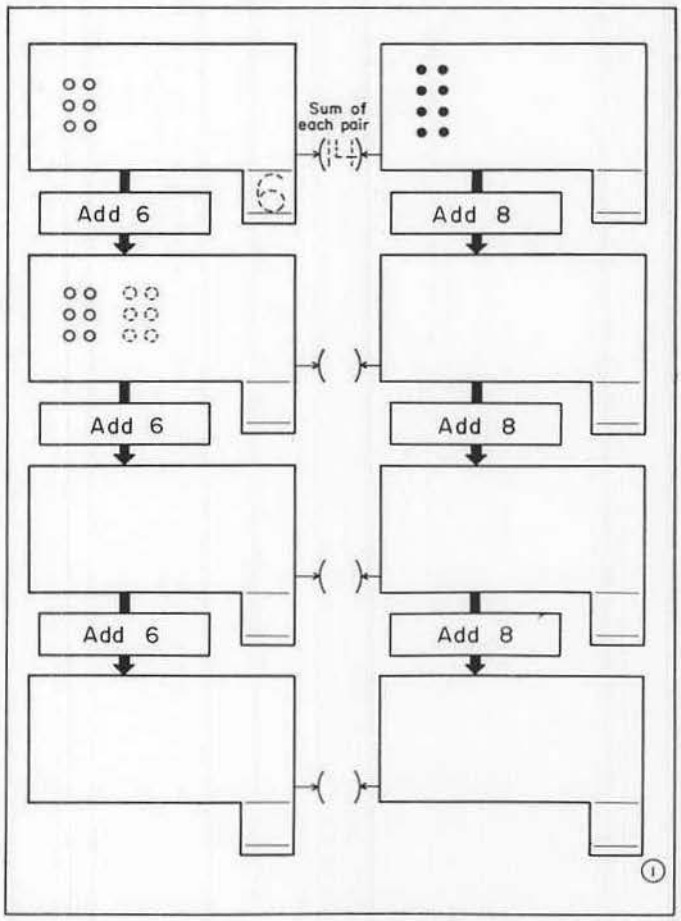
One way to help pupils break loose from parroting stereotyped word problems is to write several beginning sentences or phrases on the board as "thought starters." For example:

$$9 + \_\_\_ = 14$$

a. Harry and Jack captured some men from Mars . . .
b. Alice met the Mad Hatter . . .
c. When the Pilgrims landed . . .
d. In the United Nations . . .
e. In two weeks there are . . .
f. Mary is very fond of . . .
g. Many satellites were circling . . .
h. I was playing hopscotch . . .
i. On a baseball team, there are . . .
j. The telephone rang . . .

Drill

Consider using as a drill sheet page 1 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



See section VI on page 31 of the Introduction.

You may also wish to include among drill activities some "translation" work in which a pupil is given a story or a description of a situation and is then asked to write *at least one* mathematical sentence which could serve as a headline for the story. Here are some examples of stories and headlines which pupils might write:

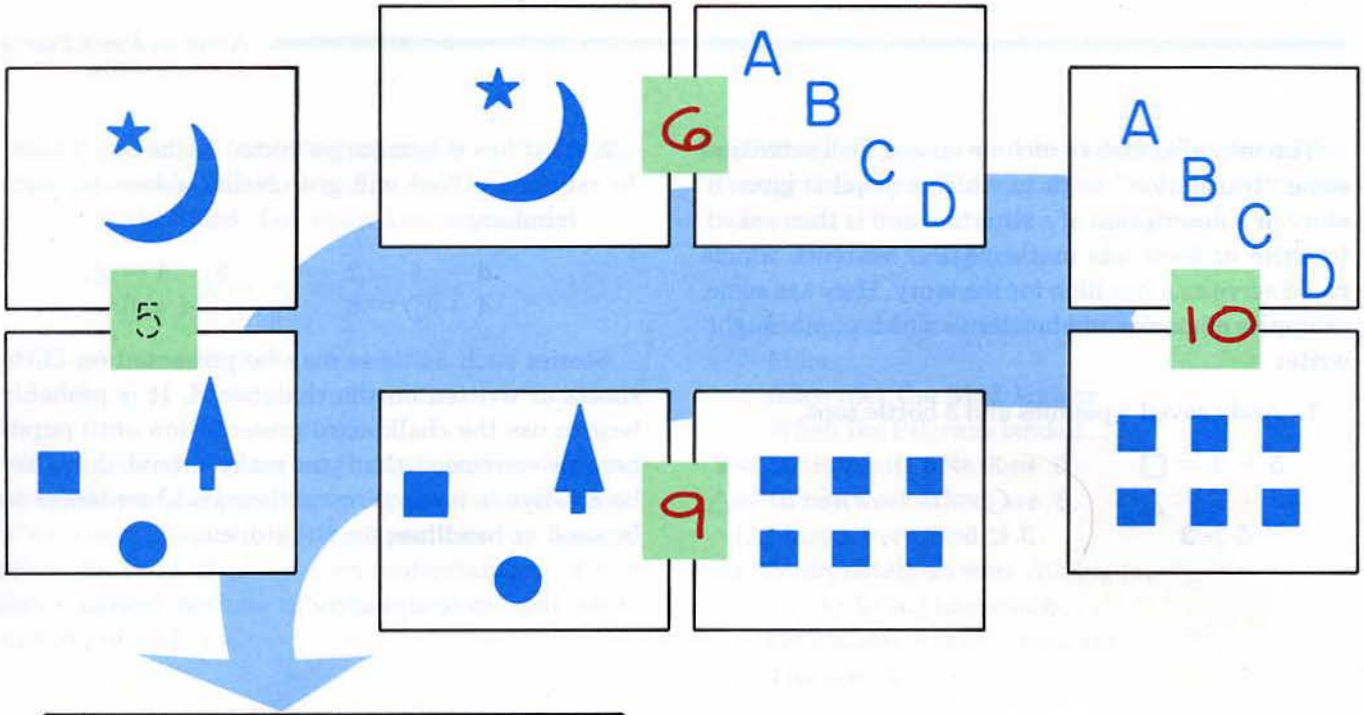
1. Andy saved 5 pennies and 3 bottle tops.

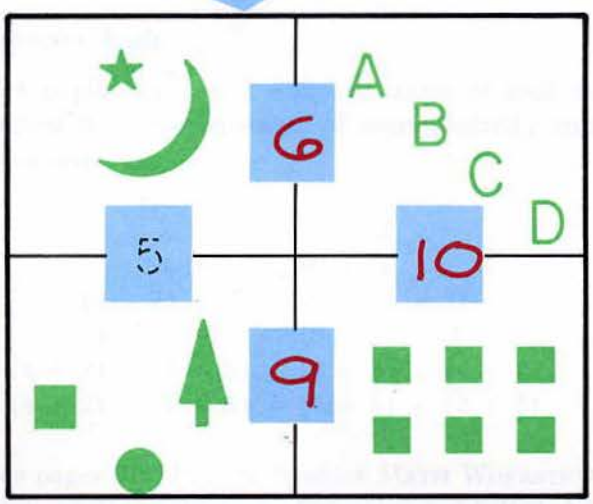
$5 + 3 = \square$ $\quad$ $5 + 3 = 8$ $\quad$ $5 - 3 = 2$
$3 + 5 = \triangle$ $\quad$ $3 + \bigcirc = 5$ $\quad$ $5 - 3 = \triangle$
$5 > 3$ $\quad$ $3 < 5$

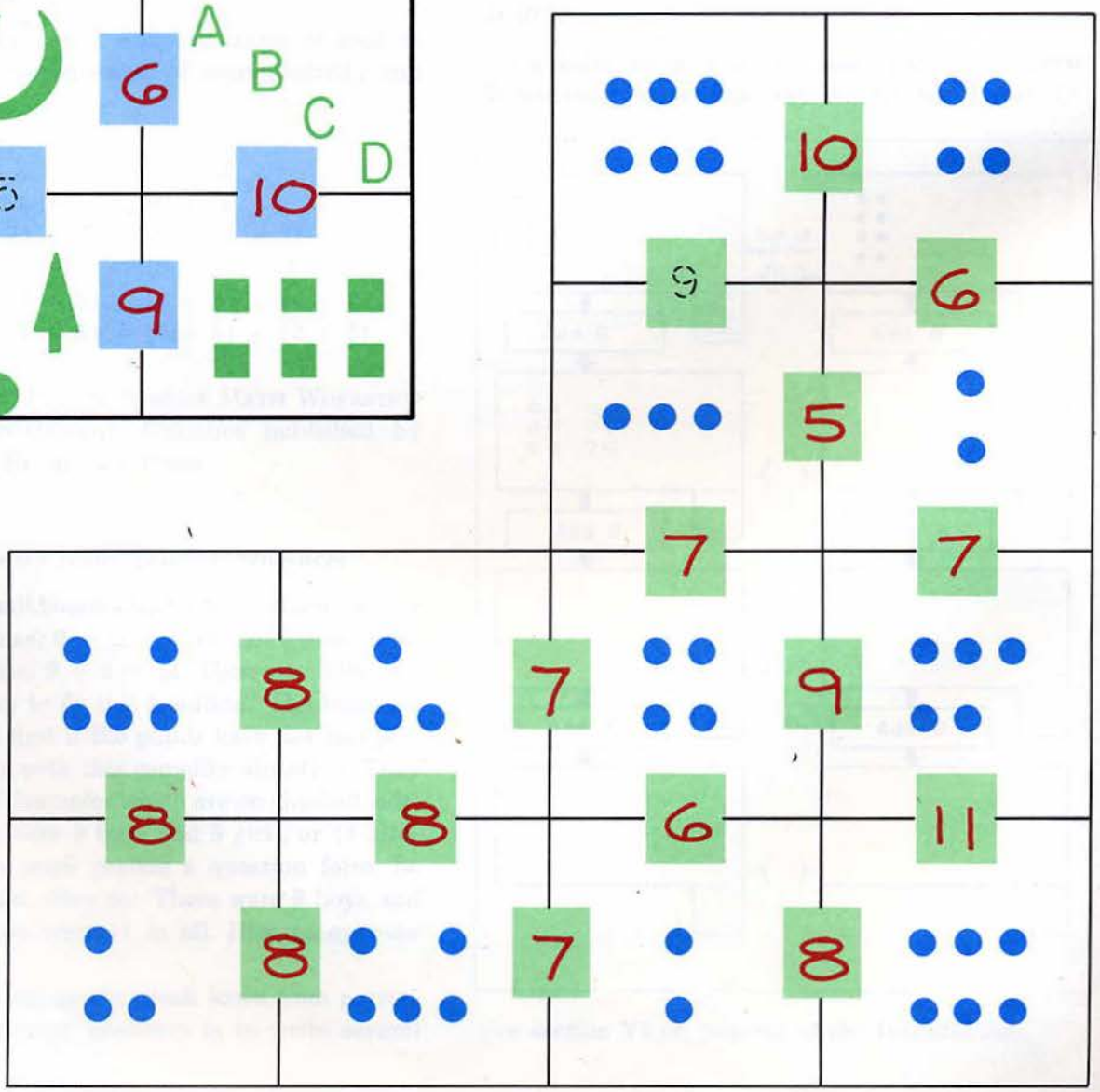
2. Ned has 6 hamburger buns. Nellie has 4 hamburgers. Ned will give Nellie a bun for each hamburger.

$6 - 4 = 2$ $\quad$ $6 - 4 = \triangle$
$4 + \bigcirc = 6$ $\quad$ $4 < 6$

Stories such as these may be presented on Ditto sheets or written on the chalkboard. It is probably best to use the chalkboard presentation until pupils become convinced that you really intend that they be creative in composing mathematical sentences to be used as headlines for the stories.


A
B
C
D
6
5
9
10


A
B
C
D
6
5
10
9


10
9
6
5
7
7
8
7
9
8
8
6
11
8
7
8

2 Counting designs - simple addition.

Tasks: *Pupils record the number of objects displayed in each pair of adjacent boxes. (Note that the four pairs of adjacent boxes at the top of the page are combined to form the 4-cell table at the left.)*

Purpose: To review counting and addition.

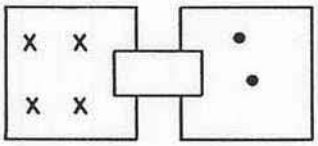
Unifying Ideas: Structure; number and counting; addition and subtraction.

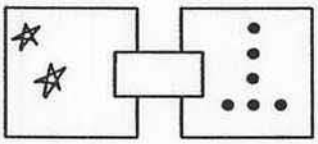
Materials for Teacher: Flash cards (as described below); Ditto masters.

Materials for Pupils: Counters; Ditto copies of the sketches described below.

The Lesson:

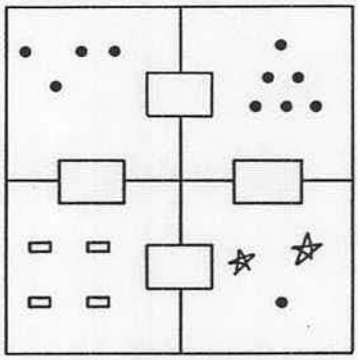
This page provides another kind of counting, adding, and recording experience. You may wish to prepare pupils for this page in the following way. Draw the following sketches on the chalkboard and on Ditto sheets:



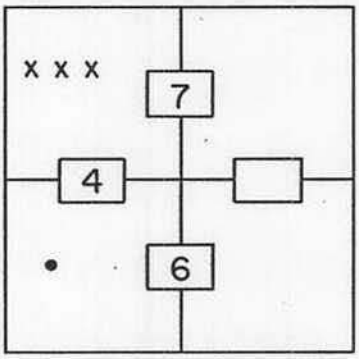


Point to the box with the ×'s and ask: How many ×'s? Then point to the adjacent box and ask: How many dots? Then ask: How many altogether? Four plus two is .... Write a 6 in the box in the middle. Pupils follow your lead by writing on their Ditto sheets.

Now point to the middle box of the second pair of adjacent boxes (stars and dots), and offer the chalk to the class. Invite a pupil to come up and fill in the missing number. When the class grasps the idea, use more complex sketches like this one:



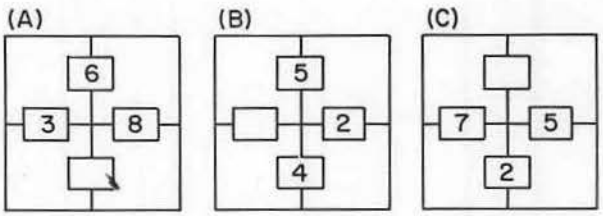
Now do reconstruction problems in which pictures are to be drawn and numbers recorded:



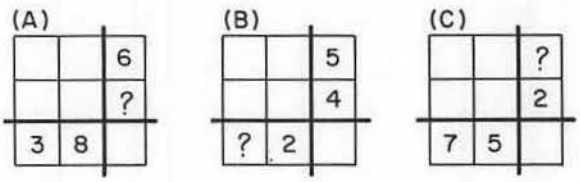
more

Headline for the Day: Jody bought some blue pencils and some green pencils. He bought 12 pencils altogether. What can you say about Jody's purchase? Make a table.

As a final touch, challenge pupils to fill in the missing numbers in sketches such as the following *without* drawing any pictures in the boxes:



These problems are equivalent to those which require a pupil to find a missing row-sum or column-sum in a *Cross Number Puzzle* (see pages 18-23 of the Introduction):



The total number of objects displayed in the four boxes is the sum of the column-sums. It is also the sum of the row-sums. Hence, the missing number is easy to find. Answers: (A) 5, because $6 + 5 = 3 + 8$; (B) 7, because $7 + 2 = 5 + 4$; (C) 10, because $10 + 2 = 7 + 5$.

Pupils who understand what is involved in the foregoing problems have a grasp of the idea that the order and the grouping in addition can be changed without changing the total.

Flash Card Activities

We describe, below, flash card activities which are useful for diagnostic and remedial work.

Flash cards are a time-tested device to help develop the ability to recall without hesitation the 100 basic facts of addition and the 100 basic facts of subtraction.

The use of flash cards as the *only* device to accomplish this job is not wise. Their use as punishment is self-defeating—a resisting mind is not a receptive mind.

Flash cards are most effectively used in a gamelike atmosphere. In a race against time the learner competes with himself.

A valuable use of flash cards is a diagnostic one. They can be used to locate the particular facts that are most troublesome. Therefore, the set of cards used ought to include all 200 facts.

Flash cards may be made with 3″ by 5″ cards. On one side of a card the fact is indicated by an open sentence: $5 + 3 = \square$. On the back of the same card, the complete statement is given: $5 + 3 = 8$.

In preparing a set, you may find the following check list helpful:

Second Number

First Number	0	1	2	3	4	5	6	7	8	9
0	a	b	c	d						
1										
2										
3										
4										
5										
6										
7										
8										
9										

a ✓ $0 + 0 =$ ___
b ✓ $0 + 1 =$ ___
c ✓ $0 + 2 =$ ___
d ✓ $0 + 3 =$ ___

Each cell points out one of the basic facts of addition. To come up with the total list of facts, simply complete the table, moving across the cells horizontally. Start with the 0-row and make cards such as $0 + \bigcirc = \_\_\_$, $0 + 1 = \square$, $0 + 2 = \triangle$, etc.

For each addition fact, such as: $3 + 5 = 8$, there is a subtraction fact: $8 - 5 = 3$.

Used diagnostically, an appropriate group of flash cards (consisting of facts pupils have already recognized and considered) are mixed up thoroughly—and the game begins.

As the pupil responds, separate the cards into two piles—those he answers without hesitation and those he does not call up automatically.

The second group delimits the pupil's problem. Separate this group into two groups as he goes through them again—those on which he only hesitates and those which he gets wrong or must "figure out." On the first group, the pupil needs a little practice; on the second, he needs to work hard. To help him master the second group, follow the procedures shown on the next page.

(a) For each fact, be sure that he draws a diagram or manipulates counters, and then reasons to a correct conclusion.

(b) Make supplementary cards for each of these facts, using as many different forms of the fact as you have time to make. Suppose that the pupil has trouble with: $3 + 9 = \square$. The following forms are recommended:

3 + 9 = □	12 − 9 = □
3 + □ = 12	12 − □ = 3
□ + 9 = 12	□ − 9 = 3
9 + 3 = □	12 − 3 = □
9 + □ = 12	12 − □ = 9
□ + 3 = 12	□ − 3 = 9

and

3 + 9	3 + 12	+ 9 12	9 + 3	9 + 12	+ 3 12
12 − 9	12 − 3	− 9 3	12 − 3	12 − 9	− 3 9

This great variety of arrangements makes the pupil think through the question in many related ways and thus reinforces his memory. Furthermore, the variety deepens his understanding of important relationships among the facts of arithmetic. By using many forms of a question you help keep the practice lively instead of allowing it to become dull and monotonous. From time to time that group of the 200 basic facts with which the pupil is having trouble ought to be presented in diagnostic fashion, working each time to eliminate more and more facts which have already been committed to memory, and focusing more and more sharply on those facts which still give trouble.

Important: Remember that this drill is most effective when it is interspersed with dissimilar activities and when a gamelike aspect is maintained.

Mastery Tests

Pupils can be tested on the 100 facts of addition and the 100 facts of subtraction by using Ditto sheets like those shown in the following columns.

Example: 2,1 | 3 / 2,3 | 5

ADD

1.	9,4	1,7	7,5	5,7	3,1
2.	5,8	4,9	1,3	7,0	7,8
3.	8,4	2,7	0,6	0,4	6,5
4.	2,3	8,5	4,6	9,3	1,9
5.	4,1	1,2	5,9	3,2	5,2
6.	9,1	0,8	8,2	4,2	9,2
7.	1,0	5,4	2,0	8,3	3,7
8.	0,5	3,3	8,7	3,8	4,7
9.	6,6	1,1	2,5	6,3	9,5
10.	2,2	4,4	6,2	3,9	7,6
11.	0,3	0,0	7,9	9,9	5,3
12.	7,4	5,1	0,2	2,1	8,6
13.	1,5	9,0	1,8	2,6	2,9
14.	6,8	1,4	9,8	6,0	0,9
15.	7,7	7,2	4,0	3,0	3,5
16.	5,0	4,8	8,1	4,5	8,9
17.	4,3	8,8	0,1	7,3	2,4
18.	9,6	7,1	6,1	0,7	3,6
19.	5,6	6,4	1,6	5,5	6,7
20.	6,9	2,8	8,0	9,7	3,4

Example: 0 | 1 | 1 / 2 | 3 | 5 / 2 4

ADD

a	0 5 / 7 1	b	2 8 / 6 5	c	0 1 / 1 1
d	2 1 / 0 9	e	5 9 / 7 2		
f	9 8 / 9 6	g	4 1 / 0 8	h	9 0 / 1 2
i	3 6 / 8 5	j	5 7 / 9 3		
k	4 3 / 4 5	l	4 8 / 6 3	m	3 9 / 7 4
n	1 7 / 4 0	o	2 4 / 2 3		
p	3 5 / 3 2	q	1 5 / 6 0	r	6 1 / 0 3
s	6 8 / 6 7	t	7 9 / 7 6		
u	5 2 / 5 4	v	4 7 / 9 2	w	8 7 / 8 9
x	8 0 / 1 3	y	2 6 / 8 4		

ADD

Example:

	1	4
2	3	6
3	4	7

A

	2	5	1	6	4	
5						5
7						7
1						1
6						6
4						4
	2	5	1	6	4	

B

	3	0	8	7	9	
5						5
7						7
1						1
6						6
4						4
	3	0	8	7	9	

C

	2	5	1	6	4	
2						2
0						0
8						8
9						9
3						3
	2	5	1	6	4	

D

	3	0	8	7	9	
2						2
0						0
8						8
9						9
3						3
	3	0	8	7	9	

SUBTRACT

Example: $\begin{array}{r} 5 \\ -2 \\ \hline 3 \end{array}$

A

$\frac{0}{0}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{17}{8}$	$\frac{8}{2}$
$\frac{12}{6}$	$\frac{6}{6}$	$\frac{9}{8}$	$\frac{12}{4}$	$\frac{13}{8}$
$\frac{8}{1}$	$\frac{13}{7}$	$\frac{10}{9}$	$\frac{4}{0}$	$\frac{8}{3}$
$\frac{11}{2}$	$\frac{8}{7}$	$\frac{2}{2}$	$\frac{9}{6}$	$\frac{7}{4}$
$\frac{8}{0}$	$\frac{10}{5}$	$\frac{4}{2}$	$\frac{13}{4}$	$\frac{5}{1}$

B

$\frac{8}{4}$	$\frac{11}{9}$	$\frac{13}{6}$	$\frac{15}{7}$	$\frac{9}{0}$
$\frac{10}{8}$	$\frac{9}{1}$	$\frac{7}{7}$	$\frac{14}{9}$	$\frac{7}{2}$
$\frac{10}{6}$	$\frac{1}{0}$	$\frac{11}{5}$	$\frac{5}{3}$	$\frac{7}{5}$
$\frac{9}{3}$	$\frac{14}{7}$	$\frac{4}{1}$	$\frac{5}{0}$	$\frac{17}{9}$
$\frac{3}{3}$	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{14}{8}$	$\frac{11}{4}$

C

$\frac{11}{3}$	$\frac{9}{9}$	$\frac{12}{5}$	$\frac{12}{9}$	$\frac{9}{2}$
$\frac{6}{4}$	$\frac{11}{8}$	$\frac{10}{1}$	$\frac{6}{0}$	$\frac{14}{6}$
$\frac{3}{1}$	$\frac{5}{2}$	$\frac{8}{6}$	$\frac{10}{4}$	$\frac{6}{3}$
$\frac{8}{5}$	$\frac{10}{7}$	$\frac{4}{4}$	$\frac{15}{8}$	$\frac{14}{5}$
$\frac{2}{0}$	$\frac{18}{9}$	$\frac{12}{3}$	$\frac{15}{9}$	$\frac{7}{1}$

D

$\frac{6}{1}$	$\frac{16}{9}$	$\frac{11}{6}$	$\frac{10}{3}$	$\frac{12}{7}$
$\frac{7}{6}$	$\frac{1}{1}$	$\frac{12}{8}$	$\frac{9}{4}$	$\frac{7}{0}$
$\frac{13}{9}$	$\frac{3}{0}$	$\frac{13}{5}$	$\frac{2}{1}$	$\frac{7}{3}$
$\frac{6}{2}$	$\frac{11}{7}$	$\frac{8}{8}$	$\frac{16}{7}$	$\frac{5}{4}$
$\frac{15}{6}$	$\frac{10}{2}$	$\frac{16}{8}$	$\frac{5}{5}$	$\frac{9}{5}$

ADD

Example: $\begin{array}{r} 2+1 \\ \hline 3 \end{array}$

A

3+4	9+7	8+0	2+8	6+9
5+6	6+4	1+6	5+5	6+7
3+6	0+7	6+1	7+1	9+6
4+3	8+8	0+1	7+3	2+4
8+9	4+5	8+1	4+8	5+0

B

7+7	7+2	4+0	3+0	3+5
0+9	6+0	9+8	1+4	6+8
1+5	9+0	1+8	2+6	2+9
8+6	2+1	0+2	5+1	7+4
0+3	0+0	7+9	9+9	5+3

C

7+6	3+9	6+2	4+4	2+2
6+6	1+1	2+5	6+3	9+5
4+7	3+8	8+7	3+3	0+5
1+0	5+4	2+0	8+3	3+7
9+2	4+2	8+2	0+8	9+1

D

4+1	1+2	5+9	3+2	5+2
1+9	9+3	4+6	8+5	2+3
8+4	2+7	0+6	0+4	6+5
7+8	7+0	1+3	4+9	5+8
9+4	1+7	7+5	5+7	3+1

SUBTRACT

Example: $\begin{array}{r} 4-3 \\ \hline 1 \end{array}$

A

9-6	4-0	14-5	11-9	7-3
10-2	5-3	8-1	12-6	16-9
1-1	3-2	17-8	2-1	6-3
13-8	9-0	12-9	9-9	8-6
12-3	6-1	6-0	7-2	5-1

B

14-9	11-4	13-4	0-0	8-4
11-7	8-2	15-7	9-1	10-9
15-8	16-7	7-7	10-7	7-0
8-8	15-9	11-2	18-9	5-5
3-1	13-6	14-7	6-4	9-8

C

10-1	4-2	8-7	5-4	6-6
4-3	11-6	1-0	9-5	10-8
9-4	14-8	6-5	13-9	10-3
2-0	10-6	4-1	10-5	12-4
13-5	5-2	12-7	4-4	8-3

D

2-2	16-8	13-7	17-9	7-5
12-5	7-6	11-8	3-0	12-8
15-6	8-5	9-7	9-2	7-4
5-0	7-1	14-6	9-3	11-5
6-2	11-3	3-3	10-4	8-0

Example: 3,1 | 2 / 3,2 | 1 — SUBTRACT

1.	4,3		3,1		5,2		12,5		9,3	
2.	0,0		6,5		11,5		8,0		11,6	
3.	8,7		5,3		7,4		12,8		12,7	
4.	2,1		8,6		11,8		10,1		6,6	
5.	9,4		2,2		7,5		11,7		14,8	
6.	13,9		9,7		2,0		10,3		6,1	
7.	1,1		4,2		13,4		12,9		9,9	
8.	7,6		8,4		10,9		10,6		5,4	
9.	8,2		1,0		9,2		5,5		13,8	
10.	9,8		14,6		11,4		8,5		4,0	
11.	10,5		11,3		10,7		5,1		13,5	
12.	7,1		15,7		4,1		14,7		7,2	
13.	17,8		8,1		9,6		4,4		13,7	
14.	5,0		10,8		7,0		6,4		12,4	
15.	3,2		3,3		18,9		10,2		9,5	
16.	15,6		16,8		6,3		13,6		11,2	
17.	12,3		10,4		15,9		3,0		12,6	
18.	16,7		16,9		9,1		7,3		9,0	
19.	7,7		6,0		15,8		11,9		14,9	
20.	17,9		14,5		8,8		6,2		8,3	

Example: 4-2 / 2 — SUBTRACT

A

9-8	8-2	7-6	3-3	13-9
11-3	15-7	8-1	10-8	4-4
5-2	11-5	7-4	11-6	7-5
10-7	4-1	9-6	7-0	18-9
9-3	11-6	12-7	2-2	14-8

B

9-4	2-1	8-7	0-0	4-3
16-8	10-4	16-9	6-0	14-5
2-0	13-4	10-9	9-2	11-4
6-3	15-9	9-1	15-8	5-5
6-1	9-9	5-4	13-8	4-0

C

13-5	7-2	13-7	12-4	9-5
5-1	14-7	8-8	6-4	10-2
12-5	8-0	12-8	10-1	11-7
3-1	6-5	5-3	8-6	1-1
10-5	7-1	17-8	5-0	3-2

D

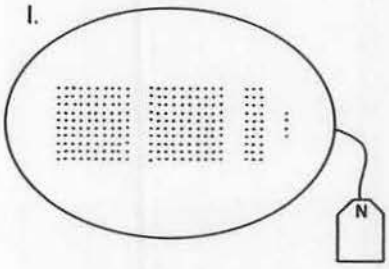
11-2	12-6	9-0	14-9	8-3
13-6	3-0	7-3	11-9	6-2
10-3	12-9	10-6	6-6	8-5
9-7	4-2	8-4	1-0	14-6
15-6	12-3	16-7	7-7	17-9

Drill

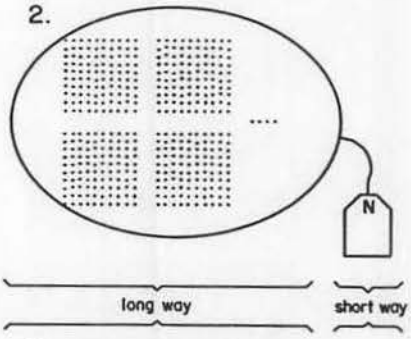
Here are suggestions for some simple drill work on place value:

	Long way	Short way
0.	3 tens and 4 ones	34
1.	2 tens and 3 ones	___
2.	5 tens and 2 ones	___
3.	___ tens and . . . ones	38
4.	1 ten and 5 ones	___
5.	6 tens and 6 ones	___
6.	2 tens and 0 ones	___
7.	___ tens and . . . ones	50
8.	___ tens and . . . ones	98
9.	___ tens and . . . ones	89
10.	8 tens and 1 one	___

1.



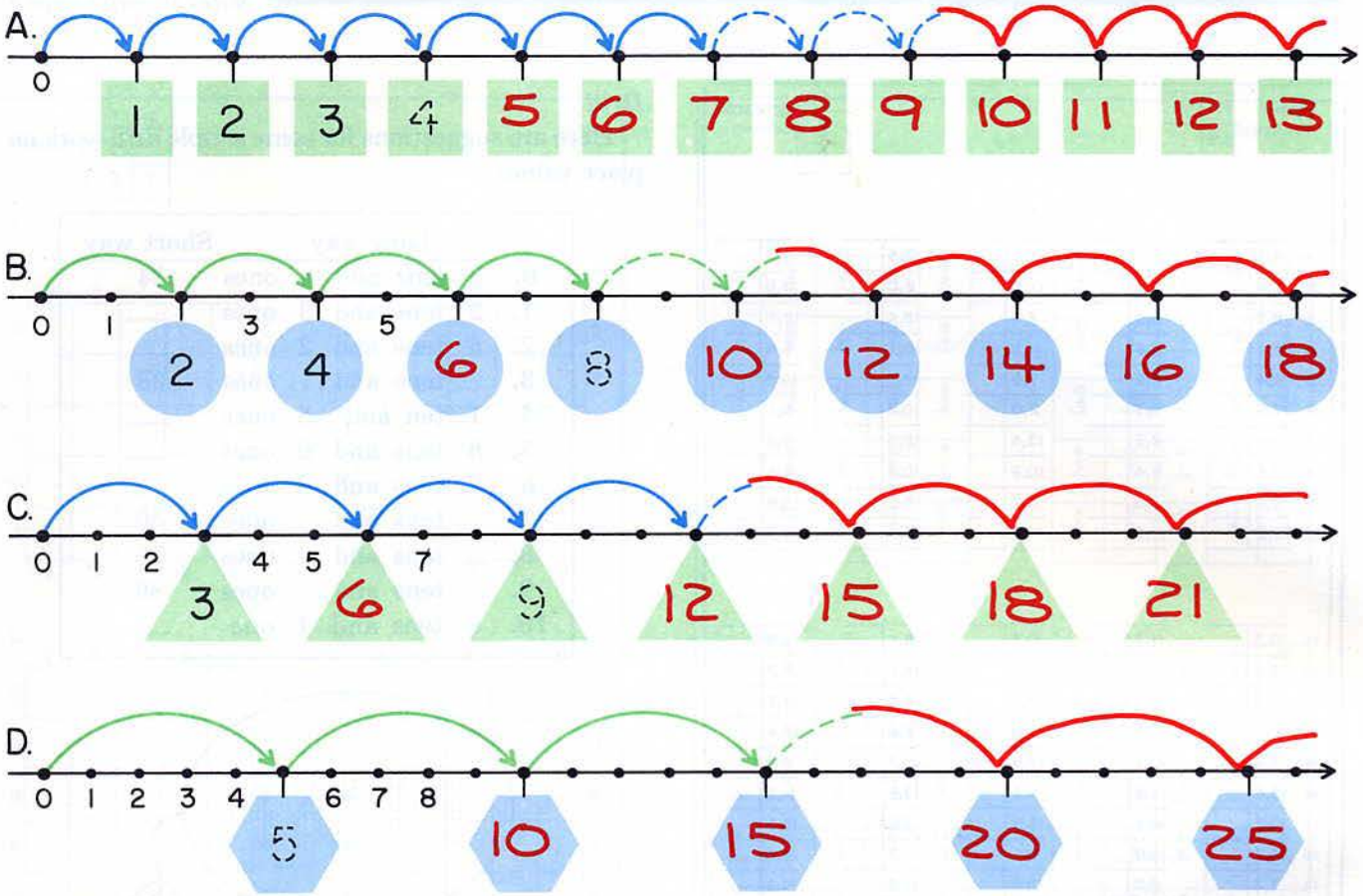
2.

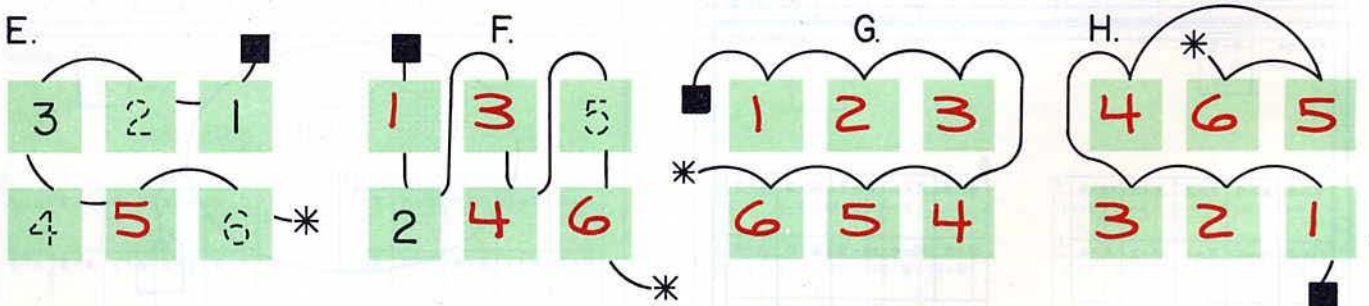


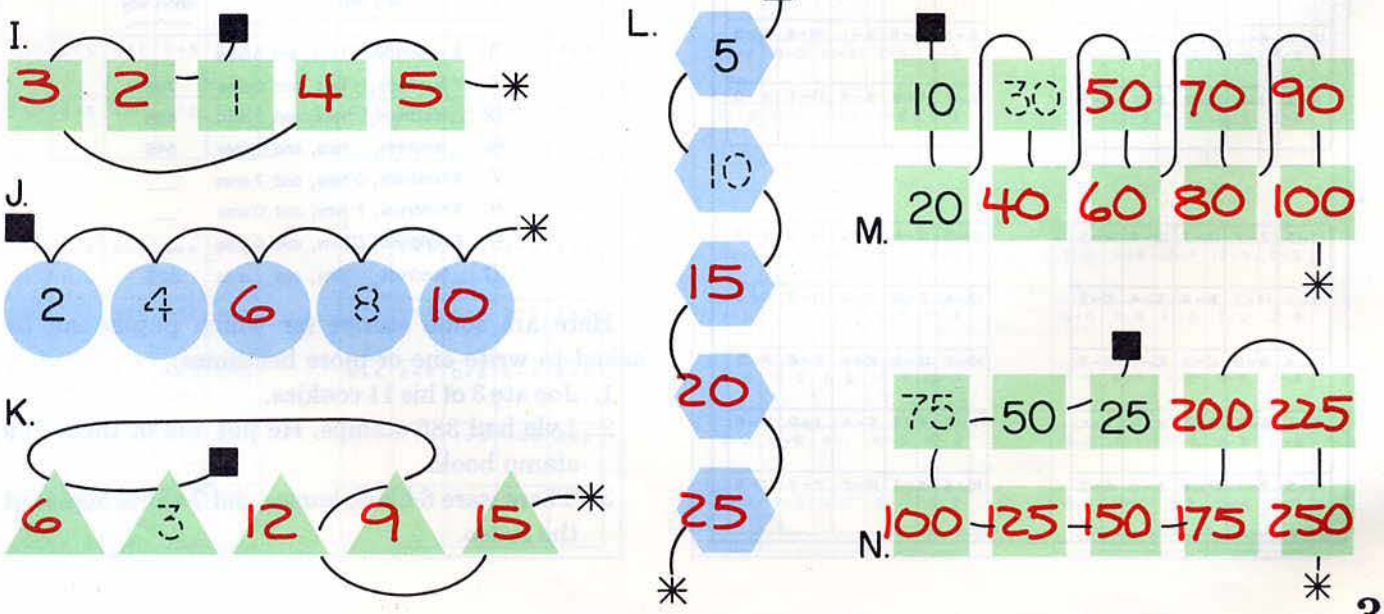
3.	5 hundreds, 6 tens, and 3 ones	___
4.	7 hundreds, ___ tens, and 9 ones	789
5.	___ hundreds, ___ tens, and ___ ones	456
6.	___ hundreds, ___ tens, and ___ ones	546
7.	3 hundreds, 0 tens, and 7 ones	___
8.	3 hundreds, 7 tens, and 0 ones	___
9.	8 hundreds, 0 tens, and 0 ones	___
10.	___ hundreds, ___ tens, and ___ ones	222

Here are some stories for which pupils can be asked to write one or more headlines:

1. Joe ate 3 of his 11 cookies.
2. Lyle had 385 stamps. He put 153 of them in a stamp book.
3. There were 6 black horses and 7 white horses at the rodeo.


A.
0
1 2 3 4 5 6 7 8 9 10 11 12 13
B.
0 1 3 5
2 4 6 8 10 12 14 16 18
C.
0 1 2 4 5 7
3 6 9 12 15 18 21
D.
0 1 2 3 4 6 7 8
5 10 15 20 25


E.
3 2 1
4 5 6
F.
1 3 5
2 4 6
G.
1 2 3
6 5 4
H.
4 6 5
3 2 1


I.
3 2 1 4 5
J.
2 4 6 8 10
K.
6 3 12 9 15
L.
5
10
15
20
25
M.
10 30 50 70 90
20 40 60 80 100
N.
75 50 25 200 225
100 125 150 175 250

3 Number line jumps - counting by l's, 2's, 3's, 5's, l0's, 25's.

Tasks: *Pupils use the "jump" clues provided to label the required landing points on the number lines in A-D. For the exercises in the lower half of the page, pupils should complete each counting sequence in accordance with the indicated path. Note that exercise A corresponds with exercises E-I, B with J, C with K, and D with L.*

Purpose: To review counting by 1's, 2's, 3's, 5's, 10's, and 25's.

Unifying Ideas: Number and counting; functions and relations.

Materials for Teacher: Two empty boxes; small objects; masking tape (for number line chunks); Ditto masters; colored chalk.

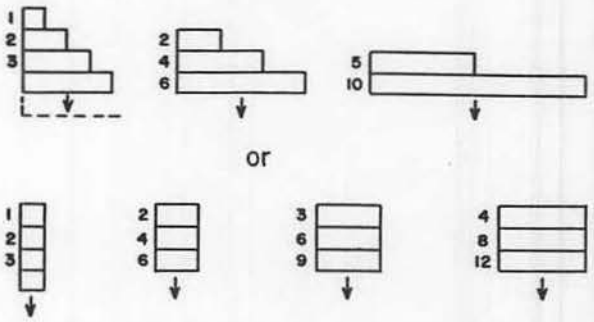
Materials for Pupils: Cuisenaire rods; Ditto sheets.

The Lesson:

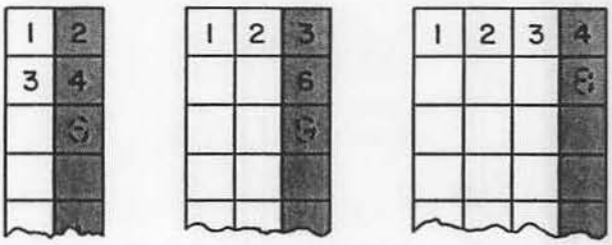
Throughout MATH WORKSHOP you will note a recurring emphasis on pattern. Pupils explore situations in which items are connected to each other logically. They will find the beginnings of patterns and their job is to use the given clues to extend or complete the patterns. Quite frequently such pages can be initiated by doing "silent arithmetic" (see the booklet *Games and Enrichment Activities*, pages 1-5 and 9-11) while you are working at the board. At other times pupils may be encouraged to try to complete the page on their own with no instructions from you. Perhaps pupil page 3 is one of the pages that you may wish to ask your pupils to try without instructions.

You may prefer to introduce the page (or follow it up) with any of these activities:

1. Using Cuisenaire rods or measured strips, have pupils build models and extend the patterns shown at the top of the next column.

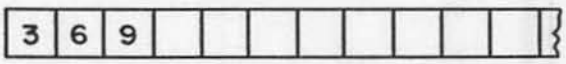


2. Have pupils number shaded areas only:



3. Have pupils color in every second (third, fourth, etc.) number block to form a pattern. The example shown below is for 3. Such colorings make interesting "wallpaper" patterns.

Number Blocks Colored



Another good way to introduce this page and almost any other in the book is to project a transparency of the page on the chalkboard or a screen.

Number Lines

In designing pupil page 3, we have assumed that pupils are already familiar with number lines from their work in an earlier grade. If this is not the case for your pupils, you will need to carry out some preliminary work. Use a cardboard strip to mark the

Headline for the Day: Connie took some jumps of the same size starting from 0 on a number line. She stopped jumping when she landed on 36. What can you say about the size of each jump Connie took?

0-point and the 1-point, and use two cardboard strips (of the same length) to mark the 2-point. It is essential that pupils understand the significance of the 2-point—the distance between the 0-point and the 2-point represents the length of a 2-car train, etc. Make up some exercises on a Ditto sheet which are like those on pupil page 3.

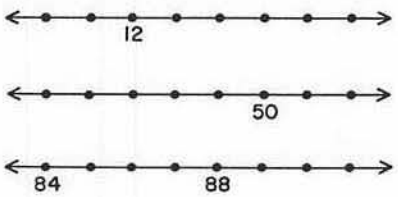
Play some games of counting forward and counting backward similar to the following:

From the number 10 count backward two numbers; then count forward three numbers.

From 60, count backward five numbers; then forward five numbers.

What are the numbers between 30 and 40?

Draw some number line chunks on the board (or make them out of masking tape) similar to the following and discuss naming the points:

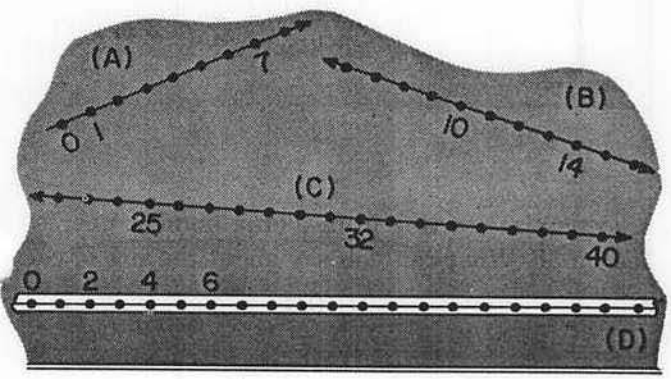


For some extremely simple number line activities, see pages 1-4 and 13 of the booklet *Games and Enrichment Activities*. Be sure to modify the activities suggested there by including a marked point for 0 on each number line drawn.

When the foregoing preliminary activities have been completed (or in case your pupils are already familiar with number lines) you may wish to carry out the following additional preparatory activities.

Activity 1

Display several number line chunks on the chalkboard; A, B, and C in chalk, and D on masking tape:

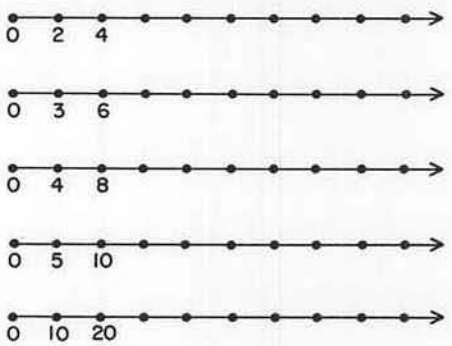


Select volunteers to complete the labeling of number lines A, B, and C. For number line D proceed as follows. Start at the 0-point and label the first few multiples of 2 in red chalk — 0, 2, 4, and 6. Offer the red chalk to a volunteer who should label the next few multiples of 2. Have him pass the red chalk to another classmate to continue the labeling a few more steps, etc. What numbers did we use? (The even numbers.) Now, using green chalk, label points 1, 3, and 5 in that order. Offer the green chalk to a volunteer to label the next of the unlabeled points. Have him pass the green chalk to another classmate to continue the game. What numbers did we use this time? (The odd numbers.)

Erase the board. Who remembers all the numbers we wrote in red chalk? List them on your paper. Billy, read your list. Make a list of the numbers we wrote in green. Jane, read your list to the class.

Activity 2

Distribute Ditto sheets showing the following and ask pupils to label the marked points:



Activity 3

Draw a number line on the chalkboard, label the points for the even numbers, and show the connection between a jump and an addition sentence. Start with jumps of 2 or 4 and with even numbers as starting points. For example, draw a jump from 4 to 6 and write the sentence: $4 + 2 = 6$. Then write the sentence: $6 + 4 = 10$. Ask a pupil to draw the jump. (Encircle the labels for the landing points.) Change to jumps from the odd-numbered points to the even-numbered points and from odd to odd.

Activity 4

Draw a number line on the chalkboard and present the compound sentence:

$$9 + 5 = \square \quad \text{and} \quad \triangle + 9 = 14$$

To find the missing sum, one starts at 9, jumps 5 to the right, and lands on 14. Then, to find the missing addend (in the second clause of the sentence), one starts at 14 and jumps 9 to the left. A pupil who is aware of the fact that addition is commutative should be able to predict the landing point for the second jump. Continue with other compound sentences such as:

$3 + 7 = \square$ and $\triangle + 3 = 10$
$4 + 11 = \square$ and $\triangle + 4 = 15$
$13 + 5 = \square$ and $5 + \triangle = 18$

How Many Ways?

Use two boxes—one red and one blue—and fourteen objects.

In how many ways can we arrange all fourteen objects in the boxes so that in each case there are at least ten objects in one of the boxes?

The condition that there must be at least ten objects in one of the boxes must be kept in mind.

As the different arrangements are made, a record is kept and appropriate mathematical sentences are noted:

	Red	Blue	
Mary	10	4	$10 + 4 = 14$
Bill	2	12	$2 + 12 = 14$
Jane	12	2	$12 + 2 = 14$
Al	0	14	$0 + 14 = 14$
Fred	4	10	$4 + 10 = 14$
Sue	13	1	$13 + 1 = 14$
Tom	14	0	$14 + 0 = 14$
Jim	11	3	$11 + 3 = 14$
Gene	1	13	$1 + 13 = 14$
Sarah	3	11	$3 + 11 = 14$

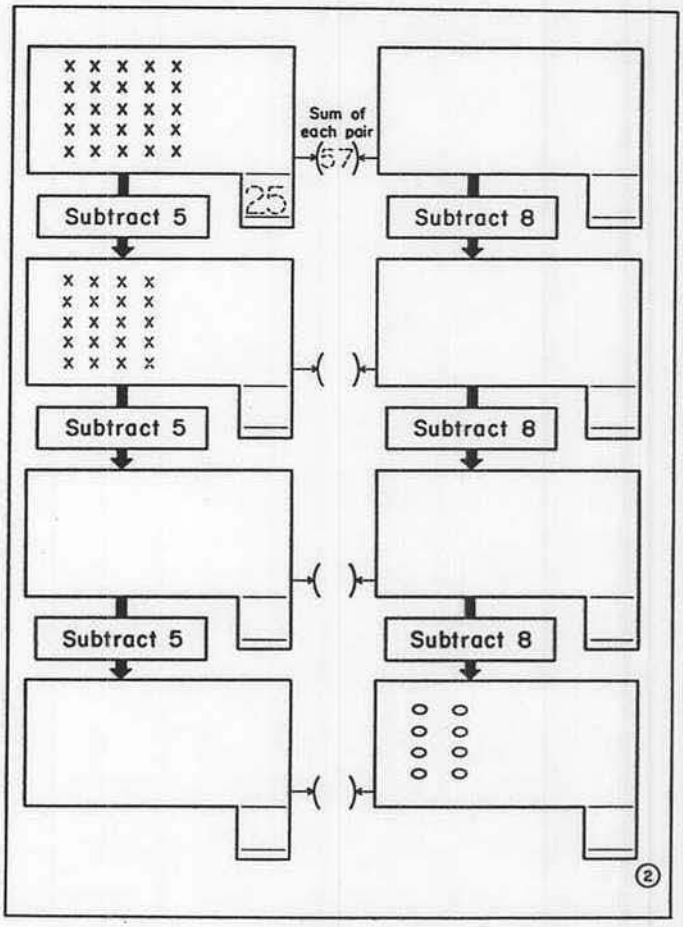
Is that all? Can you suggest a plan for making the arrangements that would help us be sure that we haven't missed a possibility?

Can there be the same number of objects in each box under the conditions of this problem?

Can one box have 10 more objects than the other? (Yes) . . . 1 more? (No) . . . 2 more? (No) . . . 3 more? (No) . . . 4 more? (No) . . . 5 more? (No) . . . 6 more? (Yes) . . . 7 more? (No) . . . 8 more? (Yes) . . . 9 more? (No) . . . 11 more? (No) . . . 12 more? (Yes) . . . 13 more? (No) . . . 14 more? (Yes)

Drill

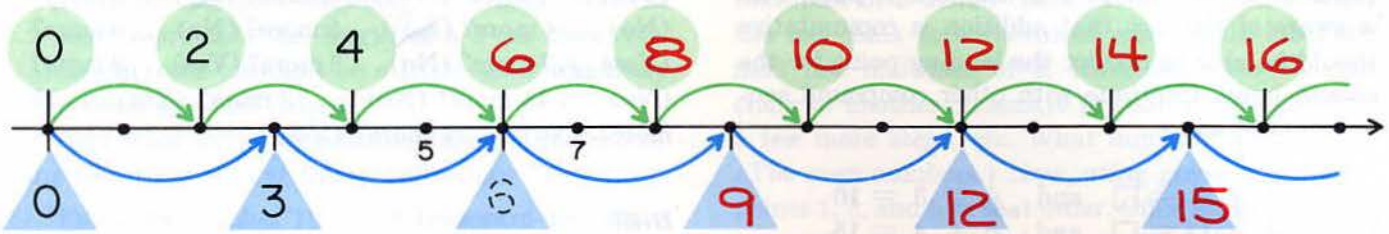
Consider using as a drill sheet page 2 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



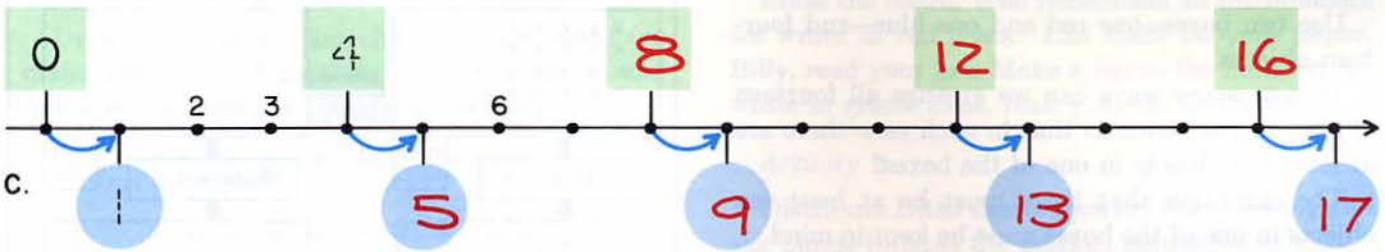
Here are some stories for which pupils can be asked to write one or more headlines:

1. Tami had 6 cents. She spent 3 cents.
2. Sam sold 10 of his 17 newspapers.
3. There were 56 cookies baked for a party at Fran's house. Her guests ate 37 of these cookies.

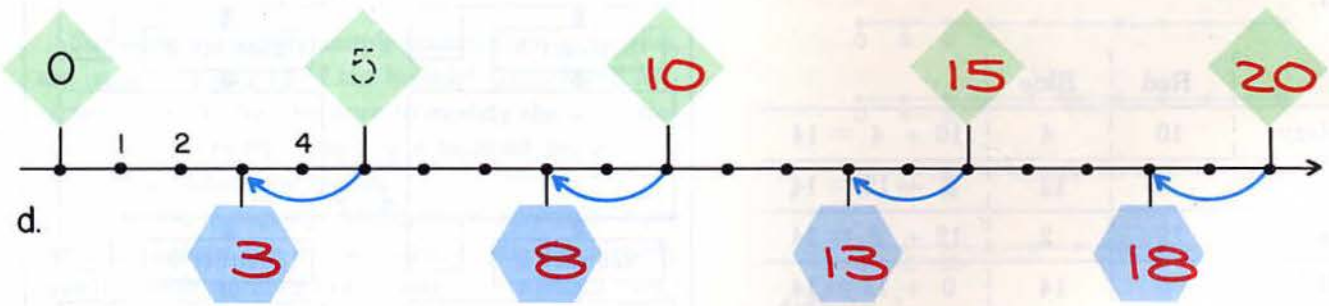
a.



b.



c.



d.

e. 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20

f. 20, 18, 16, 14, 12, 10, 8, 6, 4, 2, 0

g. 30, 27, 24, 21, 18, 15, 12, 9, 6, 3, 0

h. 0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50

i. 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21

4 Number line jumps - counting by l's, 2's, 3's, 4's, 5's-simple addition (+1) and subtraction (-2) on the number line.

Tasks: *Pupils use the clues provided to continue each sequence as required.*

Purpose: To review counting by 1's, 2's, 3's, 4's, and 5's.

Unifying Ideas: Number and counting; functions and relations.

Materials for Teacher: Masking tape for number line chunks.

Materials for Pupils: Cuisenaire rods or counters if needed.

The Lesson:

There are three aspects of the activities on this page which bear comment.

1. In rows *b*, *c*, and *d* the number patterns above and below the lines are related and pupils should be helped to observe this relationship through such questions as: In line *b*, do the 2-jumps and 3-jumps ever land on the same points? (Yes: 6, 12) If we continued the line, where would they land together next? (18) After that? (24) Record the points: 6, 12, 18, 24, ___.

2. In lines *c* and *d* the curved arrows suggest patterns. These patterns are easily discovered if we think of a curved arrow as representing the words "goes to." Thus in *c* 0 goes to 1, 4 goes to 5, 8 goes to ___, etc. What would 20 go to? 24? 28? etc. In *d* 5 goes to 3, 10 goes to 8, etc. What would 25 go to? 30? 35? etc.

A table listing these moves is helpful in discovering the relationship. Pupils should be asked to extend the pattern another five to ten steps. Use Cuisenaire rods or counters to help pupils who don't discover these relationships even after studying the tabulations.

c	*d*
0 → 1	5 → 3
4 → 5	10 → 8
8 → 9	15 → 13
12 → 13	20 → 18
___ →	___ → ....
___ →	___ → ....

3. In exercises *e* to *i* we have increasing sequences and decreasing sequences. If pupils need help with them, draw number line chunks on the chalkboard or on a permanent chart. Have pupils use their fingers or chalk to demonstrate the patterned jumps.

more

Headline for the Day: Add the numbers listed in exercise *e* on page 4 to the corresponding numbers in exercise *i*. What can you say about the sums of these numbers?

When the page is completed, ask: How are rows *e* and *f* related? Does the pattern in row *g* look familiar? Does anyone know the special names for the numbers in rows *e* and *i*? (Even numbers; odd numbers.)

Making Up Stories from Number Sentences

In MATH WORKSHOP we encourage pupils to write or tell real or fanciful stories that contain the conditions suggested by number sentences. The numbers themselves often generate ideas. Put several open mathematical sentences on the board:

a. $12 - \_\_ = 8$	*b.* $7 + 7 = \_\_$
c. $\_\_ - 25 - 10$	*d.* $100 + 25 = \_\_$
e. $\_\_ = 11 + 11$	*f.* $\_\_ + 100 = 300$

Say to the pupils: First, let's complete these sentences, and then we shall select one as an appropriate headline for a story:

a. $12 - 4 = 8$	*b.* $7 + 7 = 14$
c. $15 = 25 - 10$	*d.* $100 + 25 = 125$
e. $22 = 11 + 11$	*f.* $200 + 100 = 300$

Now make up a story about one of the headlines.

The relation between the facts of arithmetic and story problems is not obvious to pupils. They need help in determining from a series of words the conditions that can be expressed in a mathematical sentence or formula.

As pupils exercise their ability to make up stories to fit headlines, they begin to understand what goes on in the minds of others who make up story problems. The mysteries disappear. Once they can make up stories, pupils gain confidence that they can reverse the process. They begin to understand the relation between life situations and mathematical sentences. Teachers who have used this approach find that pupils become enthusiastic about being allowed to express a creativeness they never associated with arithmetic.

Often, word problems present difficulties because the pupil's reading ability is not yet sufficiently developed. He struggles with reading the words; he is unsure; he is puzzled. In such a state of mind, he is not prepared to extract a mathematical problem from words that are already a reading problem. The method outlined above—from headlines to stories—shows the connection between a mathematical idea expressed in arithmetical symbols and a life situation expressed in words. A pupil need not read at all to gain confidence in this area. Then, when he must go the other way—from stories to mathematical sentences—he is better equipped to find the cues he needs. He has tackled two very different problems but he has tackled them one at a time.

Drill

Here is a suggestion for a drill sheet:

Watch the Signs!

$\begin{array}{r} 3 \\ +\ 1 \\ \hline \bigcirc \end{array}$	$\begin{array}{r} 6 \\ -\ 5 \\ \hline \bigcirc \end{array}$	$\begin{array}{r} 5 \\ +\square \\ \hline 7 \end{array}$	$\begin{array}{r} 5 \\ -\bigcirc \\ \hline 3 \end{array}$	$\begin{array}{r} \triangle \\ +\ 5 \\ \hline 6 \end{array}$	$\begin{array}{r} \bigcirc \\ -\ 2 \\ \hline 2 \end{array}$
$2 + 3 = \bigcirc$		$3 + \triangle = 7$		$\square + 8 = 8$	
$6 - 2 = \bigcirc$		$5 - \square = 2$		$\triangle - 1 = 4$	
$\begin{array}{r} 3 \\ -\triangle \\ \hline 1 \end{array}$	$\begin{array}{r} \square \\ -\ 5 \\ \hline 0 \end{array}$	$\begin{array}{r} 5 \\ +\ 3 \\ \hline \square \end{array}$	$\begin{array}{r} \bigcirc \\ +\ 6 \\ \hline 10 \end{array}$	$\begin{array}{r} 2 \\ +\bigcirc \\ \hline 4 \end{array}$	$\begin{array}{r} 4 \\ -\ 3 \\ \hline \triangle \end{array}$
$6 - \bigcirc = 2$		$\square + 6 = 9$		$4 + \triangle = 8$	
$\square - 2 = 5$		$7 + 5 = \bigcirc$		$4 - 1 = \square$	
$\begin{array}{r} 7 \\ +\square \\ \hline 10 \end{array}$	$\begin{array}{r} 6 \\ -\bigcirc \\ \hline 3 \end{array}$	$\begin{array}{r} \triangle \\ -\ 0 \\ \hline 8 \end{array}$	$\begin{array}{r} 7 \\ +\ 8 \\ \hline \square \end{array}$	$\begin{array}{r} 6 \\ -\square \\ \hline 6 \end{array}$	$\begin{array}{r} 5 \\ +\ 7 \\ \hline \bigcirc \end{array}$
$\square + 4 = 13$		$9 - \triangle = 7$		$7 - 4 = \bigcirc$	
$4 - \triangle = 1$		$8 + \bigcirc = 12$		$\square + 9 = 11$	
$\begin{array}{r} \square \\ -\ 5 \\ \hline 4 \end{array}$	$\begin{array}{r} 3 \\ -\square \\ \hline 3 \end{array}$	$\begin{array}{r} \bigcirc \\ +\ 9 \\ \hline 13 \end{array}$	$\begin{array}{r} 10 \\ -\square \\ \hline 4 \end{array}$	$\begin{array}{r} 3 \\ +\ 8 \\ \hline \triangle \end{array}$	$\begin{array}{r} 2 \\ +\square \\ \hline 9 \end{array}$

Here are some stories for which pupils can be asked to write one or more headlines:

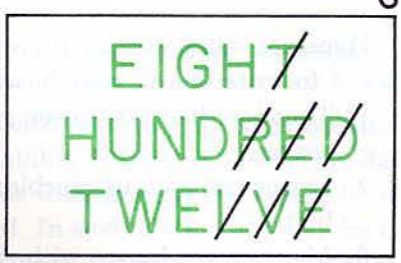
1. Theodore had fifty-seven cents. He spent fifty cents.
2. Lester has 83 blue marbles and 75 green marbles.
3. Maria saw 6 horses with silver saddles and 10 horses with brown leather saddles.
4. Miss Costanza bought forty-three chocolate cookies and twenty-eight vanilla ice-cream bars for the third-grade party.
5. Lawrence sold limeade and lemonade. He made seven quarts of limeade and six pints of lemonade.
6. A boy planted eight evergreen trees. Three of the trees died.

Look for clues that suggest what you are to do.

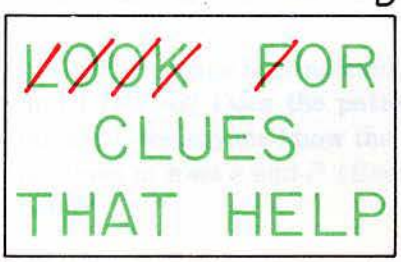
A

B

C

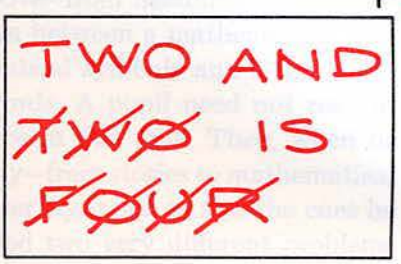


D *



E *

F *



* MANY POSSIBLE ANSWERS

	A	B	C	D	E	F
Total number of letters	8	11	18	20	10	15
Letters crossed out	3	5	7	5	4	7
Letters not crossed out	5	6	11	15	6	8

What Are My Rules?

I. 26, 24, 22, 20, 18, 16, 14, 12, 10, 8, 6, 4

II. 33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 13, 11

III. 39, 36, 33, 30, 27, 24, 21, 18, 15, 12, 9, 6

IV. 44, 41, 38, 35, 32, 29, 26, 23, 20, 17, 14, 11

V.

9	12	8	11	7	23	19	13	20	40	21	70	d
7	4	3	9	6	3	19	7	10	1	4	2	e
2	8	5	2	1	20	0	6	10	39	17	68	d-e

5 Counting letters in words - simple addition and subtraction - What Are My Rules?

Tasks: *Pupils use the clues provided in the boxes A-F or in the chart below the boxes to record the required entries. For the* What Are My Rules? *games, pupils extend each sequence as required after discovering the rules of formation. For V at the bottom of the page, pupils use the clues provided by the first, third, and sixth columns (or the pattern indicators at the right) to record the appropriate entries.*

Purpose: To review concepts of addition and subtraction.

Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations.

Materials for Teacher: Number line chunks; Ditto masters.

Materials for Pupils: Ditto sheets; Cuisenaire rods for activities described in pages 38-62 of the booklet *Games and Enrichment Activities.*

The Lesson:

The objective of the top half of pupil page 5 is for pupils to see the relationship between the display of letters in boxes A-F and the entries on the chart below the boxes. Looked at in a certain way, each display of letters represents a subtraction situation. For example, the display in box A is a representation of the fact that $8 - 3 = 5$. Obviously, recognizing the words has nothing to do with the problem. The record is concerned with counting the total number of letters, those crossed out, and those not crossed out.

Of course, pupils should be helped to see, before this lesson is completed, that these same pictures can be interpreted as suggesting other members of a fact team. In A, for example, all of the following fit:

$3 + 5 = 8$ $\qquad$ $8 - 3 = 5$
$5 + 3 = 8$ $\qquad$ $8 - 5 = 3$

So do the "reverse" statements shown at the top of the next column.

$8 = 3 + 5$ $\qquad$ $5 = 8 - 3$
$8 = 5 + 3$ $\qquad$ $3 = 8 - 5$

Chalkboard Activity 1

If you find that pupils cannot go directly to the page and work on their own, you may wish to conduct some chalkboard activities like those on pupil page 5, but using the idea of addition:

I | HE IS / HERE

II | BILL RAN / FAST

III | ———

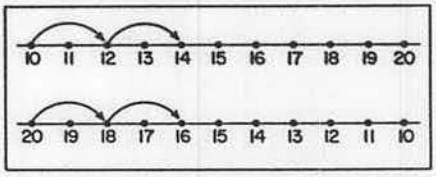
IV | ———

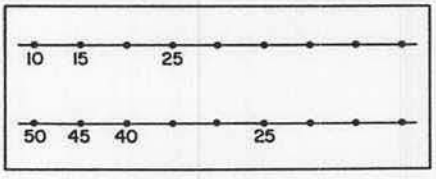
	I	II	III	IV	V	VI
Letters above the line	4					
Letters below the line	4		5	0		
Total letters		11	8			

Pupils can include their own words in III and IV from the clues given, and in V and VI they can draw additional boxes or imagine the words and boxes.

Chalkboard Activity 2

Draw number line segments and begin jumps or labeling as in the examples below. Offer the chalk to pupils to continue the patterns.





Headline for the Day: Cindy had 26¢ on Monday. She spent 2¢ each day for bubble gum. Make a chart to show how much money Cindy had left each day until she spent all her money.

Chalkboard Activity 3

2	8	5	2			10			a
7	4	3	9	20	0				b
9	12			23	19	20			$a + b$

The letter labels on the right are to be regarded as pattern indicators (or row names) and no special attention need be called to them unless pupils wish to discuss them.

You may wish to follow up this page with further, similar examples which you can quickly prepare on Ditto masters, or with such exercises as the following:

Enrichment Activity 1

Put the missing operation sign in the shaded block to make the sentence into a true one.

$2 \square 3 = 5$	$10 = 5 \square 5$
$5 \square 3 = 2$	$5 = 10 \square 5$
$5 = 3 \square 2$	$4 \square 3 = 7$
$4 = 2 \square 2$	$7 = 3 \square 4$
$4 \square 2 = 2$	$3 = 7 \square 4$
$8 \square 3 = 5$	$7 \square 3 = 4$
$8 = 5 \square 3$	$6 \square 1 = 5$
$3 \square 5 = 8$	$5 \square 1 = 6$
$3 = 8 \square 5$	$1 = 6 \square 5$

Enrichment Activity 2

Tell whether the sentences are true or false.

	T	F		T	F
$1 > 2$		F	$1 + 1 = 2$		
$2 = 1$			$2 = 3 - 1$		
$2 = 2$			$1 + 1 > 2$		
$4 = 4$			$2 < 1 + 2$		
$4 > 3$			$4 = 2 + 2$		
$3 < 4$			$4 > 2 + 3$		
$6 = 6$			$4 = 2 + 3$		
$6 > 7$			$2 + 3 > 4$		
$7 > 7$			$3 - 2 = 5$		
$8 > 7$			$6 = 3 - 3$		

Enrichment Activity 3

Pairs of Numbers is an activity that requires thinking, and does not involve large numbers. A notation that may be new for many pupils is used in asking a question that involves pairs of numbers. For example:

$$\square + \triangle = 4$$

This is an open sentence. The square and triangle are to be considered as "reserved seats" for pairs of whole numbers. If 1 approaches the square, and if 1's partner is 3, and they both sit down, they make the sentence into a true one because $1 + 3$ is 4. Another pair that would make the sentence into a true one is $(0, 4)$. A record of pairs with sum 4 can be made as follows:

	A	B	C	D	E	F
$\square$	1	0	3			
$\triangle$	3	4	1			

The pairs mentioned above have been recorded under A and B in the chart. Under C another pair has been recorded. Are there more? Yes, 4 for the square and 0 for the triangle; also, the pair $(2, 2)$. Five pairs, so far. Are there more pairs of whole numbers which can make the sentence into a true one? No!

This activity does not involve much computation but the questions it raises are not trivial. Pupils are given an opportunity to record the suitable pairs in a pattern that makes it easy to answer the question: Are there any more pairs of whole numbers that can make the sentence into a true one?

0 and 4	3 and 1
1 and 3	4 and 0
2 and 2	

The following open sentence may look like the previous one but the results are very different:

$$\square - \triangle = 4$$

Pairs of whole numbers which satisfy this sentence—that is, which turn it into a true sentence—include $(4, 0)$, $(10, 6)$, and $(5, 1)$. Are there more? Yes. And if the pairs were to be arranged in a pattern, it would become clear that it is possible to find an unlimited number of pairs for these reserved seats which satisfy the open sentence:

4 and 0	7 and 3	10 and 6
5 and 1	8 and 4	11 and 7
6 and 2	9 and 5	etc.

After pupils have become proficient at finding pairs of numbers which satisfy open sentences such

as: $\square = \triangle + 3$, $\square + \bigcirc + 1 = 12$, and: $\square - 2 > \triangle$, introduce a variation by giving them an open sentence that contains three "holes" or reserved seats, two of which are the same shape and size. For example: $\triangle + \square + \square = 15$.

The rule here is that whatever number sits in one of the squares must also sit in the other square. If 1 sits in the triangle and 7 sits in each square, the sentence is turned into a true one. We say that the pair of whole numbers (1, 7) satisfies the open sentence. Are there any more pairs of *whole* numbers that satisfy this open sentence? Yes; here is a complete list:

△, □
1, 7
3, 6
5, 5
7, 4
9, 3
11, 2
13, 1

Note well that the rule which says that frames of the same shape and size must hold places for the same number does not imply that frames of different shape or size must hold places for different numbers. In the example just considered, one of the pairs of numbers that satisfy the open sentence is (5, 5). Also, in the very first example, we found that the pair (2, 2) satisfied the open sentence.

From time to time we shall make further suggestions for *Pairs of Numbers* activities.

Enrichment Activity 4

Much practice on the basic facts of arithmetic can be concentrated in a small space in exercises like the following:

0		1		
1				
2		3		
3	3			
+	0	1	2	3

8				
6				
4				
2				
+	2	4	6	8

		10		
			10	
				10
	2			
+	1	3	5	7

Change the orientation in the grid:

+			4
	4		8
			6
	0	2	

+	7		
8		17	19
		19	
	19		

	17	16	+
19			
20		18	
10			

Change the operation in the grid:

15	10		
10	5		
5	0		
−	5	10	15

−			

			−

What Are My Rules?

The game of *What Are My Rules?* is played frequently in MATH WORKSHOP. (See pages 15-18 of the Introduction.) It is fundamentally a discovery game in which pupils search for patterns and extend the patterns according to discovered rules. We use this game whenever we wish to tell pupils to do something without having to explain in words. *What Are My Rules?* tells them to look for clues in the arrangement of pictures and numbers which indicate what needs to be done. This requires time for reflection, thinking, and study.

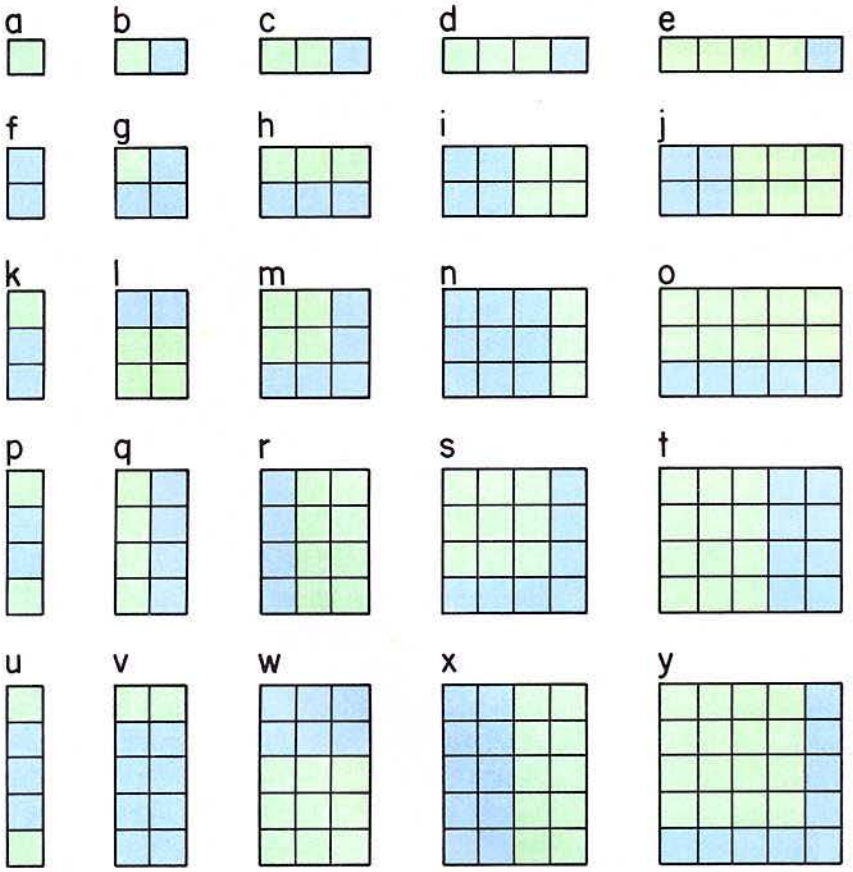
There are two *What Are My Rules?* games on pupil page 5. Both suggest the idea of subtraction as did the exercises at the top of the page. As we shall continue to point out, do not tell pupils how to do these exercises. Let them explore these and similar exercises and determine by the patterns what they are to do. Give them further clues, of course, but try to withhold actual explanations until everything else has failed. Reexamine our suggestions for doing silent arithmetic on pages 1-5 of the booklet *Games and Enrichment Activities*.

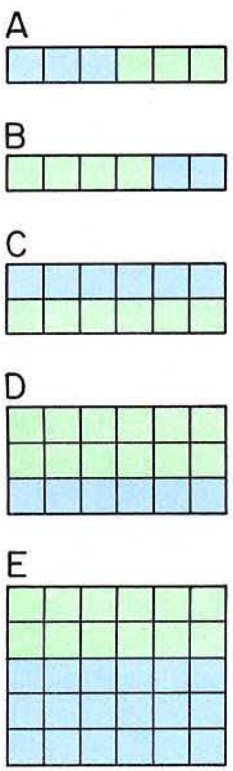
Drill

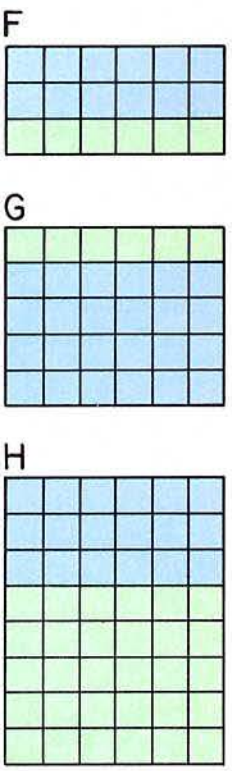
Here are some stories for which pupils can be asked to write one or more headlines:

1. The lighthouse keeper helped 7 sailboats and 6 motorboats in from the sea.
2. The popcorn vendor had 76 bags of popcorn to sell. At the end of the day he had 18 left.
3. Sandy had 6 dimes and 8 pennies. She spent 5 dimes and 5 pennies.

How many tiles are blue? How many tiles are green? How many tiles in all?







	Blue	Green	Total
x	10	10	20
y	9	16	25
A	3	3	6
B	2	4	6
C	6	6	12
D	6	12	18
E	18	12	30
F	12	6	18
G	24	6	30
H	18	30	48

	Blue	Green	Total
a	0	1	1
b	1	1	2
c	1	2	3
d	1	3	4
e	1	4	5
f	2	0	2
g	3	1	4
h	3	3	6
i	4	4	8
j	4	6	10
k	2	1	3
l	2	4	6
m	5	4	9
n	9	3	12
o	5	10	15
p	2	2	4
q	4	4	8
r	4	8	12
s	7	9	16
t	8	12	20
u	3	2	5
v	8	2	10
w	6	9	15

6-7 Counting rectangular arrays - encouraging shortcuts.

Tasks: *Pupils count or use shortcuts to counting to record in the tables the number of blue tiles and green tiles, and the total number of tiles for each of exercises* a-y *and A-H.*

Purpose: To encourage the use of shortcuts to counting (i.e., multiplication); to give practice in recognizing rectangular representations of numbers; to review addition and subtraction.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Flash cards showing rectangular arrays of square tiles and dots.

Materials for Pupils: Graph paper.

The Lesson:

On this page pupils record in the table the number of tiles colored green, those colored blue, and the total number of tiles in each rectangular array.

You will notice that arrays *a-y* represent the multiplication tables through 5 × 5.

×	1	2	3	4	5
1	1 *a*	2 *b*	3 *c*	4 *d*	5 *e*
2	2 *f*	4 *g*	6 *h*	8 *i*	10 *j*
3	3 *k*	6 *l*	9 *m*	12 *n*	15 *o*
4	4 *p*	8 *q*	12 *r*	16 *s*	20 *t*
5	5 *u*	10 *v*	15 *w*	20 *x*	25 *y*

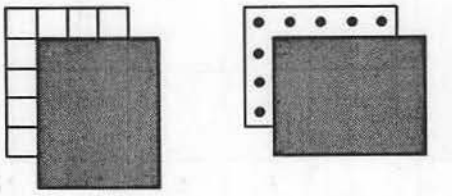
Pupils who develop skill in looking at a rectangular array and quickly responding with the total are well on the way toward mastering the basic facts of multiplication and division.

Many pupils who studied LEVEL B of MATH WORKSHOP are likely to have mastered finding the numbers for the rectangular arrays up to 5-by-5 arrays. To make the tie to multiplication strong, include practice with flash cards for these 25 facts of

more

Headline for the Day: Circle the numbers listed in the total column for exercises *a*, *g*, *m*, *s*, and *y* on page 6. What can you say about these numbers?

multiplication. Mask each card so that only a pair of adjacent edges are exposed:



How many little squares altogether?

How many dots altogether?

Vary the masking procedure—sometimes expose the right and bottom edges, sometimes the left and bottom edges, etc.

Let pupils use graph paper to draw rectangular arrays. Use expressions like "a four-by-five rectangle" and "a one-by-three rectangle." For each drawing, the pupil should write the total number of small squares included in the array. Reverse the exercise by asking pupils to draw rectangular arrays to represent given numbers. Let pupils become aware of the class of numbers which can be represented only by "one-by" arrays. (Except for the number 1, these are the prime numbers. See page 21 of the booklet *Games and Enrichment Activities* and suggestions below for activities designed to help pupils explore prime numbers.) Help them become aware of which numbers can be represented by △-by-△ arrays—these are the square numbers.

Encourage pupils to fill in the *Total* column in the tables on pupil page 6 before filling in the other columns. Note which pupils recognize the numbers of tiles in arrays *a*-*y* and don't need to count. These pupils already have a firm grasp of the idea of multiplication. Those who count one at a time need much help in counting by rows and columns (i.e., by 2's or 3's or 4's, etc.) until they have mastered at least the 5×5 table.

In recording the numbers of green and blue tiles, pupils again need to be reminded of shortcuts.

Help pupils notice the symmetry within the arrays *a*-*y* through such questions as:

1. Which array is like *b*? (Answer: *f*) *c*? (*k*) *h*? (*l*) *n*? (*r*) etc.
2. What is special about arrays *a*, *g*, *m*, *s*, and *y*? (Same number of columns and rows—they are squares.)

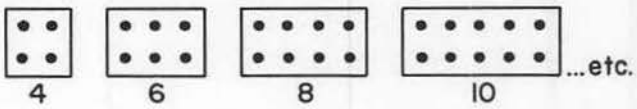
Before letting pupils complete work on the arrays labeled A-H, ask them to compare the arrays. Help them notice that the number of blocks in C is the same as the number of blocks in A and B together. This observation may be recorded by writing: $N(\text{C}) = N(\text{A}) + N(\text{B})$. Similarly:

$$N(\text{D}) = N(\text{B}) + N(\text{C})$$
$$N(\text{E}) = N(\ ) + N(\ )$$
$$N(\text{F}) = N(\ )$$
$$N(\text{G}) = N(\ )$$
$$N(\text{B}) = N(\ ) + N(\ )$$

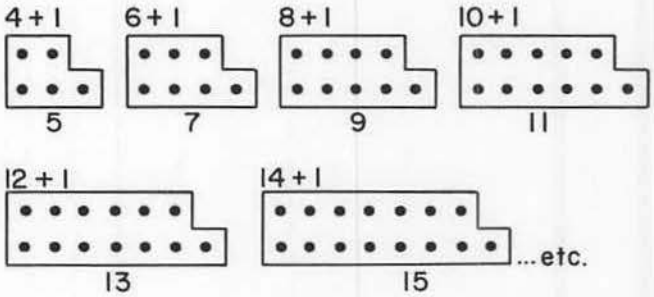
Have pupils complete the above.

Discovering Prime Numbers

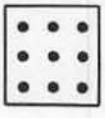
Which numbers greater than 3 can be represented by rectangular arrays of dots with more than one dot in each row and each column? Certainly, there are such arrays for all even numbers greater than 3. Let's draw pictures to show this:



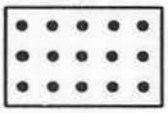
But how about the odd numbers greater than 3? First, let's draw a sequence of pictures that follow a pattern so that we do not skip any odd numbers. We shall copy the pictures of even numbers we have already drawn and then we will add one dot to each:



Which of these displays of dots can be rearranged in rows and columns so that all rows will contain the same number (greater than 1) of dots and all columns will contain the same number (greater than 1) of dots? What about the display of five dots? Seven dots? Try as we will, we cannot find any such arrangement with five or seven dots! Can nine dots be so arranged? (Yes)



How about 11 dots and 13 dots? (Neither can be so arranged.) How about 15 dots? (Yes)



Can 17 dots and 19 dots be so arranged? (No)

Now the point has been reached for individual experiment. Let's make a list of all the numbers greater than 3 that we have found for which we cannot find a rectangular array of dots with each row and each column containing more than 1 dot:

5 7 11 13 17 19

We call these numbers *prime numbers*. What is the next larger number that is a prime number? Have pupils find all the numbers less than 50 that are prime numbers. Be sure they include 2 and 3.

A prime number is a whole number greater than 1 whose only representation by a rectangular array is a 1-by-☐ or a ☐-by-1 array. In other words, a prime number is a whole number which has exactly two factors, 1 and itself.

Drill

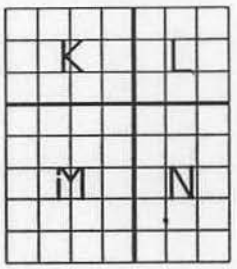
Here are some stories for which pupils can be asked to write one or more headlines:

1. Celia had 89 cents. She spent 53 cents.
2. In the first row of Dana's class there are 4 girls and 3 boys.
3. Karen and Jason counted cars as they walked to school. Karen counted 68 cars and Jason counted 87 cars.

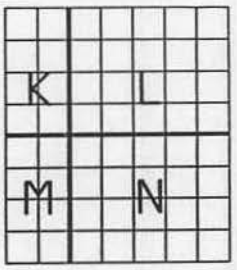
Consider using as a drill sheet page 5 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:

Find How Many

Color all "K" blocks red, all "L" blocks blue, all "M" blocks green, all "N" blocks yellow.



K has 12 boxes
L has ____ boxes
M has ____ boxes
N has ____ boxes
K + L has ____ boxes
M + N has ____ boxes
K + M has ____ boxes
L + N has ____ boxes
K + L + M + N has ____ boxes



K has 8 boxes
L has ____ boxes
M has ____ boxes
N has ____ boxes
K + L has ____ boxes
M + N has ____ boxes
K + M has ____ boxes
L + N has ____ boxes
K + L + M + N has ____ boxes

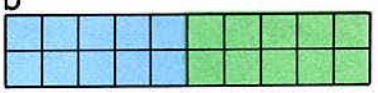
How many tiles are green?

How many tiles are blue?

How many tiles in all?

a

b



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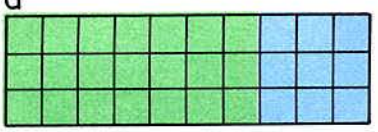

e

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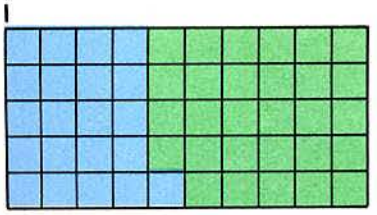

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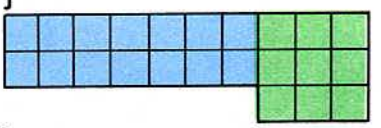

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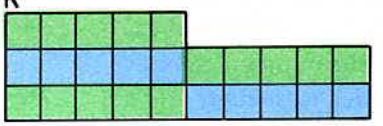

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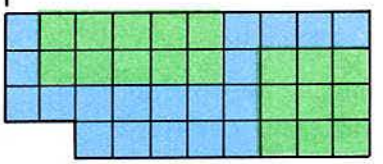

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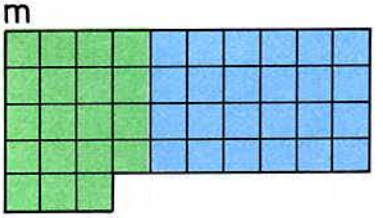

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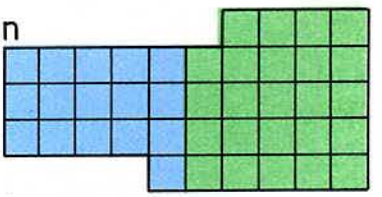

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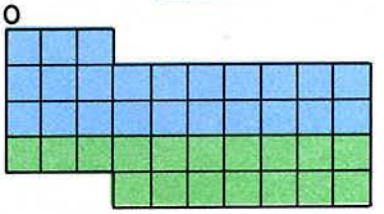

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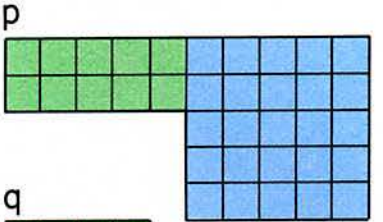

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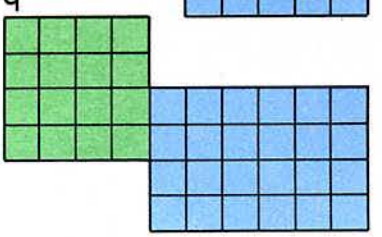
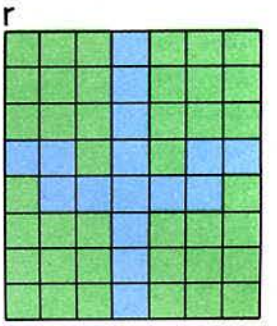


	Green	Blue	Total
a	6	4	10
b	10	10	20
c	12	8	20
d	21	9	30
e	18	12	30
f	20	20	40
g	16	24	40
h	24	26	50
i	29	21	50
j	9	14	23
k	15	10	25
l	19	19	38
m	19	24	43
n	24	16	40
o	17	23	40
p	10	25	35
q	16	24	40
r	40	16	56

r



Notes on Pupil Page 7

Tasks: *Pupils are to count or use shortcuts to record the number of blue tiles, green tiles, and the total number of tiles for each of the exercises* a-r.

Purpose: To review and practice adding and partitioning numbers with some emphasis on multiples of ten; to encourage the use of shortcuts to counting.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Flash cards; Ditto master.

Materials for Pupils: Cuisenaire rods; graph paper; Ditto sheets.

The Lesson:

This is a lesson in working with multiples of ten. As recommended for pupil page 6, emphasis on this page should be placed on filling the *Total* column first. Ask pupils if they can see a simple way to count the total number of tiles in each array. We wish them to look for shortcuts to counting! The chief discovery to be made on this page is that each of the arrays except *r* is 10 tiles wide. Thus, we can count by 10's as one shortcut. In *a* through *e* this is all we need to do to get the total. In *j* through *m* we have multiples of 10 and some left over. In *n*, *o*, and *q*, if we can imagine regrouping, we come up with rectangular arrays with 10 in each row.

The counting of the blue and green tiles should be done by looking for familiar arrays. In *r* one of the four green regions consists of $(3 \times 3) + 1$ tiles.

Before the pupils work on this page it may be wise to include further practice in recognizing arrays using both the flash cards and the activities with graph paper discussed in Notes on Pupil Page 6.

Adding Tens

When pupil page 7 has been completed, have pupils complete the following sentences on a Ditto sheet.

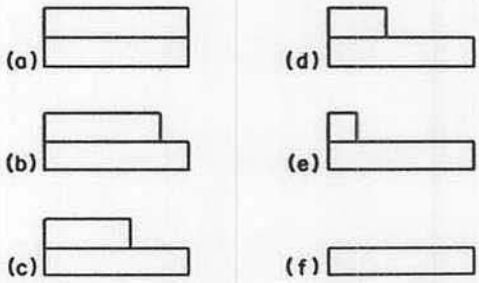
50 + □ = 50	30 + □ + □ = 50	20 + □ + □ + □ = 50
40 + □ = 50	□ + 30 + 10 = 50	50 = 10 + □ + 10 + 10
30 + □ = 50	10 + 10 + □ = 50	50 = □ + □ + 20 + 10
20 + 30 = □	20 + 20 + 10 = □	□ = 10 + 10 + 10 + 20
□ + 40 = 50	□ + 20 + 10 = 50	□ + □ + □ + □ + □ = 50
	□ + □ + 10 = 50	

Remember that in a sentence with frames, you must write the same number in all frames of the same size and shape. For example, the first sentence in the second column must be completed as 30 + 10 + 10 = 50 and not as 30 + 5 + 15 = 50. (This rule of "same number for frames of the same size and shape" does *not* mean that frames of different size or shape must be filled with different numbers. See "Enrichment Activity 3" in Notes for Pupil Page 5.)

Cuisenaire Rods

Establish the agreement that the white Cuisenaire rod represents 10. Then have pupils complete the following number sentences and construct models based on this agreement. The diagram below shows models for sentences (a) through (f):

(a) 50 − 50 = △
(b) 50 − 40 = △
(c) 50 − 30 = △
(d) 50 − 20 = △
(e) 50 − △ = 40
(f) 50 − △ = 50



Extend the Patterns

10, 20, 30, 40, , , ,
15, 25, 35, , , , ,
10, 30, 50, , , , ,
0, 20, 40, , , , ,

Headline for the Day: Frances said she could make a square with the pictures for exercises *h* and *i* on page 7. How could she do this? What other pictures could you use from exercises *a* through *i* to make squares?

Numbers Have Many Names

Game 1

Here is a game for the entire class. On the chalkboard, draw the following chart:

Numbers have many names!				
other names for 3+2	other names for 10	other names for 11-8	other names for 7+5	other names for 9-9

Use an odd number of columns and select an even number of teams, preferably two. A member of the first team must write in the appropriate column another name for 3 + 2. A member of the next team must write another name for 10, and so on.

The odd number of columns and the even number of teams assure that the team that had 3 + 2, 11 − 8, and 9 − 9 the first time across will have 10 and 7 + 5 the next time and vice versa. Each team scores whenever one of its members gives a correct "other name." Whenever one of its members gives an incorrect other name, the opposing team scores, provided that one of its members spots the error.

Games such as this let each pupil participate successfully at his own level. Activities based on the idea that numbers have many names (i.e., that there are many ways to "get" a number) help the teacher to estimate the ability of each pupil. They help the teacher to discover who in the class is slow or quick, timid or confident. For example, a front-runner may reveal himself by using 3000 ÷ 1000 or $\sqrt{9}$ as other names for 11 − 8. (It is always possible to let the more apt pupils show off their prowess by asking them to write the most difficult looking names for a given number that they can think of.)

Game 2

A variation can be used to make Game 1 more difficult. A member of one team starts a name for the number in question, and the other team must complete it. Another name for 3 − 2 might be started as 0 + ___ or as 19 − ___. The next player has to complete the name: 0 + 1, or 0 + 10 − 9, or 0 + 4 − 1 − 2, etc.

Sometimes, *you* may wish to begin the other names in order to match the challenge of the situation to the recognized ability of the next player. This is important to the pupil: to be asked a question that is not too easy nor too hard, a question he can answer if he makes an effort.

Game 3

How many names for each of the numbers from 0 through 10 can you write in the form of ___ + ___, using whole numbers only?

0	**1**	**2**
0 + 0	0 + 1	2 + 0
(1 name)	1 + 0	1 + 1
	(2 names)	0 + 2
		(3 names)

Have pupils record the results in a chart:

Number	Names in form of ___ + ___
0	1
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9
9	10
10	11

This leads to an interesting pattern. It suggests that there will be 21 names for 20 in the form of ___ + ___. Test the pattern. Does it keep on giving you reliable predictions?

Game 4

Pupils who finish early the exploration in Game 3 may be asked to explore the same question for names that are in the form ___ + ___ + ___.

0	**1**	**2**
0 + 0 + 0	0 + 0 + 1	0 + 0 + 2
(1 name)	0 + 1 + 0	0 + 2 + 0
	1 + 0 + 0	2 + 0 + 0
	(3 names)	1 + 1 + 0
		1 + 0 + 1
		0 + 1 + 1
		(6 names)
3	**4**	**5**
0 + 0 + 3	0 + 0 + 4	0 + 0 + 5
0 + 3 + 0	0 + 4 + 0	0 + 5 + 0
3 + 0 + 0	4 + 0 + 0	5 + 0 + 0

0 + 1 + 2	0 + 1 + 3	0 + 1 + 4
1 + 0 + 2	0 + 3 + 1	1 + 0 + 4
0 + 2 + 1	1 + 0 + 3	0 + 4 + 1
1 + 2 + 0	1 + 3 + 0	1 + 4 + 0
2 + 1 + 0	3 + 0 + 1	4 + 1 + 0
2 + 0 + 1	3 + 1 + 0	4 + 0 + 1
1 + 1 + 1	2 + 2 + 0	0 + 2 + 3
(10 names)	2 + 0 + 2	2 + 3 + 0
	0 + 2 + 2	0 + 3 + 2
	1 + 1 + 2	2 + 0 + 3
	1 + 2 + 1	3 + 0 + 2
	2 + 1 + 1	3 + 2 + 0
	(15 names)	3 + 1 + 1
		1 + 3 + 1
		1 + 1 + 3
		2 + 2 + 1
		2 + 1 + 2
		1 + 2 + 2
		(21 names)

The results, once more, are to be recorded in a chart:

Numbers	Names in form of __ + __ + __
0	1
1	3
2	6
3	10
4	15
5	21
6	?

Multilevel Motivated Practice

Mathematics is full of problems that require exercise in recalling combination and putting them to work. Some of these problems can be stated simply and their solutions require little maturity. They can be explored by pupils using the abilities they are developing. The solutions are within their reach.

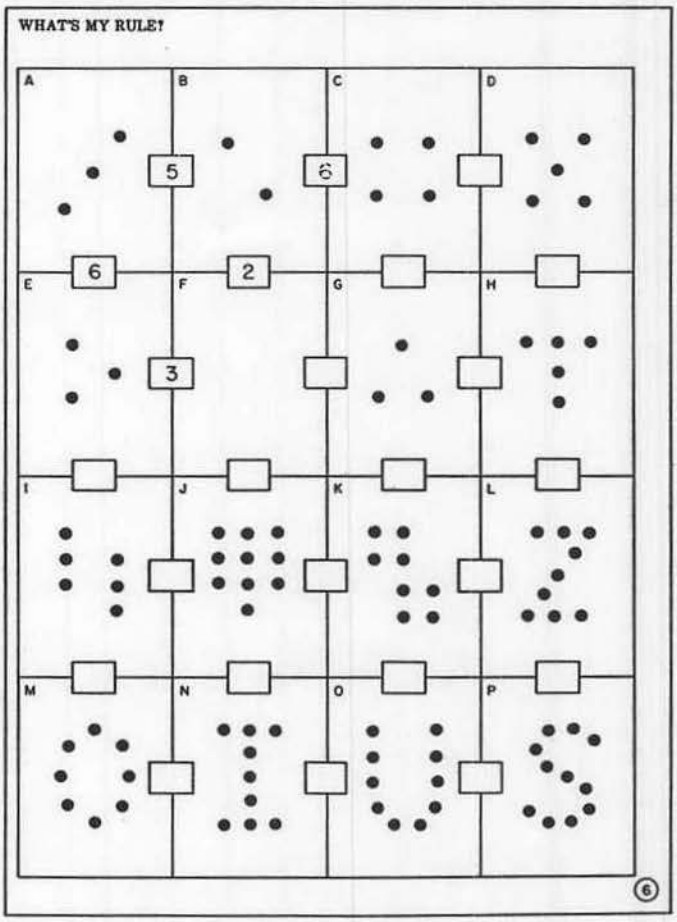
Many of the activities suggested in this guide are truly "multilevel." Some grow out of situations created by manipulative materials used by beginners. Less than obvious questions are posed that can be answered by newcomers to arithmetic.

Most of the suggested activities begin simply. As they are extended, the questions that are asked require more background. If carried as far as possible, many would involve quite sophisticated mathematical ideas. However, the sequence of activities is designed so that a teacher can begin and end with work that is within the capacity of his particular class.

Each activity is presented with emphasis on many ways of providing pupils with highly motivated drill in the fundamentals of arithmetic. While mathematical ideas that are important in their own right form the bases for these activities, their main purpose is the development of the pupil's skill in basic arithmetic, for it is only in doing arithmetic *correctly* that the excitement and fun comes through. We do not wish pupils to become "mathematical stutterers."

Drill

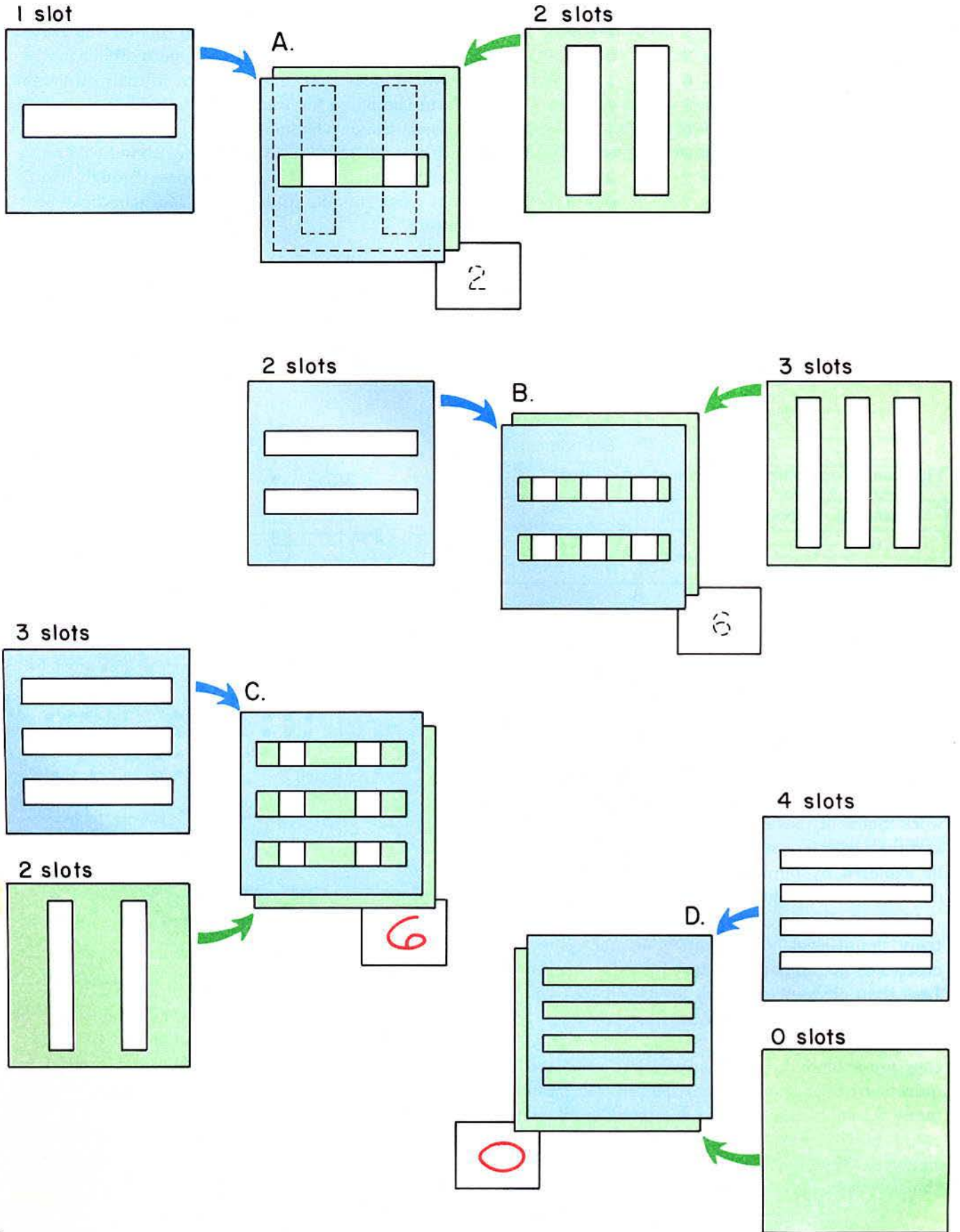
Consider using as a drill sheet page 6 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Here are some stories for which pupils can be asked to write one or more headlines:

1. Cal placed 32 books on the first shelf of his bookcase, 27 on the second shelf, and 33 on the third shelf.
2. 5 tugboats towed 8 big ships into port.
3. Monte picked 7 boxes of berries while Milo picked 5 boxes of berries.

Imagine a series of cards — with slots cut in them. Put one on top of the other. In how many places does light show through?



8 Introduction (review from Levels A and B) of multiplication in terms of intersections of vertical and horizontal strips.

Tasks: *Pupils count and record in each white box the number of places light would show through.*

Purpose: To examine multiplication by using intersecting vertical and horizontal strips.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Demonstration-size, slotted cards of tagboard as described below; overhead projector.

Materials for Pupils: Cuisenaire rods; desk-size slotted cards.

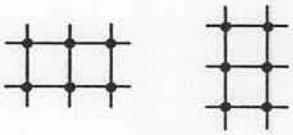
The Lesson:

This and the next several pages review the idea of multiplication (and division) in terms of models which were introduced in LEVELS A and B. We have already encountered (pupil pages 6 and 7) the significance of rectangular arrays as representations of multiplication. This idea will be strengthened in the pictures and experiences that follow. Remember that we wish pupils to become more and more skillful in looking at a rectangular array and giving the number of objects in the array. Before LEVEL C is finished, the average pupil should know the multiplication and division facts through 10×10.

The following models are used in this book as representations of the members of the fact team:

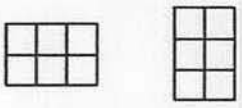
$2 \times 3 = 6$ $\quad$ $6 \div 3 = 2$

$3 \times 2 = 6$ $\quad$ $6 \div 2 = 3$

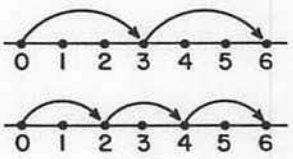


Crossing points of horizontal and vertical lines

Rectangular arrays of counters or dots



Areas of rectangles



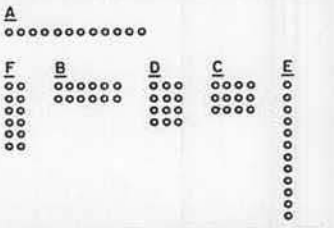
Number line jumps

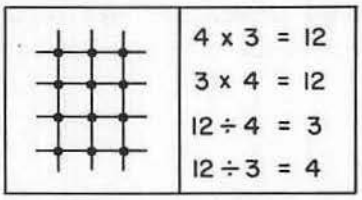
Special emphasis is given on pupil pages 8 through 18 to the crossing points of lines as a model of multiplication.

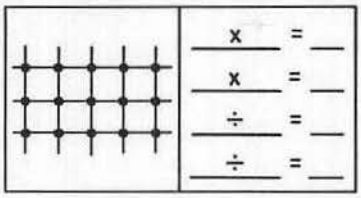
When pupils have finished these pages, they should be able to interpret such pictures as those on the following page in terms of multiplication-division fact teams.

more

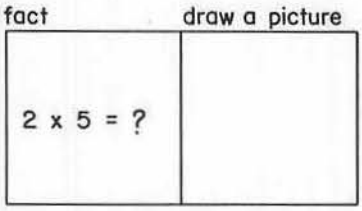
Headline for the Day: Flavio said there were the same number of counters in each of these pictures. What do you think?



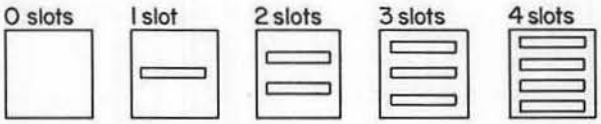




Moreover, if a pupil doesn't remember a particular fact, he should draw a picture to help:



In preparation for pupil page 8 make two sets of slotted cards:



When one card is placed over another so that the slots of one card are horizontal and the others vertical, light will show through the crossing points. Keep a record on the chalkboard of the number of places where the light shows through.

Horizontal slots	Vertical slots	Crossing points
1	2	2
2	1	2
0	1	0
0	2	0
2	2	4
3	2	
2	3	
3	3	

These actions can be carried out very effectively by holding each card of a pair against a windowpane, and then holding the pair against the pane. Or, if an overhead projector is available, place one card on the platform (note the number of streaks of light on the screen) and then superpose the second card. This demonstration will emphasize to the pupil what it is that should be counted in the exercises on pupil page 8.

See pages 62-79 of the booklet *Games and Enrichment Activities* for multiplication activities involving the use of Cuisenaire rods.

Drill

Here is a suggestion for a drill sheet:

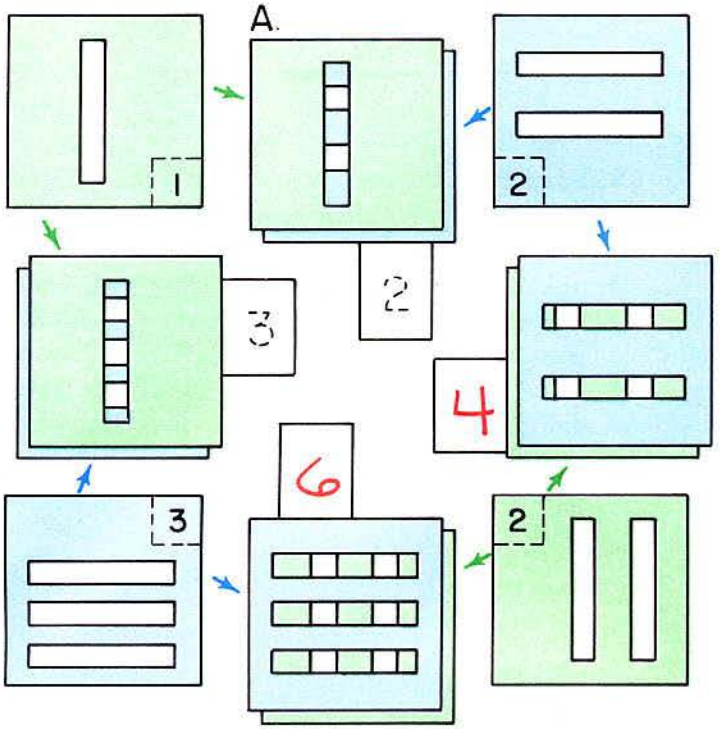
	I	II	III	IV	V	VI
A	4 + ○ = 7	8 + □ = 10	□ + 5 = 9	○ + 3 = 6	2 + 5 = △	8 + 3 = ○
B	5 + 0 = □	7 + ○ = 11	△ + 5 = 8	6 + ⬡ = 12	○ + 4 = 9	2 + 6 = △
C	8 + ⬡ = 11	△ + 4 = 10	○ + 7 = 11	7 + △ = 14	⬡ + 3 = 12	8 + 8 = □
D	○ + 7 = 13	9 + □ = 14	5 + △ = 10	△ + 6 = 14	8 + 9 = ○	5 + 6 = □
E	9 + △ = 16	⬡ + 5 = 11	□ + 9 = 12	6 + 9 = ⬡	7 + 6 = △	⬡ + 9 = 17

	VII	VIII	IX
F	6 + 3 = ○	4 + ○ = 6	□ + 3 = 10
G	2 + ○ = 6	△ + 8 = 12	5 + 8 = □
H	△ + 0 = 8	8 + 7 = ○	5 + △ = 14
I	7 + 9 = ○	8 + △ = 13	□ + 6 = 6
J	□ + 6 = 15	6 + ○ = 14	9 + 9 = △
K	6 + ○ = 15	□ + 6 = 14	8 + 7 = ○
L	2 + △ = 10	□ + 0 = 17	2 + 9 = □

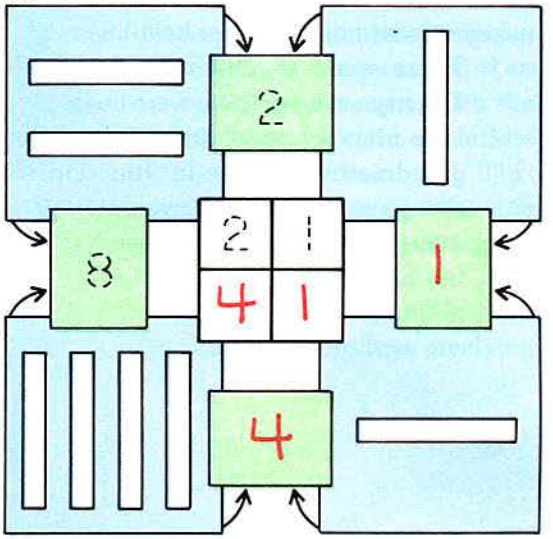
Here are some stories for which pupils can be asked to write one or more headlines:

1. There were 45 children at the playground. Twenty-seven of them were boys.
2. It is fifty-two miles from Irene's home to her grandmother's house in Junction City. When she goes to visit her grandmother she passes a large drive-in theater twenty-five miles from her home.
3. Samantha climbed two ladders. One of the ladders was four feet high. The other ladder was seven feet high.
4. Jake's class had a spelling bee. The boys' team made 32 points while the girls' team made 27 points.
5. Mike had eighteen pieces of candy. He gave one piece to each of fifteen friends and kept the rest for himself.

A.



B.



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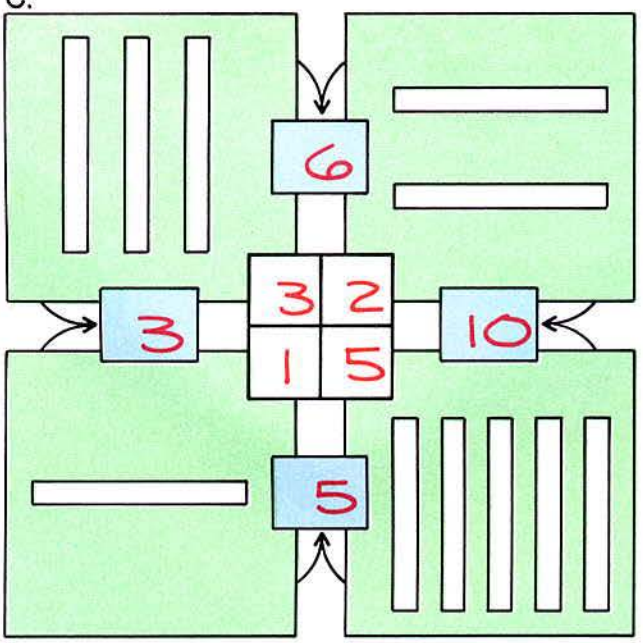
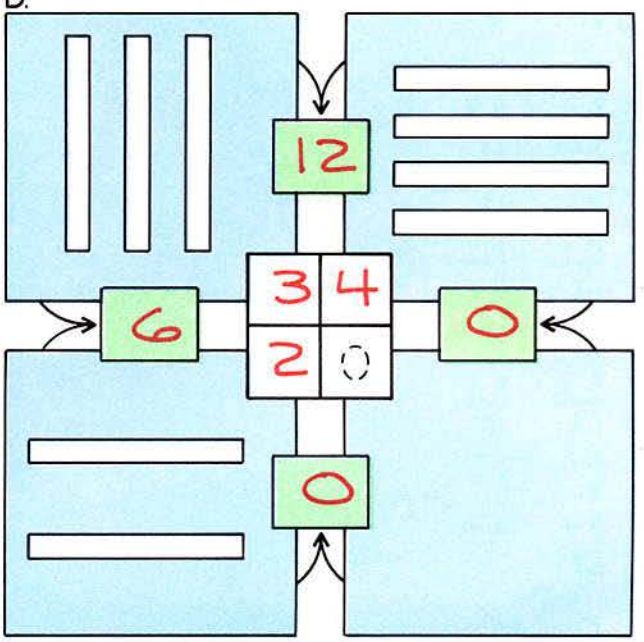
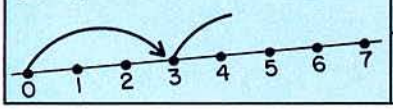


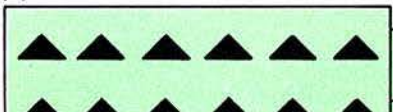
D.



E.

jumps of 3				
	two jumps	6	three jumps	9
	five jumps	15	ten jumps	30

F.

rows of 6				
	in three rows	18	in four rows	24
	in half a row	3	in five rows	30

9 Multiplication in terms of intersections of vertical and horizontal strips, number line jumps (x3), multiples of 6 (x6).

Tasks: *In exercises B-D pupils count and record in the white frames the number of vertical or horizontal strips. In the tinted frames pupils record the number of places light would show through. (For exercise A see the discussion below.) In exercises E and F pupils use the representations provided to record in the tinted boxes either the landing points (exercise E) or the number of triangles (exercise F) specified by the directions.*

Purpose: To review and extend understanding of multiplication by exploring several representations of this operation.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Demonstration-size, slotted cards (as described in Notes on Pupil Page 8); number line chunks; tagboard cards picturing objects like those in exercise F; overhead projector.

Materials for Pupils: Slotted cards.

The Lesson:

Exercise A provides clues useful in determining what is called for in exercises B-D. In each of the latter exercises we are given four slotted cards. The number in a tinted block is the product of the numbers of slots in the cards joined by the tinted block. The number in a white block (in B-D) is the number of slots in the card touched by the white block.

Each of these exercises (A-D) represents a multiplication *Cross Number Puzzle:*

A

1	2	2
3	2	6
3	4	

B

2	1	2
4	1	4
8	1	

C

3	2	6
1	5	5
3	10	

D

3	4	12
2	0	0
6	0	

The upper part of this page shows elaborate records of the crossing points of slotted cards. As preparation for this page continue the work suggested in Notes on Pupil Page 8, having pupils demonstrate the situations suggested in the chart below. Have pupils anticipate how many crossing points they will find before actually arranging the cards.

Horizontal	Vertical	Crossing points
1	2	
1	3	
3	2	
2	2	

Record keeping on pupil page 9 involves (1) recording the number of slots on each card, and (2) indicating for every pair of adjacent cards the number of crossing points that would be observed if the cards were overlapped.

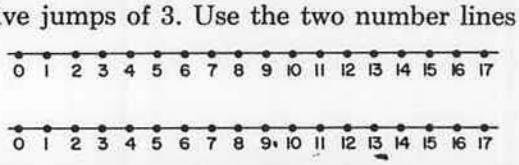
The bottom section of pupil page 9 introduces a technique we shall use often in LEVEL C. We begin with a pictured situation—in the case of this page, jumps of 3 on a number line and two rows of 6 triangles in each row. Pupils are then asked to regard the situation in different ways and to record the number for each way.

Begin this activity with similar examples on the chalkboard until you are satisfied pupils have the idea.

(5¢) (1¢)	money shown	6¢	twice as much	
	3¢ more		half as much	

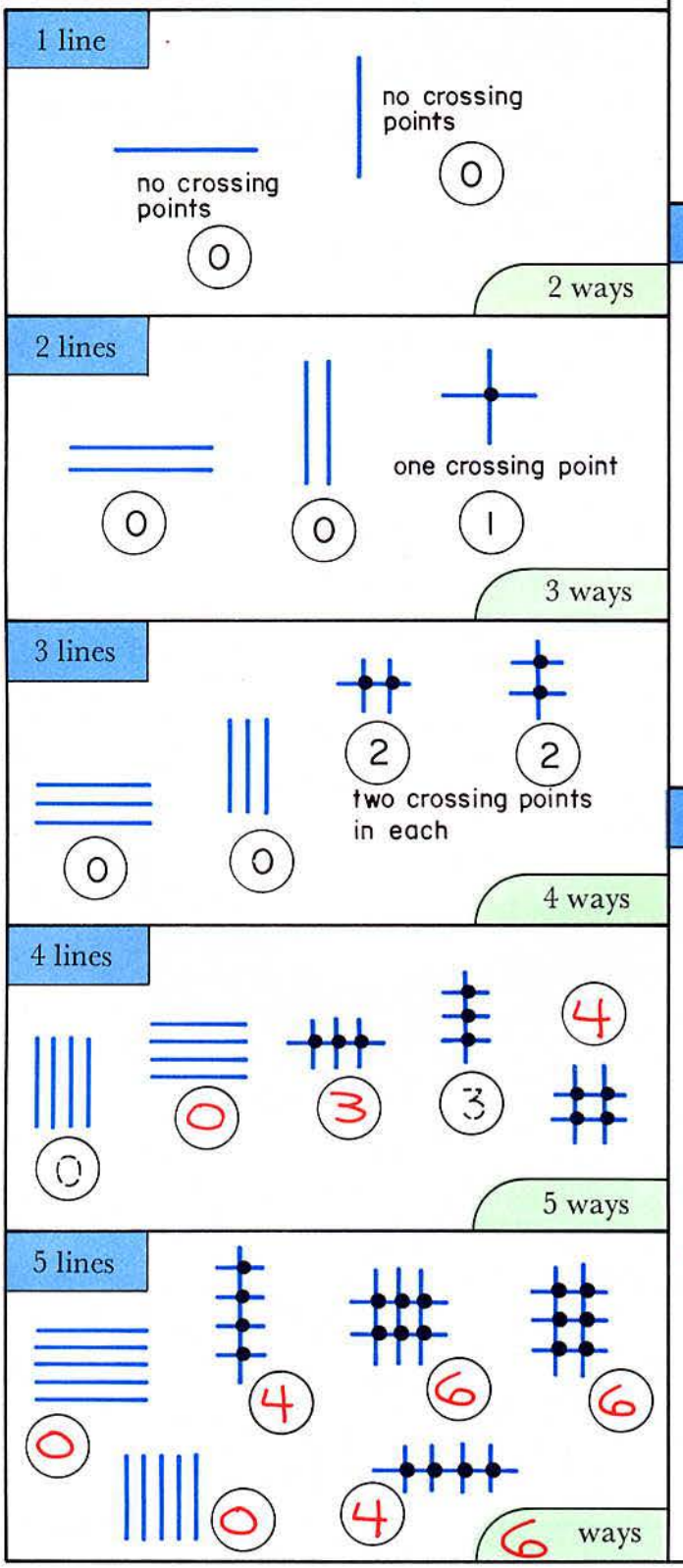
apples, 5¢ ea.	2 apples	10¢	2 of each	
oranges, 10¢ ea.	1 of each		2 apples and 3 oranges	

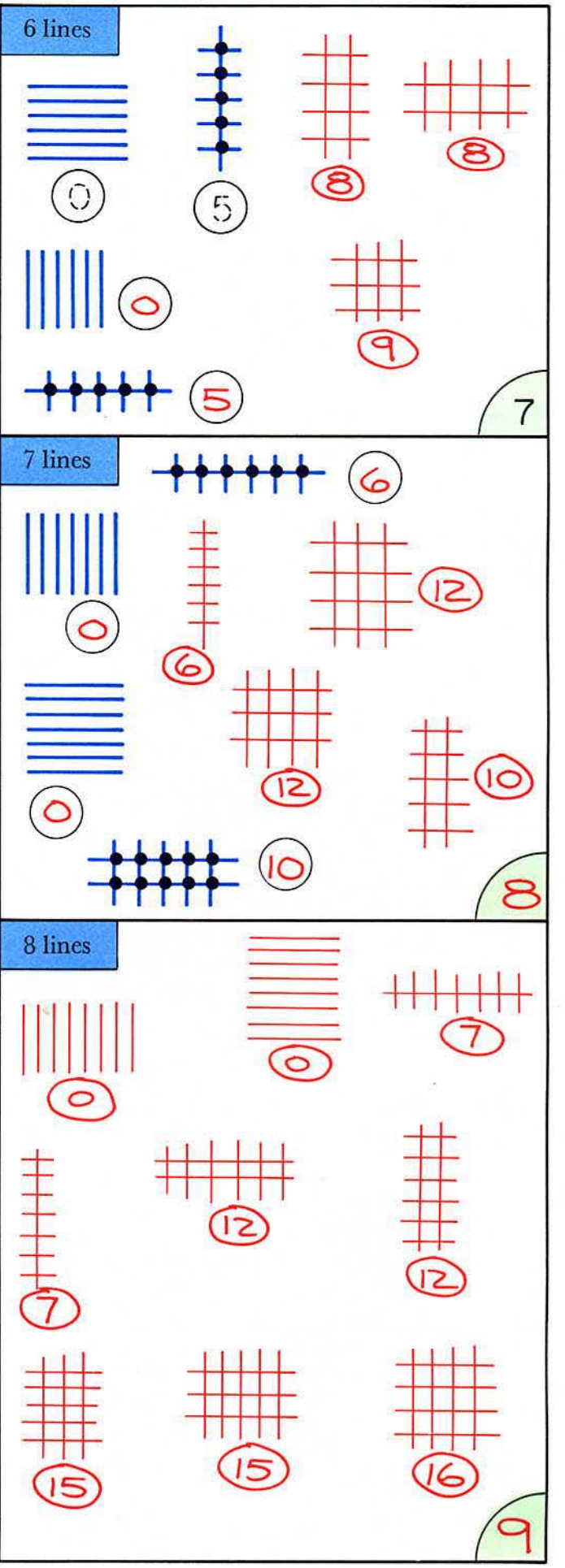
Headline for the Day: John said he could show that two jumps of 3 followed by three jumps of 3 brings you to the same landing point as taking five jumps of 3. Use the two number lines below to check to see if John is correct.



Here is some work Numo was doing. He didn't finish it.

Can you do it for him?





10-11 Using vertical and horizontal lines to explore related sums and products.

Tasks: *Pupils use the clues provided in the first four exercises to complete the remaining exercises either by constructing representations or recording in the circles the pertinent information supplied by the given representations.*

Purpose: To explore multiplication (and division) using crossing lines as representations.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Overhead projector; grease pencil; colored chalk; flash cards (see below).

Materials for Pupils: None.

The Lesson:

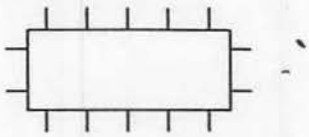
On pupil page 10 (and 11) we have a record of some of Numo's explorations of the number of different ways 1 to 10 lines can be arranged horizontally or vertically, or in some combination of these. Where there is at least one horizontal line and one vertical line, he regarded them as crossing.

One good way to introduce this work is to divide your class into groups of two to four pupils each to work as teams on pupil pages 10 and 11 with no instructions except these: On these two pages are some entries from Numo's math notebook. It appears that some entries are missing. Study what he has done. What are Numo's rules? Can you supply the missing information?

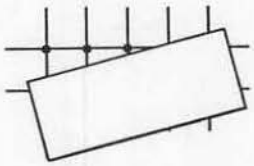
Make sure that one pupil in each group is an able one. This team approach to arithmetic offers pupils a change of pace in their work. It involves cooperation and teamwork, and at the same time provides the excitement of competition.

Uncovering Flash Cards

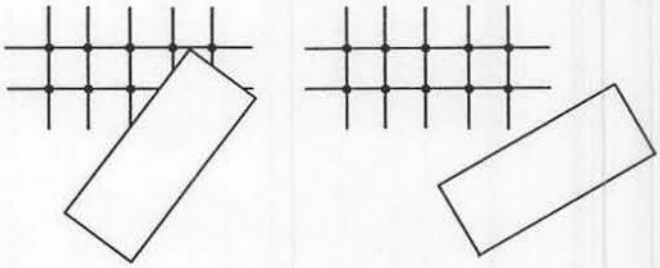
As a follow-up to this page, we suggest work with a set of flash cards depicting crossing lines. Cover the crossing points so that only the ends of the horizontal and vertical lines can be seen. Slowly take off the cover card. When can pupils predict the number of crossing points?



How many lines?



How many crossing points?

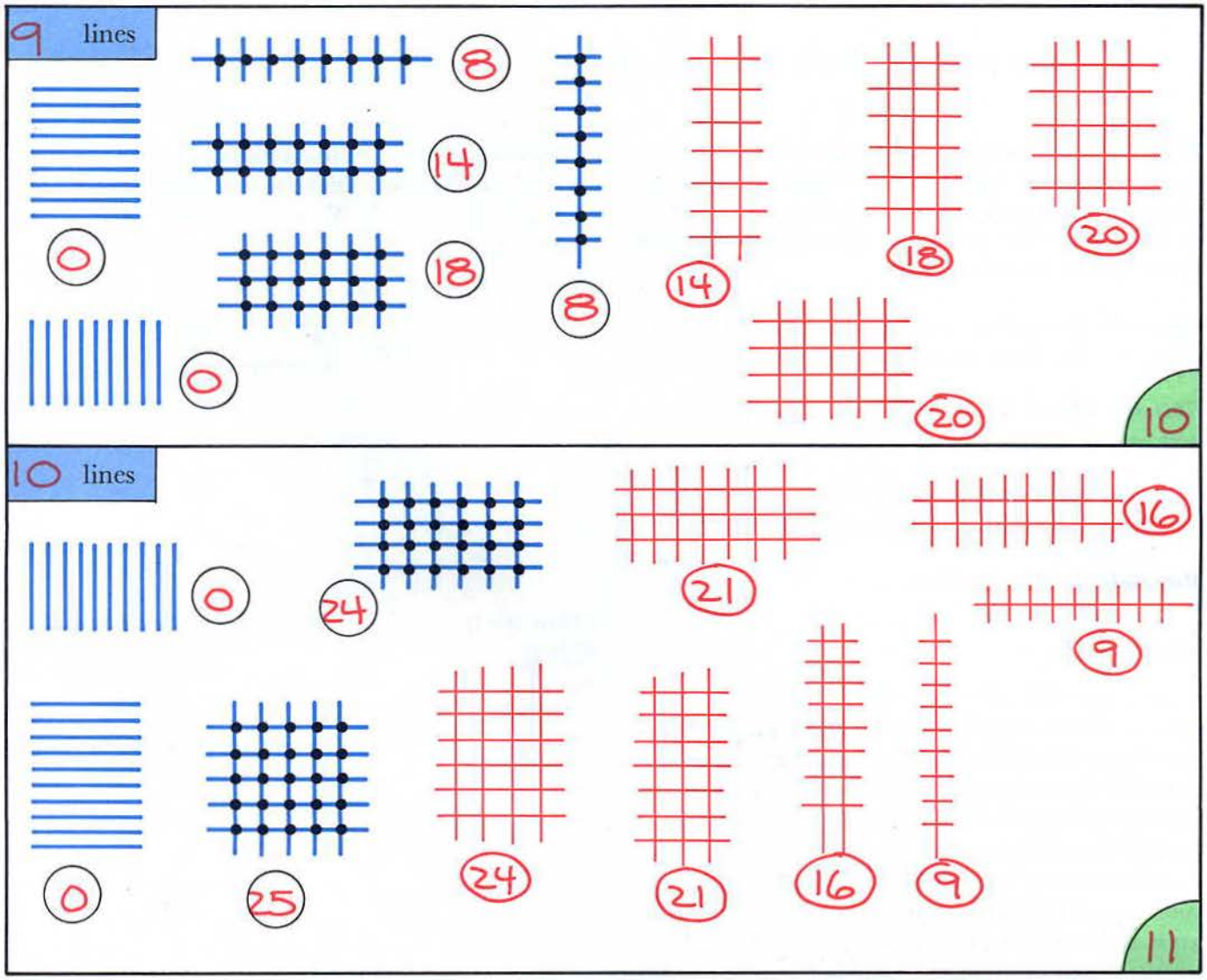


Drill

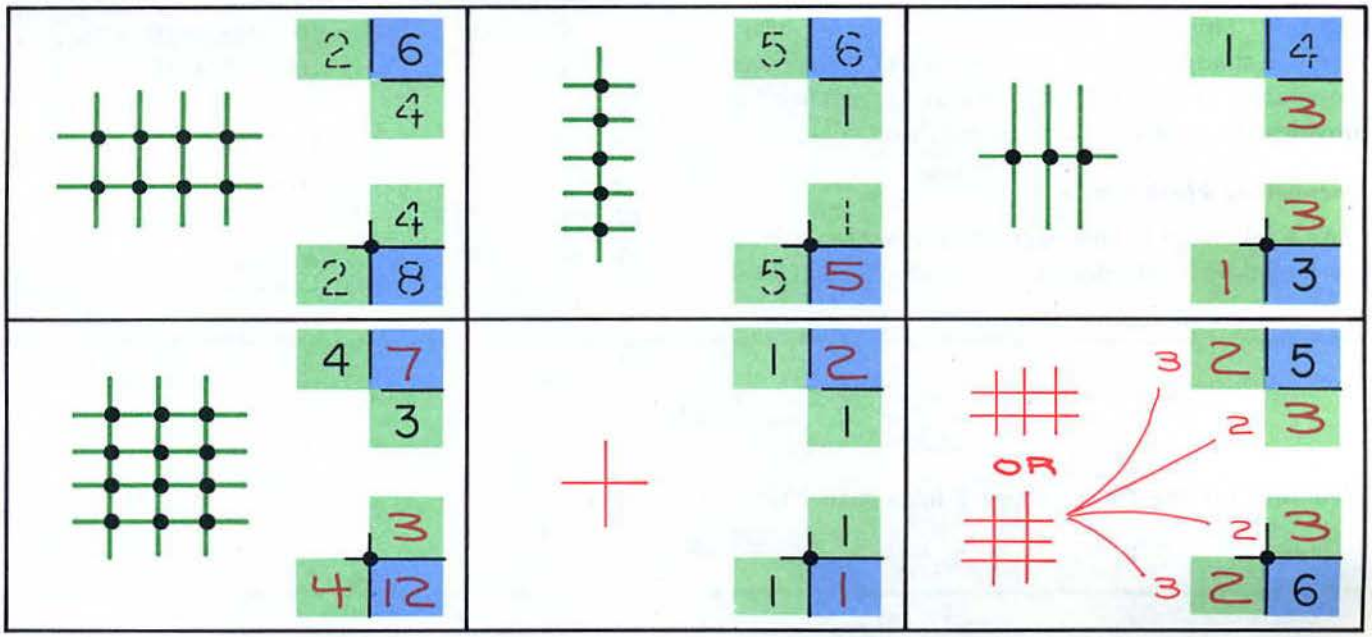
Here is a suggestion for a drill sheet:

1. $48 + 10 = \square$	2. $59 - 10 = \square$
3. $70 + 10 = \square$	4. $268 + 10 = \square$
5. $268 + 100 = \square$	6. $268 + 1 = \square$
7. $683 - 100 = \square$	8. $683 - 10 = \square$
9. $683 - 1 = \square$	10. $3 + 10 = \square$
11. $3 + 100 = \square$	12. $3 + 1000 = \square$
13. $728 + \square = 748$	14. $351 + \square = 451$
15. $927 - \square = 907$	16. $732 - 632 = \square$
17. $841 - 831 = \square$	18. $962 - 762 = \square$
19. $573 - 73 = \square$	20. $683 - 603 = \square$

Headline for the Day: Draw 7 lines with 10 crossing points.



Here is more of Numo's work. Can you find the rules he is following? Help him finish his sketches and records.



Notes on Pupil Page 11

Tasks: *For the upper part of the page pupils record in the tinted box the number of lines to be used in constructing the required representations. They record in the circles the number of crossing points pictured, and construct the missing representations. For the lower part of the page pupils record in the blue boxes the number of lines used in each representation. In the green boxes they record the number of crossing points for each representation. In the last two exercises pupils must also supply the missing representations.*

Purpose: To further explore multiplication (and division) using crossing vertical and horizontal lines as representations.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Overhead projector; grease pencil; colored chalk; flash cards (as described in Notes on Pupil Page 10).

Materials for Pupils: Strips for pupils who are having difficulty in working just with pictures.

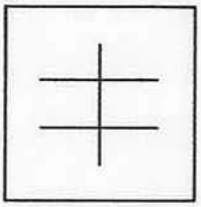
The Lesson:

Draw a table on the board without comment:

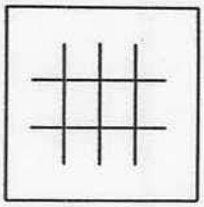
	Horizontal lines	Vertical lines	Number of lines	Number of crossing points
A				
B				
C				
D				

Now draw pictures like the following, recording information in the table. (You may wish to use the overhead projector for these drawings.)

A



B



	Horizontal lines	Vertical lines	Number of lines	Number of crossing points
A	1	2	3	2
B	2	3	5	6
C				
D				

When you draw the next picture, offer the chalk to the class to see who can fill in the record. Continue with other information and with several pictures.

As a variation, present the numbers in the table and have pupils draw the pictures.

Finally, present problems in which only the number of lines or the number of crossing points, or both, are given. Pupils need to complete the table and draw the appropriate pictures.

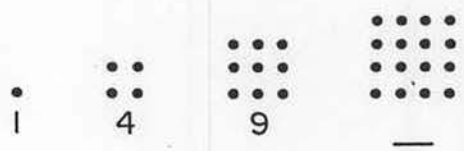
	Horizontal lines	Vertical lines	Number of lines	Number of intersections
E			5	
F				12
G			6	8

Drill

Here are some stories for "headline practice":

1. Toni invited 8 boys and 8 girls to her birthday party. She was able to seat everyone including herself.
2. The milkman delivered 6 gallons of milk and 32 quarts of eggnog to the grocer.
3. Blanche bought 2 sizes of ice-cream cones. She also bought 6 different flavors of ice-cream.

Headline for the Day: Extend the pattern two more steps:



Think of 4 things. Call them *a*, *b*, *c*, and *d* . . . so we can talk about them.

Think of 3 other things. Call them *x*, *y*, and *z*.

$\{a, b, c, d\}$ and $\{x, y, z\}$

Put them together and count them. We find 7 in all.

When we think this way, we write:

$$4 + 3 = 7$$

We say, "Four plus three equals seven." This is an example of ADDITION.

We say that seven is the *sum* of four and three and that four and three are *addends* of seven. But, we might think of the four things and the three things in another way.

We could think of making combinations by pairing a thing from the first group with a thing from the second. We could find all the pairs that can be made this way.

$\{a, b, c, d\}$ and $\{x, y, z\}$

ax	bx	cx	dx
ay	by	cy	dy
az	bz	cz	dz

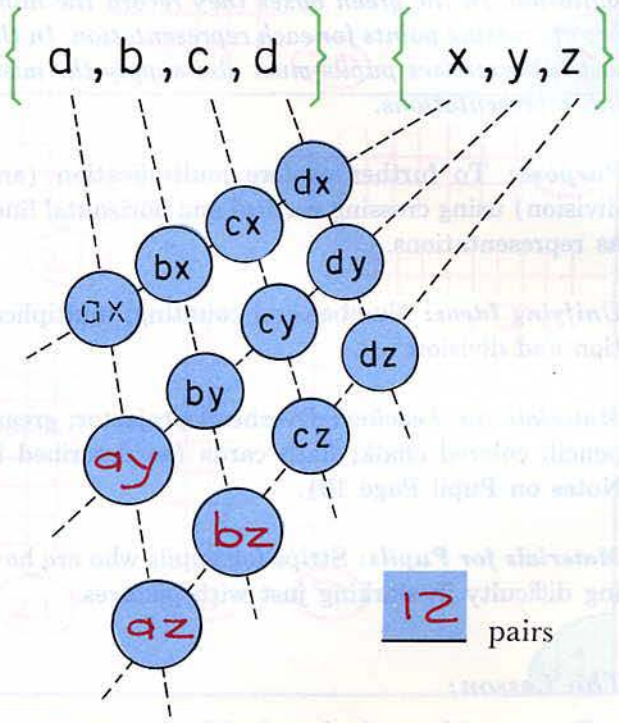
We find 12 pairs in all. When we think this way, we write:

$$4 \times 3 = 12$$

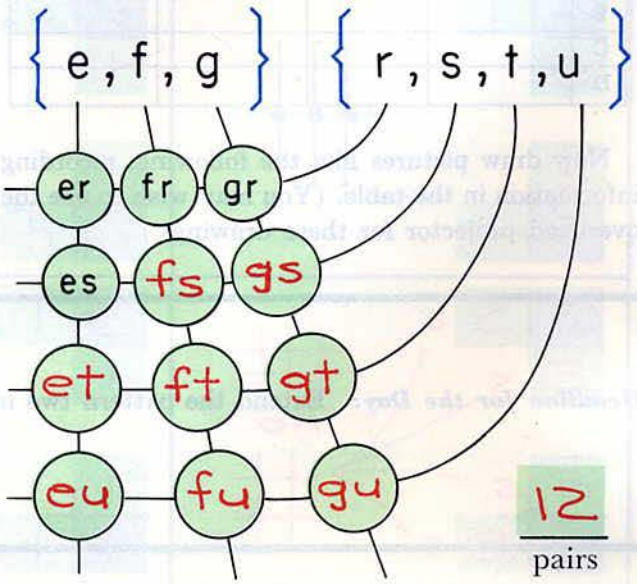
We say, "Four times three equals twelve, or four multiplied by three equals twelve." This is an example of MULTIPLICATION.

We say that "twelve is the *product* of four and three" and that "four and three are *factors* of twelve."

We could have used this drawing to find the pairs in the example above:



Suppose that we had had only 3 things in the first group and 4 in the other. We would have had:



12-13 Introduction (review from Levels A and B) of multiplication sign - multiplication in terms of Cartesian products of sets.

Tasks: *Pupils record in the tinted regions the missing pairs of letters or the total number of pairs.*

Purpose: To examine multiplication in terms of pairing (Cartesian product of sets).

Unifying Ideas: Sets; multiplication and division.

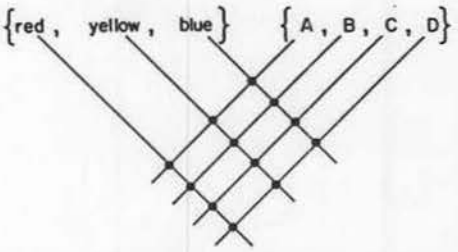
Materials for Teacher: Ditto masters; flash cards.

Materials for Pupils: Ditto sheets of simple pairing exercises similar to those on the pupil page and those described below.

The Lesson:

Given a first number and a second number, consider two sets, one of which has the first number for its count and the other of which has the second number for its count. Consider next a third set whose members are all the pairs whose first components belong to the first set and whose second components belong to the second set. The third set is called *the Cartesian product* of the first set by the second set. The number of pairs in the third set is called *the product* of the first number by the second.

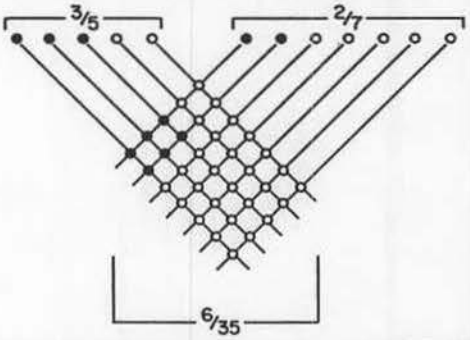
This abstract definition of multiplication is reflected in the illustrations on pupil pages 12 and 13. Let's consider an example. Suppose that we wish to find the product of 3 by 4. According to the definition, we should consider a 3-set and a 4-set, and then consider a third set whose members are all the pairs with first component from the 3-set and second component from the 4-set. For example, let the 3-set be the set consisting of the colors red, yellow, and blue, and let the 4-set be the set consisting of four cardboard squares, one with an A marked on one side, one with a B, one with a C, and one with a D. What are all the possible pairs? Well, you could color the A-square red. Or you could color it yellow. Or you could color it blue. So far, we have three possible pairings of a color and the A-square. As you can see, we also have three possible pairings using the B-square. Likewise, there are three pairings using the C-square and three using the D-square. We can represent all of the members of the third set as follows:



Each dot in the array represents one of the pairs in the third set. Since there are 12 such dots, it follows from the definition that 12 is the product of 3 by 4. The same illustration also shows that the product of 4 by 3 is 12.

This definition of multiplication can be used to show that the product of 0 by any number is 0 and that the product of any number by 0 is 0. In either of these cases it is obvious that the number of pairs in the Cartesian product is 0 since pairs cannot be formed if there is no supply of first components or no supply of second components.

One of the reasons we have selected this definition for multiplication is that it can be easily extended to multiplication of fractions. For example, here is a way to discover the product of 3/5 by 2/7:



The repeated addition approach to multiplication works well with whole numbers but cannot be reasonably extended to fractions.

Headline for the Day: Tami bought five pencils for a quarter. One pencil cost 5 cents more than the others. How much did the more expensive pencil cost?

It is not our intention that pupils give any verbal description of multiplication in terms of sets. If a pupil wishes to discover the product of 3 by 4, he might first draw 3 parallel lines; then he should draw 4 parallel lines which cross the first 3; then he should count the crossing points. Or, he might draw a 3-by-4 rectangle on graph paper and count the little squares. Or, he might make 4 jumps of 3 on the number line. Etc.

Pairing Skirts and Blouses

You might wish to prepare pupils for the exercises on pupil pages 12 and 13 by having them consider the selection of outfits. These activities are familiar to pupils who have used MATH WORKSHOP, LEVEL B. Make a cardboard Numa for each girl in the class and a Numo for each boy. Give each girl a Ditto sheet with pictures of three blouses and three skirts on it; give each boy a Ditto sheet with pictures of three shirts and three pairs of trousers. Have pupils color each blouse (or shirt) and skirt (or pair of trousers) a different color, and then have the pupils cut out the costumes.

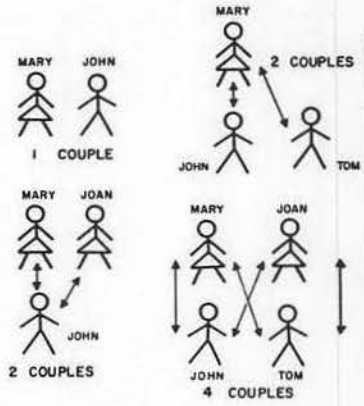
How many complete outfits can you make with one blouse and no skirt? Or with one shirt and no trousers? The answer, of course, is 0. Can you make any complete outfits with one skirt, or one pair of trousers, and no blouse, or shirt?

How many complete outfits can you make with one top and one bottom? (One) With one top and two bottoms? (Two) With two tops and one bottom? (Two) With two tops and two bottoms? (Four) Let's make a record to show what we have discovered:

Tops	Bottoms	Outfits
1	0	0
0	1	0
1	1	1
1	2	2
2	1	2
2	2	4
2	3	6
3	2	6
3	3	?

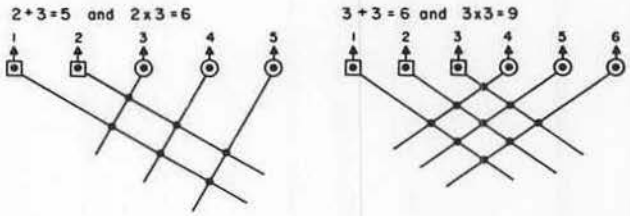
Pairing Boys and Girls

Have Mary come up. Is Mary a couple? (No) Ask John to come up, too. Now, can we make a couple (Yes) Ask two more girls and two more boys to come forward. There are three girls (Mary, Joan, and Sue) and three boys (John, Tom, and Jim). Now how many choices for couples do we have? How many couples can we choose from two girls (Mary and Joan) and one boy (John)? How many couples can we choose from two girls (Mary and Joan) and two boys (John and Tom)? Draw on the board:



GIRLS	BOYS	COUPLES
1	0	0
0	1	0
1	1	1
2	1	2
1	2	2
2	2	4
3	2	?

As pupils act out situations like these, they get an increasingly clear picture of what we mean when we say that $3 \times 2 = 6$ and that $3 \times 3 = 9$. Pupils can grasp the difference between multiplication and addition if illustrations are used to show that:



A Shorthand Notation

Develop the agreement that a way to describe what happens in the pairing problems is to write multiplication number sentences:

$$3 \times 2 = 6$$

Read this as "three times two is six" or as "three multiplied by two is six."

Factors-Product

None of the big words or complicated rules that usually accompany the ideas of multiplication or division is mentioned in MATH WORKSHOP. Pupils

used to be told that there was a multiplicand and a multiplier and that these led to a product. They were also told that in division there was a divisor that had to be divided into a dividend in order to get a quotient. No wonder pupils were so often confused. The terminology is enough to discourage most adults.

In MATH WORKSHOP, however, pupils work with the ideas without being burdened with big words. It is possible to refer to these ideas with two words, *factor* and *product*. These terms are included in the pupil pages and you should make use of them during class discussions.

A FACTOR
ANOTHER FACTOR
$2 \times 3 = 6$ ← THE PRODUCT

$2 \times \_ = 6$ – We have a factor and the product. We can find the missing factor.

$2 \times 3 = \_$ – We have two factors given. We can find the product.

$\_ \times \_ = 12$ – We have the product given. We can find several pairs of factors that would make this a true statement – (1 and 12, 2 and 6, 3 and 4).

THE PRODUCT
A FACTOR
$6 \div 2 = 3$ ← THE OTHER FACTOR

$6 \div 2 = \_$ – We have the product and a factor. We can find the missing factor.

$6 \div \_ = 3$ – (Same language).

$\_ \div 2 = 3$ – We have both factors given. We can find the missing product.

Automatic Mastery of Basic Facts

Regular sessions of 3-5 minutes should be spent reviewing the basic multiplication facts learned in connection with the various representations. Such drill will increase the number of automatic responses of pupils. Aim for automatic mastery of the multiplication facts through 10×10 by the close of LEVEL C.

Use flash cards like those suggested earlier for addition and subtraction. (See Notes on Pupil Page 2.) Pupils who miss facts should be asked to rediscover them via representations.

During drill sessions, as the facts are stated, keep a record in a table like the one at the top of the next column.

Second factor							
5	0	5	10	15	20	25
4	0	4	8	12	16	20
3	0	3	6	9	12	15
2	0	2	4	6	8	10
1	0	1	2	3	4	5
0	0	0	0	0	0	0
X	0	1	2	3	4	5

First factor

Drill

Here is a suggestion for a drill sheet:

	I	II	III	IV	V	VI
A	8 − ⬡ = 6	9 − □ = 8	⬡ − 5 = 2	○ − 4 = 6	11 − 2 = △	8 − 6 = ⬡
B	12 − 3 = □	10 − ○ = 5	△ − 0 = 6	8 − ⬡ = 5	○ − 6 = 3	11 − 9 = △
C	10 − ⬡ = 8	△ − 7 = 5	○ − 1 = 9	11 − △ = 8	⬡ − 8 = 4	14 − 9 = □
D	○ − 4 = 9	14 − □ = 6	13 − △ = 7	12 − 9 = △	13 − 5 = ○	14 − 5 = □
E	13 − △ = 6	⬡ − 5 = 7	□ − 6 = 6	16 − 7 = ⬡	12 − 4 = △	⬡ − 5 = 6

Here are some stories for headline practice:

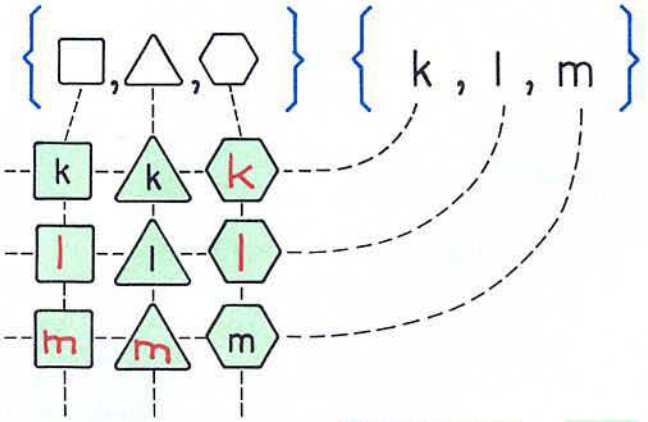
1. Linda bought 2 tickets for rides at the carnival. She had 4 different rides to choose from: the roller coaster, the ferris wheel, the merry-go-round, and the airplanes.

   $2 \times 4 = \square$ $\quad$ $4 \times 2 = 8$

2. Monica picked flowers for her friend in the hospital. She picked 49 daisies and a dozen tulips.
3. Mrs. Simmons bought 10 gallons of gasoline for her car. She averages 15 miles per gallon of gasoline.

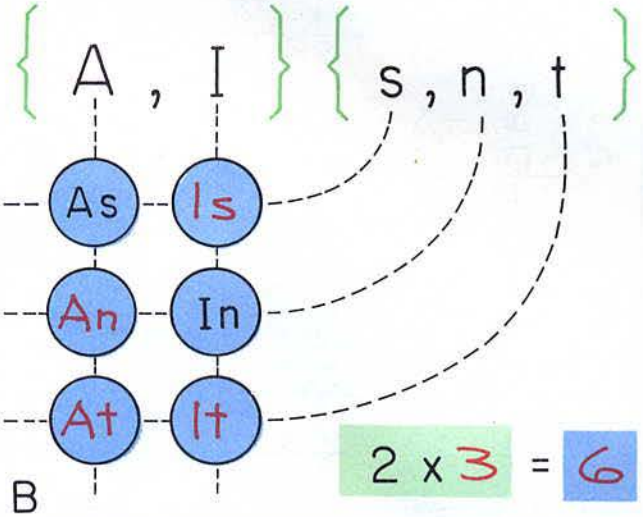
   $10 \times 15 = \square$ $\quad$ $15 \times 10 = 150$

Let's do some more pairing. Remember, each pair of things must get its first thing from the first group and its second thing from the second group.



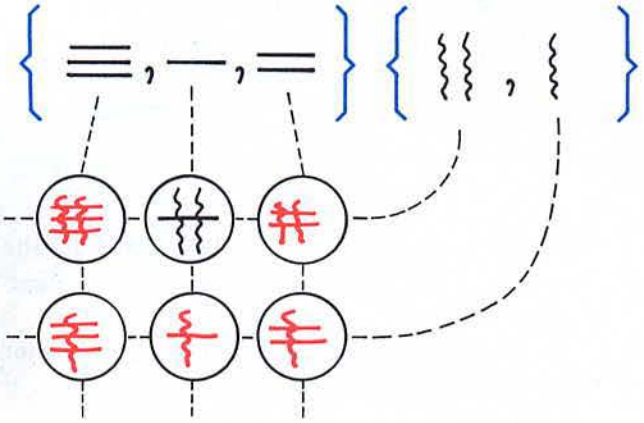
A

3 x 3 = 9



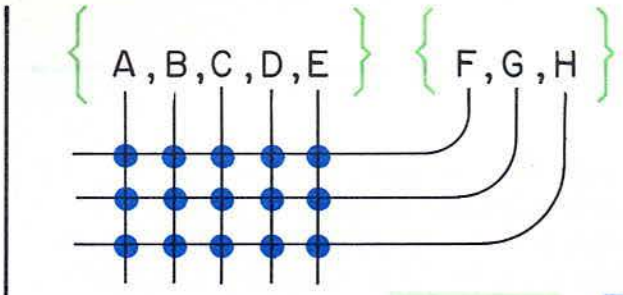
B

2 x 3 = 6



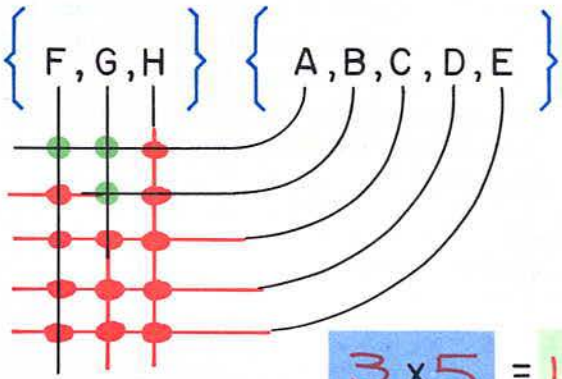
C

3 x 2 = 6



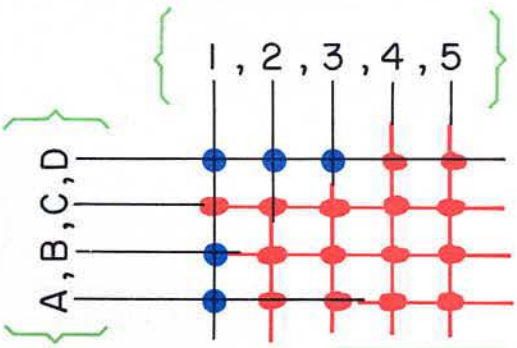
D

5 x 3 = 15



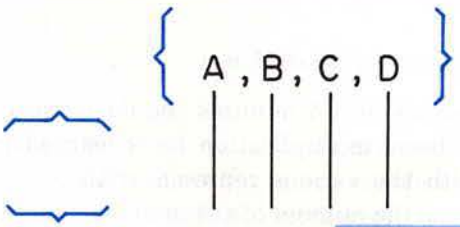
E

3 x 5 = 15



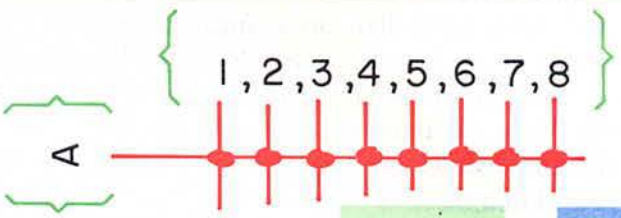
F

4 x 5 = 20



G

0 x 4 = 0



H

1 x 8 = 8

Notes on Pupil Page 13

Tasks: *Pupils complete each representation as required and record in the tinted boxes either missing factors or a missing product.*

Purpose: To further explore multipliaction (and division) in terms of pairing.

Unifying Ideas: Sets; multiplication and division.

Materials for Teacher: Ditto masters.

Materials for Pupils: Cuisenaire rods for pages 62-79 of the booklet *Games and Enrichment Activities;* Ditto sheets of exercises similar to those on pupil page 13; Ditto sheets of blank 10 × 10 tables.

The Lesson:

Continue with the pairing activities suggested in Notes on Pupil Page 12, recording multiplication number sentences that fit the pictures.

Do examples similar to those on pupil page 13, using the silent arithmetic approach we have referred to earlier (see pages 1-5 of the booklet *Games and Enrichment Activities*).

As examples are completed, have pupils keep a record of the two ways of thinking about working with a pair of displays of objects—addition and multiplication:

Table of sums

Second addend \ +	0	1	2	3	4	5
5						
4						
3		4	5		7	
2						
1				4		
0			2			

First addend

Table of products

Second factor \ ×	0	1	2	3	4	5
5						
4						
3		3	6		12	
2						
1				3		
0			0			

First factor

To simultaneously develop such tables strengthens the idea that there are different ways to obtain a third number from two numbers.

Simplifying the 100 Facts of Multiplication

Pupils should be given experience in recording the facts of multiplication in a two-dimensional table. Prepare Ditto sheets of blank tables for such studies as indicated below.

The table is designed to include the 100 basic multiplication facts that are essential to progress in arithmetic. Our purpose here is to impress upon children that they really don't have such a big job, *if* they understand some very important ideas.

Zero as a Factor

If one factor is 0, what do we know about the product? Here are some examples:

$0 \times 3 =$ ___	$9 \times 0 =$ ___
$7 \times 0 =$ ___	$0 \times 8 =$ ___
$0 \times 0 =$ ___	$117 \times 0 =$ ___

Pupils have found that there are 0 pairs if there are 0 first things or 0 second things to be paired. Thus we now know 19 of the 100 facts of multiplication and can fill in that part of the table easily.

One as a Factor

If one of the factors is 1, what do we know about the product?

$1 \times 8 =$ ___	$0 \times 1 =$ ___
$2 \times 1 =$ ___	$7 \times 1 =$ ___
$1 \times 1 =$ ___	$1 \times 87 =$ ___

If one of the factors is 1 then the other factor and the product are the same number. Again, pictures of pairing have made this idea clear. Now we know another 17 facts of multiplication, and these are listed in the table at the top of the next page.

Headline for the Day: · Sal paid 3¢ for each of 4 pencils. He gave the clerk 2 dimes and 1 nickel.

⑨	0	9								
⑧	0	8								
⑦	0	7								
⑥	0	6								
⑤	0	5								
④	0	4								
③	0	3								
②	0	2								
①	0	1	2	3	4	5	6	7	8	9
⓪	0	0	0	0	0	0	0	0	0	0
X	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨

Square Numbers

Ten of the products to be located in the table are often referred to as *square numbers*. Of course, numbers have no shape at all. But if we think of them in relation to rectangular arrays of dots, we observe that whenever such an array has the same number of rows as columns and all are of the same width, the array has a square shape. The multiplication fact suggested by such an array is one in which both factors are the same number. There are 10 cells in the table that should contain square numbers. Two of these, 0×0 and 1×1, we have already considered.

$0 \times 0 =$ ___ $\quad 5 \times 5 =$ ___
$1 \times 1 =$ ___ $\quad 6 \times 6 =$ ___
$2 \times 2 =$ ___ $\quad 7 \times 7 =$ ___
$3 \times 3 =$ ___ $\quad 8 \times 8 =$ ___
$4 \times 4 =$ ___ $\quad 9 \times 9 =$ ___

Only 36 Facts to Memorize

Find the appropriate cells in the table for the products in the above examples. They occupy one of the main diagonals of the table. Since we already know that $0 \times 0 = 0$ and $1 \times 1 = 1$, there are only 8 new facts for us to memorize. That leaves us only 56 more facts to consider. One of these is: $3 \times 7 =$ ___, and it has a partner: $7 \times 3 =$ ___. The order in which a pair of numbers is multiplied doesn't change the product. Each remaining fact has its partner on the other side of the diagonal of square numbers. Make this clear by giving a few examples.

Have pupils fill in the cells as the table is developed. Have them color in lightly all cells that appear *above* the diagonal of square numbers and all that have 0 or 1 as a factor. The 36 multiplication facts to be memorized are now clearly revealed. They are shown in the table at the top of the next column.

All the 100 basic facts of multiplication and division.

⑨	0	9	18	27	36	45	54	63	72	81
⑧	0	8	16	24	32	40	48	56	64	72
⑦	0	7	14	21	28	35	42	49	56	63
⑥	0	6	12	18	24	30	36	42	48	54
⑤	0	5	10	15	20	25	30	35	40	45
④	0	4	8	12	16	20	24	28	32	36
③	0	3	6	9	12	15	18	21	24	27
②	0	2	4	6	8	10	12	14	16	18
①	0	1	2	3	4	5	6	7	8	9 ← 1x□=□
⓪	0	0	0	0	0	0	0	0	0	0 ← 0x□=0
X	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨

The 36 facts in the box and under the steps are the heart of the problems involving memory.

Inside box, above steps — simply change order of factors. For example, since $6 \times 5 = 30$, $5 \times 6 = 30$.

Simplifying the 100 Facts of Addition

Just as we can simplify our approach to the 100 multiplication facts by the use of a two-dimensional table and by noting certain properties of multiplication, so can we simplify our approach to the 100 addition facts.

Zero as an Addend

When 0 is one of the addends, what do we know about the sum? Examples:

$0 + 4 =$ ___ $\quad 0 + 0 =$ ___
$9 + 0 =$ ___ $\quad 2 + 0 =$ ___
$0 + 1 =$ ___ $\quad 0 + 78 =$ ___

When one addend is 0 then the other and the sum are equal. (Exactly like multiplication by 1.)

One as an Addend

When 1 is one of the addends, what do we know about the sum?

$1 + 7 =$ ___ $\quad 6 + 1 =$ ___
$9 + 1 =$ ___ $\quad 1 + 0 =$ ___
$1 + 1 =$ ___ $\quad 2 + 1 =$ ___

When one addend is 1, the sum is that number which comes after the other addend in the order of counting.

So far we have treated 36 of the 100 facts as instances of general principles. There are 64 more.

Doubles

Often we use the expression "double it." The results of doubling are numbers that are listed along one of the main diagonals in the table.

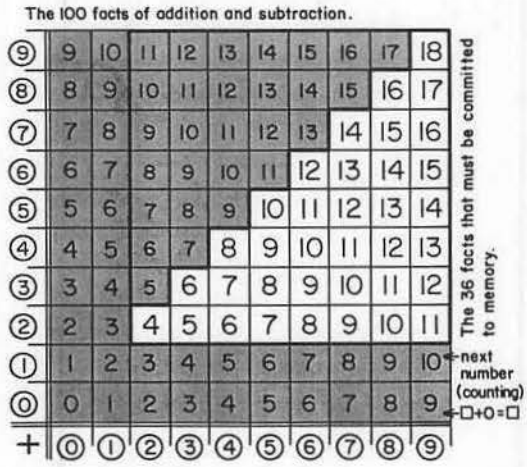
$0 + 0 =$ ___ $5 + 5 =$ ___
$1 + 1 =$ ___ $6 + 6 =$ ___
$2 + 2 =$ ___ $7 + 7 =$ ___
$3 + 3 =$ ___ $8 + 8 =$ ___
$4 + 4 =$ ___ $9 + 9 =$ ___

Only 36 Facts to Memorize

Of these ten doubling facts, the first two have already been considered. The others must be memorized. How many more to go? ($64 - 8 = 56$. This reminds us of our experience with multiplication.)

Let's look at the rest of the facts, one at a time. For example, consider: $3 + 7 =$ ___. Does this have a partner on the other side of the diagonal of doubles? (Yes: $7 + 3 =$ ___.) Does every cell not on this main diagonal have a partner on the other side of the diagonal? (Yes)

Have pupils fill in all the rest of the cells. Now color in lightly all cells that appear above the diagonal and all that have 0 or 1 as an addend. Again, we have reduced a mass of 100 facts to a manageable group, 36 in all, to be memorized!



Try to direct the activity so that the parallelisms between the multiplication and addition tables are obvious. The important fundamental similarities are:

I. Multiplying a number by 1 or adding 0 to it give the number itself as product or sum.

$1 \times 5 = 5$ $5 + 0 = 5$
$6 \times 1 = 6$ $0 + 6 = 6$
$1 \times 7 = 7$ $7 + 0 = 7$

Later, in algebra, pupils will learn a language that will help them express these two ideas as follows:

For each x, $x \times 1 = 1 \times x = x$
and $x + 0 = 0 + x = x$.

II. The order in which a pair of factors or addends is considered does not affect the outcome.

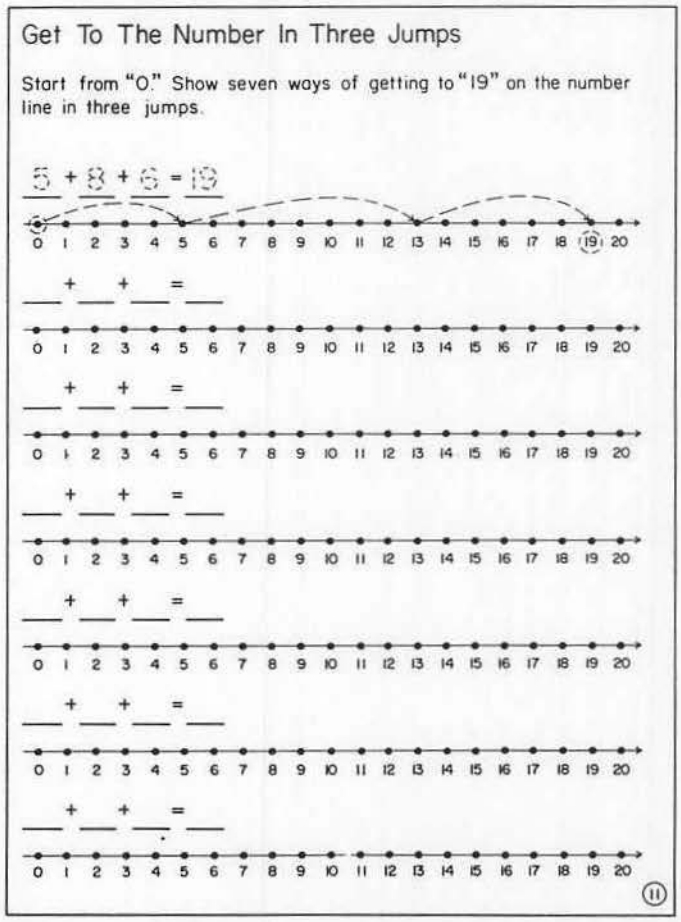
$2 \times 3 = 3 \times 2 = 6$ $2 + 3 = 3 + 2 = 5$
$6 \times 7 = 7 \times 6 = 42$ $6 + 7 = 7 + 6 = 13$

In algebra we express these ideas as follows:

For each x, for each y, $x \times y = y \times x$
and $x + y = y + x$.

Drill

Consider using as a drill sheet page 11 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:


Get To The Number In Three Jumps
Start from "0." Show seven ways of getting to "19" on the number line in three jumps.
5 + 8 + 6 = 19
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
11

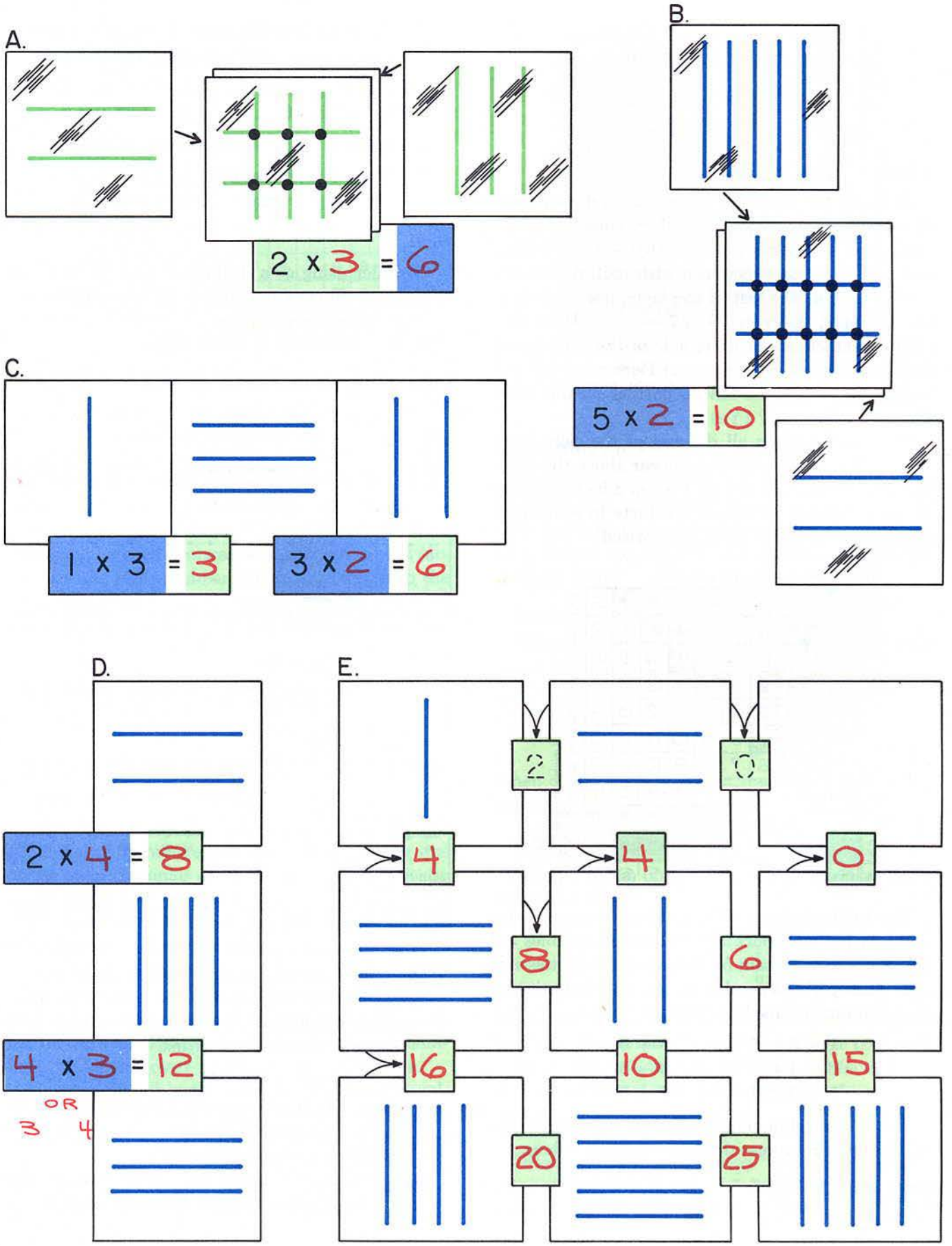
Here are some stories for headline practice:

1. Mr. Jointly needs 1 gallon of oil for the engine of his automobile. The gas station attendant brought the oil in quart cans.

   □ quarts = 1 gallon $1 \times 4 = 4$

2. Fernando took a speed test in handwriting. He wrote 49 letters in the first minute. He wrote 57 letters in the second minute.
3. Lydia planted some pine trees. She planted 3 rows of trees. She put 8 trees in each row.

Imagine pieces of clear plastic with lines drawn on them. Put one on top of another. In how many places do the lines cross?



14 Multiplication in terms of intersections of vertical and horizontal lines.

Tasks: *In each of exercises A-D pupils use the representations to record in the tinted boxes either the missing factor or the missing product. In exercise E pupils use the clues provided (ghosted numbers and arrows) to record the required products in the tinted boxes.*

Purpose: To further extend understanding of multiplication (and division) using crossing vertical and horizontal lines as representations.

Unifying Ideas: Multiplication and division.

Materials for Teacher: Overhead projector; transparencies as described below.

Materials for Pupils: None.

The Lesson:

Throughout this book there are many opportunities provided to reexamine ideas that have been studied earlier. Each time the setting is changed, however, so that pupils really need to *think* about the new situation. Memory alone will not suffice. Another important effect of such diversity of activity is that interest is kept high as pupils look for clues in trying to understand the new situations. Shouts of "Ohhh, I get it" and similar "Aha!" expressions of discovery occur frequently in such a stimulating environment.

Pupil page 14 suggests a visual aid if you wish to make one. The mathematical sentences in A-E are records of stacking clear plastic or glass sheets marked with horizontal and vertical lines and determining the number of crossing points. Pupils are asked in C-E to imagine these actions. In E they are asked to record the numbers of crossing points if each two adjacent sheets were placed one on the other. You can prepare them to use their imagination by presenting some of the exercises on an overhead projector. Draw parallel lines on acetate sheets and use these sheets as transparencies.

You may wish to ask pupils to write all of the multiplication sentences represented in E on another sheet of paper.

Practice on Multiplication Facts

Give practice in multiplication in tables like the following:

4			
2			
0			
×	0	2	4

5			
3			
1			
×	1	3	5

5			
3			
1			
×	0	2	4

Notice that in these examples we are also exploring the questions:

1. Given a pair of even factors, what can you say about the product?
2. Given a pair of odd factors, what can you say about the product?
3. Given one even and one odd factor, what can you say about the product?

Drill

Here is a suggestion for a drill sheet:

Fill in the unshaded squares in the addition table.

+	0	1	2	3	4	5	6	7
0								
5								
10								
15								
20								
25								
30								
35								

Headline for the Day: Circle the number which doesn't belong in the pattern:

1, 2, 4, 8, 16, 32, 62

Bill had some marbles. He had two boxes — one square and one round. He put all the marbles in the two boxes.

What can we say about this situation? Here are two things we might say:

1. If Bill had an even number of marbles, he could have put the same number of marbles in each box.
2. If he had an odd number of marbles, he could not have put the same number in each box.

The records below show some other things we can say about the situation.

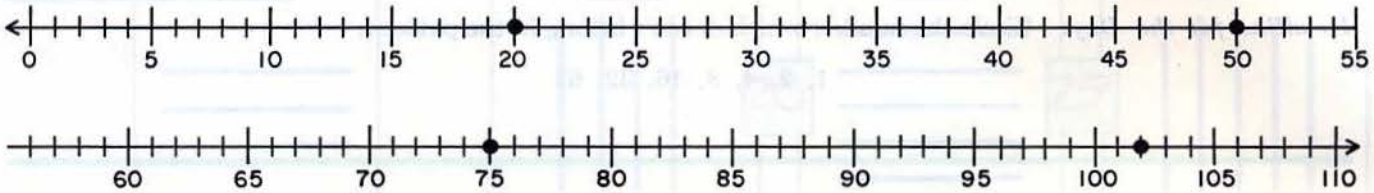
If Bill had 20 marbles	
and if he put ____ in the square box	then he put ____ in the round box
15	5
5	15
10	10
1	19
3	17
0	20
2	18
14	6
4	16
13	7

If Bill had 50 marbles	
and he put ____ in the square box	then he put ____ in the round box
10	40
20	30
25	25
2	48
41	9
45	5
12	38
21	29
17	33
37	13

If Bill had 75 marbles	
and he put ____ in the square box	then he put ____ in the round box
70	5
10	65
45	30
50	25
31	44
6	69
67	8
16	59
35	40
37	38

If Bill had 102 marbles	
and if he put ____ in the square box	then he put ____ in the round box
100	2
3	99
5	97
10	92
20	82
50	52
62	40
9	93
89	13
47	55

You may wish to use the two number line chunks below to check some of your results.



15 Listing the various partitions for given numbers of marbles - optional use of number line chunks to determine appropriate partitions.

Tasks: *Pupils complete the table for each given condition by recording the missing addends.*

Purpose: To review subtraction in terms of imagining the partitioning of displays of objects.

Unifying Ideas: Structure; addition and subtraction.

Materials for Teacher: Two demonstration-size boxes (square and circular); marbles or counters; number line chunks; graph board.

Materials for Pupils: Cuisenaire rods or counters.

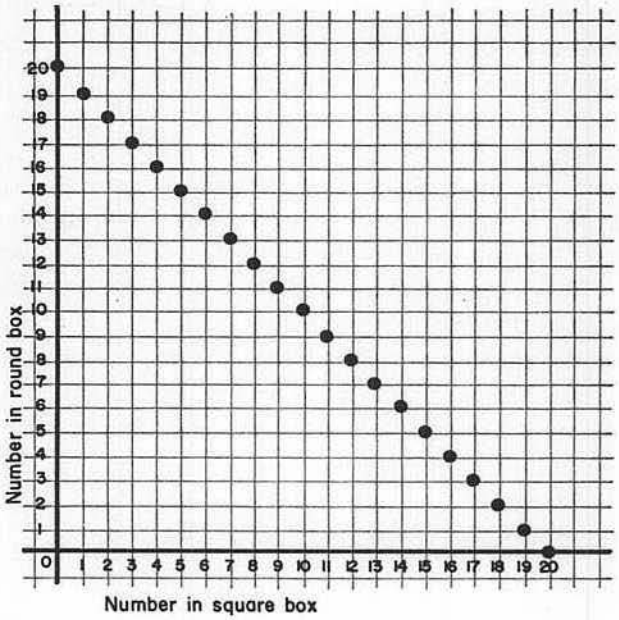
The Lesson:

The exercises on this page provide excellent practice in subtraction and many opportunities to gain insights into subtraction. As preparation for the first exercise (If Bill had 20 marbles . . .), you may wish to draw a graph on a graph board. Start by drawing a horizontal number line and a vertical number line as shown below. Then suppose that Bill put 15 marbles in the square box and 5 marbles in the round box. Put one finger of your right hand on the 15 on the horizontal number line and one finger of your left hand on the 5 on the vertical number line. Move the left hand horizontally to the right and the right hand vertically upward. The paths will cross at a point. Put a large dot at that point. Let pupils "plot" other points, each point representing one of the partitions of 20 objects. What the graph looks like when all 21 points have been plotted is shown at the top of the following page.

The graph provides interesting insight into addition (and subtraction). For example, the fact that the graph slopes downward from left to right shows that, given a pair of variable quantities whose sum is constant, if one quantity increases then the other

more

Headline for the Day: What are some of the points that you could land on if you start at 2 on the number line and make jumps of 3 to the right?



must decrease, and conversely. Also, if one increases by a certain amount, the other must decrease by that same amount, and conversely.

This last idea is useful in doing the exercises on pupil page 15. For example, consider the second table. Suppose that the pupil has written a 40 for his first answer. He then notes that the second problem tells him that there are 20 marbles in the square box, an increase of 10 over the previous case. Hence, there should be a decrease of 10 marbles in the round box. To find the second answer, the pupil merely subtracts 10 from 40.

When pupils have completed pupil page 15, they should be ready for questions such as the following:

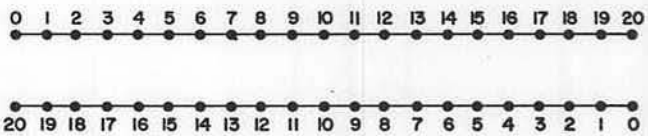
> Suppose that Bill had 20 (50, 75, 102) marbles and wished to put them all in the two boxes so that the square box would contain 2 more than the round box. How many should be put in each? Answers: (11, 9); (26, 24); can't be done; (52, 50)

The first of these questions (the case of 20 marbles) can be solved mechanically by using the graph discussed above. Plot some of the points which represent situations in which the square box has 2 more marbles than the round box: (2, 0), (3, 1), (4, 2), (5, 3), etc. The line of plotted points will head toward the point (11, 9), a point which belongs to the previous graph. This common point represents two situations—a total of 20 marbles for both boxes, and 2 more marbles in the square box.

Can a pupil predict that in the case of 75 marbles it is impossible to have 2 more in the square box? Suppose that there were an even number of marbles in the square box. Then there would be an even number in the round box. Since the sum of two even numbers is even and since 75 is not even, there can't be an even number of marbles in the square box. Suppose that there were an odd number of marbles in the square box. Then . . .

Number Line Chunks

The problems on pupil page 15 can also be solved by using pairs of number line chunks properly aligned. For example, in the case of the first table, all we need is a pair of chunks such as the following:



Each pair of dots in vertical alignment represents a partition of the 20 marbles.

Drill

Here are some stories for headline practice:

1. Mrs. Toll changed the oil in her 4 trucks. Each truck needs 8 quarts of motor oil.

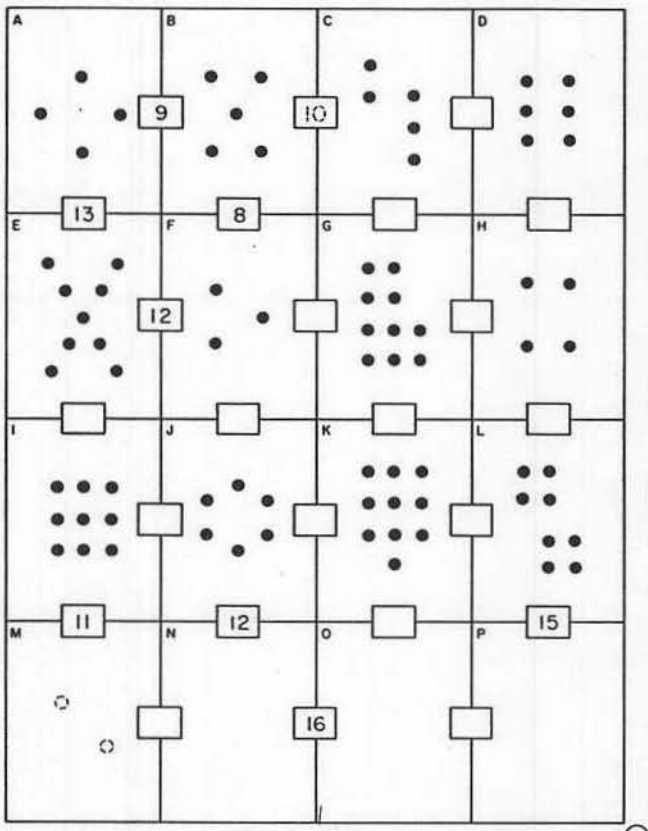
   $8 + 8 + 8 + 8 = \square$ $\qquad$ $4 \times 8 = 32$

   $8 \times 4 = \square$

2. Loni bought eight 5-cent stamps.
3. Theodore put twenty-five candles on his father's birthday cake. His father is forty-nine years old.
4. There are 29 children in a third grade class at Zabranchburg Elementary School. 14 of the children are girls.
5. Anabru Junior High School's football team scored 6 touchdowns. Each touchdown is worth 6 points.
6. When mother weighed herself while holding the baby, she weighed 151 pounds. When she put the baby down and weighed herself, she weighed 119 pounds.

Consider using as drill sheets pages 7 and 12 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown on the next page.

WHAT'S MY RULE?

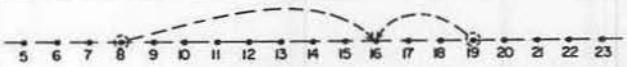


⑦

NUMBERS HAVE MANY NAMES

Trips	A	B	C	D	E	F	G	H	I	J
Starting point	7		3		5	6		2	4	
Size of jump to the right ↷	3	1		0	5		2	8		10
Landing point	10	10	10	10		10	10		10	10

	K	L	M	N	O	P	Q	R	S	T
Starting point	7	8	10	15	9		16		11	6
Size of jump to the left ↶	1	2			3	7	10	6	5	
Landing point	6		6	6		6		6		6



	U	V	W	X	Y	Z	a	b	c	d
Starting point	19	8		9		13	23	16	20	
Size of jump to the right ↷	—		—	7	—		—	0	—	3
Size of jump to the left ↶	3	—	6	—	0	—	7	—		—
Landing point	16	16	16		16	16			16	16

⑫

How many 2-digit numbers are there with
the tens digit from the green box
and
the ones digit from the blue box?

A

4, 7	0

40 70 ~~XX~~

2 numbers

B

8	9, 5

89 85 X

2 numbers

C

6	

X X X

0 numbers

D

4, 5	0, 1, 2, 3

40 41 42
43 50 51
52 53 X

8 numbers

E

1, 9, 6, 7	3, 8

13 73 68
93 18 78
63 98 X

8 numbers

F

1, 2, 3	1, 2, 3

11 21 31
12 22 32
13 23 33

9 numbers

G

9, 7	1, 2, 3, 4, 5

91 95 74
92 71 75
93 72 X
94 73 X

10 numbers

H

1, 2, 3, 4, 5	9, 7

19 59 47
29 17 57
39 27 X
49 37 X

10 numbers

I

1, 2, 3	1, 2, 3, 4

11 21 31
12 22 32
13 23 33
14 24 34

12 numbers

J

Alice had 4 blouses: pink, orange, white, and brown. She had 3 skirts: black, purple, and blue. How many blouse-skirt combinations does she have?

pink blouse with black skirt
pink blouse with purple skirt
orange blouse with black skirt
PINK blouse with BLUE skirt
ORANGE blouse with PURPLE skirt
ORANGE blouse with BLUE skirt
WHITE blouse with BLACK skirt
WHITE blouse with PURPLE skirt
WHITE blouse with BLUE skirt
BROWN blouse with BLACK skirt
BROWN blouse with PURPLE skirt
BROWN blouse with BLUE skirt

12 different combinations

16 Multiplication in terms of combining digits to form 2-digit numbers - forming blouse-skirt combinations.

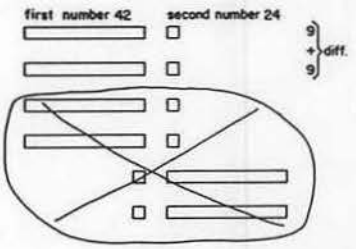
Tasks: *In each of exercises A-J pupils follow the directions at the top of the page and record in the tinted boxes all possible 2-digit numbers which can be named with the digits in the given lists. Pupils also record in the blanks the number of 2-digit numbers they have named. In exercise J pupils record in the blanks all possible blouse-skirt pairs and the number of such combinations.*

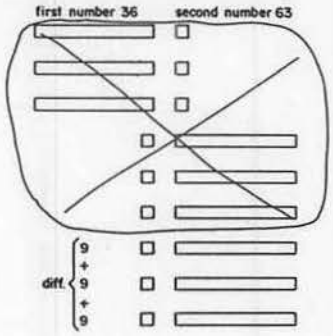
Purpose: To explore multiplication via pairing.

Unifying Ideas: Numeration; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: Cuisenaire rods.





The Lesson:

(See Notes on Pupil Page 17.)

When pupils have completed this page, you may wish to ask questions such as the following:

(1) How many 2-digit numbers are there altogether? (Answer: 90, because $9 \times 10 = 90$)

(2) How many 3-digit numbers are there altogether? (Answer: 900, because $90 \times 10 = 900$)

(3) Write a 2-digit number and then write the 2-digit number obtained by reversing the digits of the first one. (Exercises G and H should help prepare for this one.) Subtract the smaller from the larger. Is the result a number in the multiply-by-9 table? (Answer: It should be.)

You can get insight into (3) by using Cuisenaire rods to represent the numbers. In the following column are two examples.

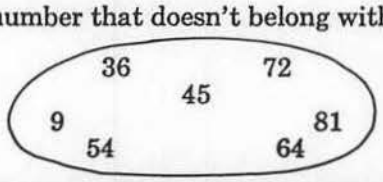
Also, question (3) suggests a fourth question relating to the result of *adding* the two numbers. We leave this for you to explore.

Drill

Here is a suggestion for a drill sheet:

1. $\square + 5 = 8$	2. $3 + \square = 10$	
3. $5 + \square = 9$	4. $\square + 3 = 8$	
5. $10 - 3 = \square$	6. $4 + \square = 9$	
7. $6 + \square = 11$	8. $12 - 9 = \square$	
9. $7 + \square = 7$		
10. $10 - 7 = \square$	11. $10 - 4 = \square$	12. $8 - 3 = \square$
13. $9 - 5 = \square$	14. $5 - 5 = \square$	15. $\square + 6 = 13$
16. $\square + 2 = 10$	17. $3 + \square = 5$	18. $6 + \square = 6$
19. $4 + \square = 5$	20. $(17 + 9) - \square = 17$	

Headline for the Day: Circle the number that doesn't belong with the others:



How many 2-letter words are there with the first letter from one list and the second letter from the other? Find as many different pairs as you can.

I.

M	e, y

Me My

2 words

II.

S, W	

0 words

III.

H, B	e

He Be

2 words

IV.

T, D, S, N	o

To Do So No

4 words

V.

A	m, n, s, t

Am An As At

4 words

VI.

B, M	y, e

By Be My Me

4 words

VII.

I	f, t, n

If It In

3 words

VIII.

O, I	n, f

On Of In If

4 words

IX.

A, I	s, n, t

As An At Is In It

6 words

X.

How many combinations?

Hamburger	with	mustard
Fish	with	nothing
Fish	with	onions
Fish	with	Catsup

Fish	with	Mustard
Hamburger	with	Onions
Hamburger	with	Catsup
Hamburger	with	Nothing
Hot Dog	with	Onions
Hot Dog	with	Catsup
Hot Dog	with	Mustard
Hot Dog	with	Nothing
———	with	———

There are 12 possible combinations.

17 Multiplication in terms of combining letters to form two-letter words - forming combinations.

Tasks: *In exercises I-IX pupils follow the directions at the top of the page and record in the tinted boxes all 2-letter words which can be constructed with the letters in the given lists. Pupils also record in the blanks the numbers of 2-letter words they have constructed. In exercise X pupils record in the blanks all possible food combinations described in the "advertisement" and the total number of such combinations. (Since the sandwiches are free, it is reasonable to agree that one should not be piggish and use more than one condiment on a sandwich.)*

Purpose: To continue the exploration of multiplication via pairing.

Unifying Ideas: Multiplication and division.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

Pupil pages 16 and 17 provide four more experiences with the idea of multiplication. We are again involved with finding all pairings of the members of two sets. (Compare exercise J on pupil page 16 with exercise X on pupil page 17.)

It would be well to divide the class into teams of 3 or 4 pupils to work on these pages on their own while you supervise their work. During this time you may prefer to work with individuals or small groups needing closer supervision or more direct instruction on work studied up to this point. While we encourage the discovery approach in MATH WORKSHOP, it is evident that there will be times when some pupils need more direct explanations. Do not hesitate to give such help when you feel a need for it.

Practice With Addends and Sums; Factors and Products

Each addition or multiplication fact involves a triple of numbers—(addend, addend, sum) or (factor, factor, product). With just two exceptions, if you are given any two components of the triple involved in a fact, you can find the third component. For example, if you are given the addition triple (2, ___, 8),

more

Headline for the Day: Jacques decided to have some dessert. He could select any one of 4 flavors of ice cream—licorice, asparagus, tomato, or cabbage. He also had his choice of anyone of 3 syrups—watermelon, lettuce, or cauliflower. Make a list of the combinations he could choose from.

you know that the missing component is 6 because $2 + 6 = 8$ and there is only one number which can be added to 2 to give the sum 8. However, if you are given either of the multiplication triples (0, ___, 0) or (___, 0, 0), you cannot determine the missing component because the product of *any* number with 0 is 0.

Put several tables on the chalkboard or on Ditto sheets for pupils to complete:

	Addend	Sum	Addend
A		8	7
B		6	3
C	2	9	
D	5		6
E	5	15	

	Sum	Addend	Addend
I	3		2
II		0	8
III	20	18	
IV		5	10
V	5	0	

	Factor	Product	Factor
a	3		4
b	2	6	
c		10	5
d	8		2
e		1	

(On rare occasions, we need only be given one component of the triple to determine the other two. This is the case in the last example above. Provided that we are limited to whole numbers, we can complete the last line above because (1, 1, 1) is the only triple of whole numbers in which the product is 1. The same is true of addition if we know that the sum is 0, since (0, 0, 0) is the only addition triple of whole numbers in which the sum is 0.)

	Product	Factor	Factor
(1)	18		
(2)		4	5
(3)	36	6	
(4)		1	2
(5)	7	1	

Here are some examples in the mathematical shorthand of addition and multiplication. In each case, one component of the triple is missing: an addend, a sum, a factor, or a product. Which is it? What number would convert the sentence into a true one?

$3 + 4 = \_\_\_$ $\quad$ $2 \times 5 = \_\_\_$

$7 + \_\_\_ = 9$ $\quad$ $4 \times \_\_\_ = 8$

$\_\_\_ + 8 = 12$ $\quad$ $\_\_\_ \times 3 = 9$

$9 = 4 + \_\_\_$ $\quad$ $\_\_\_ = 6 \times 2$

$12 = \_\_\_ \times 3$ $\quad$ $\_\_\_ = 0 + 7$

$\_\_\_ = 15 \times 1$ $\quad$ $20 = \_\_\_ \times 10$

$7 = 1 + \_\_\_$ $\quad$ $2 \times \_\_\_ = 18$

$3 \times 5 = \_\_\_$ $\quad$ $12 = \_\_\_ + 7$

$\_\_\_ = 4 \times 4$ $\quad$ $10 + \_\_\_ = 15$

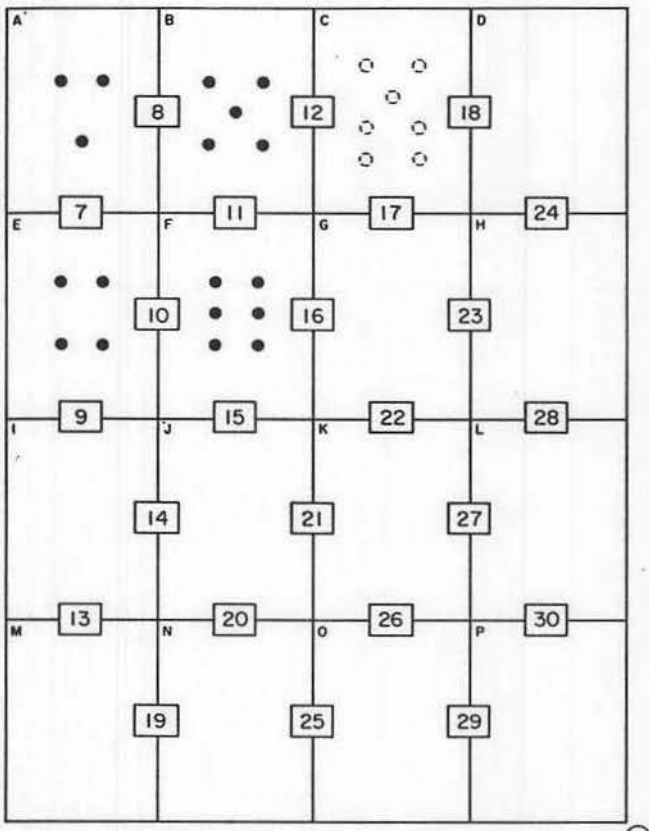
Drill

Here are some stories for headline practice:

1. Mr. Hanson's third graders cut out 58 pictures of storybook people and places. They colored 37 of them with crayons and painted the rest with water colors.
2. In a ringtoss game Joshua had the best score. He made 4 on his first turn, 4 on his second turn, and 3 on his third turn.
3. Rudy placed 24 books on 3 shelves of his bookcase so that each shelf contained the same number of books.

Consider using as a drill sheet page 8 and as a test on basic addition facts page 14 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown on the next page.

WHAT'S MY RULE?



8

Example: $\frac{5}{\frac{2}{7}}$ → 5 + 2 = 7

ADD

A

2/2	6/6	0/5	1/0	9/1
7/6	9/5	4/7	3/7	9/2
4/4	1/1	3/3	5/4	0/8
6/2	2/5	8/7	2/0	8/2
3/9	6/3	3/8	8/3	4/2

B

4/1	2/3	8/4	5/8	9/4
5/2	1/9	6/5	7/8	3/1
1/2	8/5	2/7	4/9	1/7
5/9	4/6	0/6	1/3	7/5
3/2	9/3	0/4	7/0	5/7

C

6/9	5/6	9/6	4/3	5/8
3/4	6/7	3/6	2/4	8/9
2/8	6/4	7/1	8/8	4/8
8/0	1/6	6/1	0/1	8/1
9/7	5/5	0/7	7/3	4/5

D

7/7	6/8	1/5	7/4	0/3
3/5	0/9	2/9	8/6	5/3
7/2	1/4	9/0	5/1	0/0
4/0	9/8	1/8	0/2	7/9
3/0	6/0	2/6	2/1	9/9

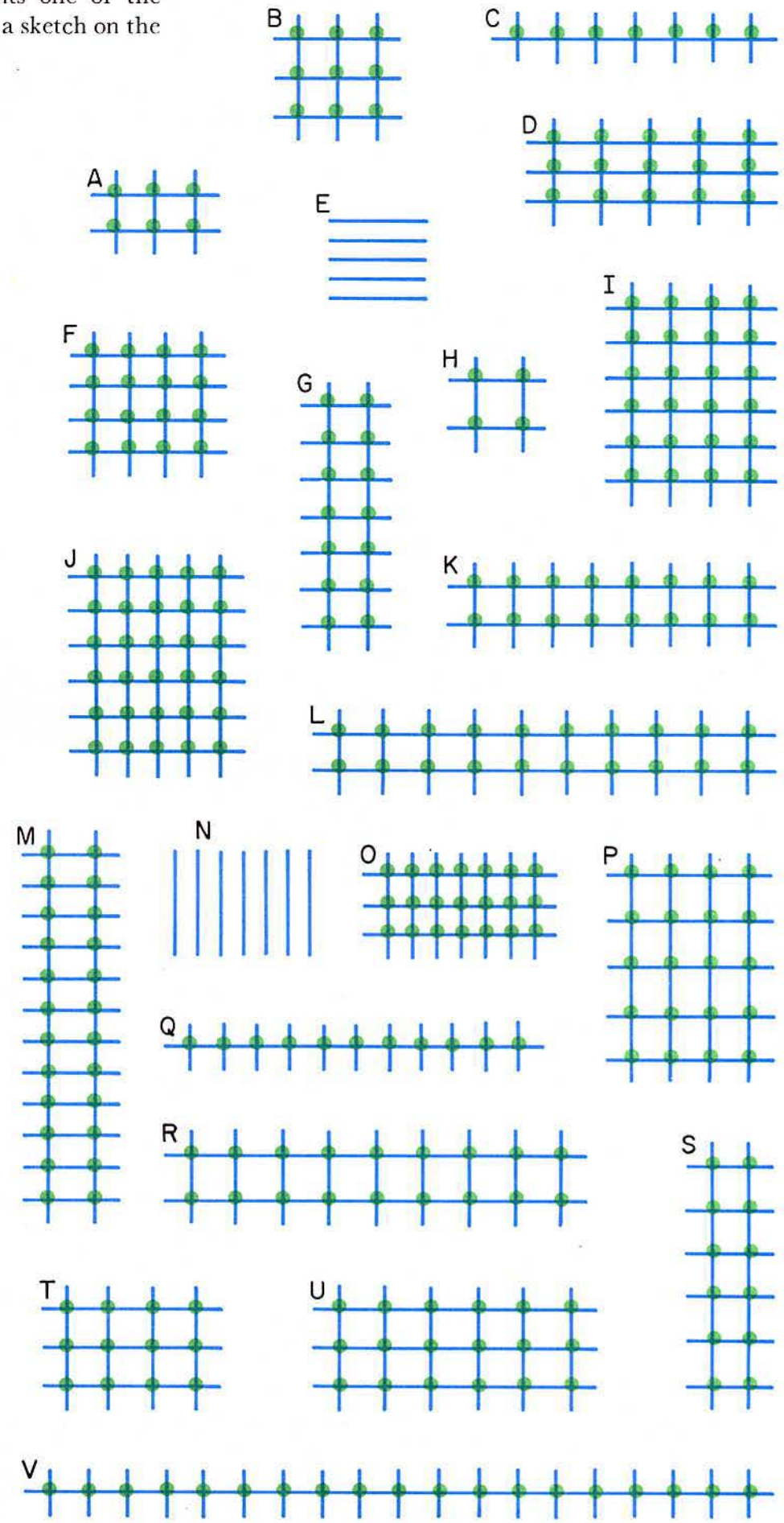
Name _____ Grade _____ Date _____

Number right _____ Time: minutes _____

14

Each line of the table fits one of the sketches. Write the letter of a sketch on the line which fits the sketch.

Sketch	Number of lines	Number of crossing points
D	8	15
A	5	6
H	4	4
E	5	0
B	6	9
I	10	24
C	8	7
M	14	24
G	9	14
K	10	16
F	8	16
P	9	20
J	11	30
L	12	20
T	7	12
S	8	12
V	20	19
O	10	21
Q	12	11
R	11	18
N	7	0
U	9	18



18 Additions and multiplication via matching entries in a table (related sums and products) with pictures of intersecting vertical and horizontal lines.

Tasks: *Pupils use the information provided by each line of the table to record in the white blank on that line the letter of the array whose description is given.*

Purpose: To practice partitioning and multiplying, and factoring and adding.

Unifying Ideas: Addition and subtraction; multiplication and division.

Materials for Teacher: Flash cards; Ditto masters.

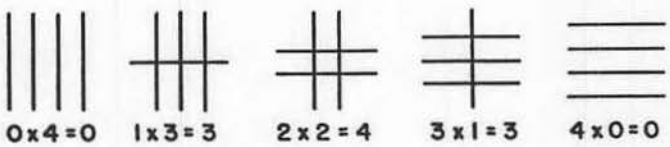
Materials for Pupils: Ditto sheets.

The Lesson:

In preparing pupils to get the most from this page give them practice exercises such as the following (perhaps on Ditto sheets):

Using a total of 4 lines, draw as many different multiplication pictures as you can. State the multiplication fact for each picture.

Answer:



Or, give them oral exercises such as the following:

I have used a total of 8 lines in drawing a multiplication picture. What are the possible numbers of crossing points? (Answer: 0, 7, 12, 15, 16)

I drew a multiplication picture with 20 crossing points. How many lines might I have used? (Answer: 21, 12, 9)

A pupil might solve the problems on pupil page 18 by writing next to each sketch the number of horizontal lines and vertical lines and the number of crossing points. Then, to match the sketch with the tabular data, all he needs to do is look for the sketch's pair in the table. Of course, this is an acceptable procedure but it doesn't afford the pupil as much opportunity in creative thinking as he would have if he worked from the table to the picture. For example, in finding a sketch for the pair (10, 24), he can run through in his mind all the 2-component partitions of 10 ($10 + 0, 9 + 1, 8 + 2, \ldots$) in order to find one the product of whose components is 24 ($10 \times 0 = 0$, $9 \times 1 = 9$, $8 \times 2 = 16$, $7 \times 3 = 21$, $6 \times 4 = 24$), and then look for a 6-by-4 sketch. Or, he might run through the 2-component factorizations of 24 ($1 \times 24, 2 \times 12, 3 \times 8, 4 \times 6, \ldots$) in order to find one the sum of whose components is 10 ($1 + 24 = 25, 2 + 12 = 14, 3 + 8 = 11, 4 + 6 = 10$), and then look for a 4-by-6 sketch. Or, he might run through the partitions and the factorizations simultaneously.

Note that the prime numbers in the exercises (7, 19, and 11) can be represented only by 1-by-____ sketches (C, V, and Q). Ask pupils if they can suggest other numbers which can be represented only by 1-by-____ sketches.

Test on Multiplication Facts

You may wish to prepare several Ditto sheets to test mastery of the multiplication facts. Any of the following approaches might be used:

1. Fill in a multiplication table.

X	0	1	2	3	4	5	6	7	8	9
9										
8										
7										
6										
5										
4										
3										
2										
1										
0										

Headline for the Day: Diane and her father planted 4 rows of tulip bulbs. In each row they planted 5 tulip bulbs. Some of the tulips turned out to be red and the rest of them were yellow. Make a table to show how many could have been red and how many yellow.

2. Fill in a multiplication table in which the order of factors is mixed or in which the factors are listed in decreasing order.

X	0	2	4	6	8	1	3	5	7	9
4										
5										
6										
3										
7										
0										
1										
8										
2										
9										

X	9	8	7	6	5	4	3	2	1	0
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

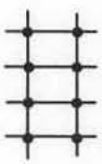
3. Test the facts presented in this form:

a. $2 \times 4 =$ ___
b. $4 \times$ ___ $= 8$
c. ___ $\times 8 = 0$
etc.

In all such tests pupils should have scratch paper or use the back of the sheet to draw pictures when they are unsure or make an error which you detect. Remind them of these three models, all of which represent 4×2 and 2×4:

(a)

(b)



(c)

Flash Cards for the Multiplication Facts

Suggestions given earlier (see Notes on Pupil Page 2) for mastering the addition and subtraction facts with the help of flash cards may be applied now to multiplication and to division.

Various Notations for Division

Here are common notations for division:

Example: 12 divided by $3 = ?$

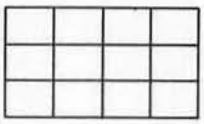
$3\overline{)\overset{?}{12}}$ $\quad 12 \div 3 = ?$ $\quad \frac{12}{3} = ?$

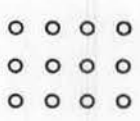
We have, of course, already anticipated division in our experience with multiplication. Actually, a division problem involves a multiplication triple (see Notes on Pupil Page 17) for which a factor component is missing. Each of the sentences: $3 \times$ ___ $=$ 12 and: ___ $\times 3 = 12$ asks the same question as is posed in each of the various notations given above.

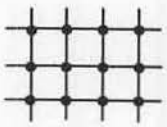
It is important in considering the question asked in each of these notations to realize that in each case we are looking for a missing factor, and to recognize that any picture which represents a multiplication fact with factors other than 0 also represents the other three members of the fact team. Thus, each of the sentences:

$3 \times 4 = 12$ $\qquad$ $12 \div 4 = 3$
$4 \times 3 = 12$ $\qquad$ $12 \div 3 = 4$

is a proper headline for any of the following pictures:







$3 \times 4 = 12$ . . . If we have a rectangular array of 3 rows and 4 columns, we have 12 blocks (or dots or crossing points).

$4 \times 3 = 12$ . . . If we have a rectangular array of 4 columns and 3 rows, we have 12 blocks (or dots or crossing points).

$12 \div 4 = 3$ . . . If we have a rectangular array of 12 blocks and there are 4 columns, we have 3 rows.

$12 \div 3 = 4$ . . . If we have a rectangular array of 12 blocks and there are 3 rows, we have 4 columns.

Drill

Here is a suggestion for a drill sheet:

Rules:

(a) Sub 2 21, 19, 17, , , , , ,

(b) Sub 5 48, , , , , , , ,

(c) Sub 10 83, , , , , , , ,

(d) Sub 3 37, , , , , , , ,

(e) Sub 4 60, , , , , , , ,

(f) Sub 6 71, , , , , , , ,

(g) Sub 5 126, , , , , , , ,

(h) Sub 7 100, , , , , , , ,

(i) Sub 6 100, , , , , , , ,

(j) Sub 8 100, , , , , , , ,

(k) Sub 7 145, , , , , , , ,

(l) Sub 9 100, , , , , , , ,

(m) Sub 8 539, , , , , , , ,

–	12	17	10	14	19	15	11	16	13	18
5	7									
3		14								
0										
4								12		
7										
1										
8										
2										
6										
9										

Here are some stories for headline practice:

1. Eighteen children went to a party. In one of the games there were 9 on each team.
2. Bill gave 12 cookies to his 6 friends. Each friend received the same number of cookies.
3. Thirty-seven people were awaiting the arrival of the express train. Sixteen of these people were children.

Mary had 23¢. She spent more than 15¢.
What can you say about this situation?

We can say that ..... if she spent 23 ¢ then she had 0 ¢ left;

if she spent 22 ¢ then she had 1 ¢ left;

if she spent 21 ¢ then she had 2 ¢ left; etc.

We can make a chart to show many of the possibilities:

If she spent ■ ¢	23	22	21	20	19	18	17	16					
then she had ■ ¢ left.	0	1	2	3	4	5	6	7					

Comment: There are 8 different amounts Mary could spend in the situation given.

Bill had a dime bank with 4 dimes in it.
He received some dimes for his birthday.
He put all the dimes he received into his bank. What can you say?

We can say that ..... if he received 1 dime then he had 50 ¢ in his bank;

if he received 3 dimes then he had 70 ¢ in his bank;

if he received 0 dimes then he had 40 ¢ in his bank; etc.

Below we can keep a record. In it we can record the results of many of the possibilities.

If Bill received ■ dimes	1	3	0	2	5	4	6	8	7	9	
then he had ■ ¢ in his bank.	50	70	40	60	90	80	100	120	110	130	

Comment: Did we consider all of the amounts that Bill might have received?

NO. HE MIGHT HAVE RECEIVED MORE THAN 9 DIMES.

WE DID EXPLORE ALL WAYS IF HE RECEIVED LESS THAN 10 DIMES.

19 Listing various amounts of money determined by speculating on What Can You Say? situations involving money.

Tasks: *Pupils record in the tinted boxes and blanks the various partitions implied by the stories and the accompanying suggestions.*

Purpose: To use addition and subtraction in several applications.

Unifying Ideas: Addition and subtraction; measurement; functions and relations.

Materials for Teacher: Colored chalk; graph board; coins (see below).

Materials for Pupils: Graph paper.

The Lesson:

What Can You Say? problems present situations in which the answer is not obvious. Conditions are given that render it possible to make a number of true statements. If the situation is explored and a record kept, it is possible to make rather conclusive statements, but not until a pattern is discovered.

As preparation for this page, put some coins in your hand and close your fingers over them. Say: In my hand I have 9¢ in coins. Can you be sure what coins I am holding? (No; you could have 9 pennies or you could have a nickel and 4 pennies.) Can you say anything for sure about the coins I am holding?

more

Headline for the Day: Duane had a quarter. He spent less than 13 cents. What can you say about this situation? Make a table.

If pupils have not played this kind of game before, you might ask some of the following questions:

Can you say for sure whether or not I have any dimes? (Yes)

Can you say for sure that I have at least 5 coins? (Yes)

Can you say for sure that I do not have more than 9 coins? (Yes)

Can you say for sure that I have at least one nickel? (No)

Can you say for sure that I have at least 4 pennies? (Yes)

(In this game of *What Can You Say?* attention to precise expression is very important.)

Can you say for sure that I have either 5 coins or 9 coins? (Yes)

Change the coins in your hand: Now, the coins I am holding are worth 13¢ altogether. What can you say for sure about these coins? Some acceptable answers are:

You have at least 3 pennies.

You have at least 4 coins.

You do not have more than 13 coins.

If you have 5 coins, they are 2 nickels and 3 pennies.

If you have more than 9 coins then you have 13 pennies.

If you have 4 pennies then you have at least 8 pennies.

Without telling the pupils what coins you actually have in your hand, change the question: Tell me some combinations of coins I could have.

After a few have been suggested, ask: How many different combinations of coins are there that are worth 13¢? Make a chart:

Dimes	Nickels	Pennies	Total No. of Coins
		13	13
	1	8	9
	2	3	5
1		3	4

There are 4 possible combinations.

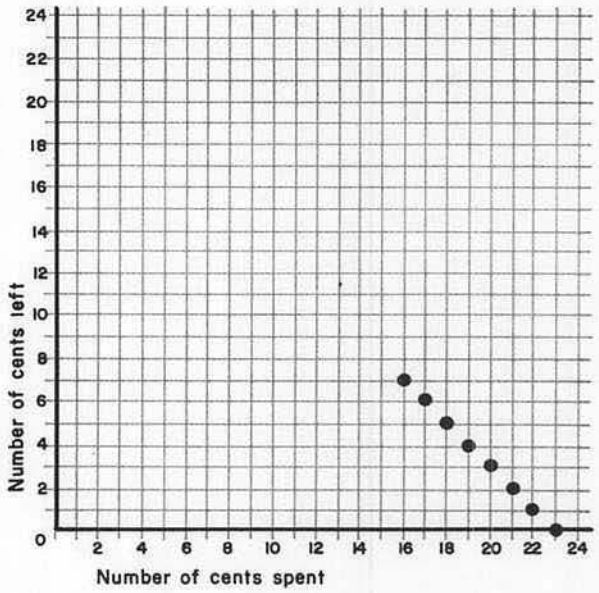
Similar problems that you may wish to explore with the class as preparation for pupil page 19 include:

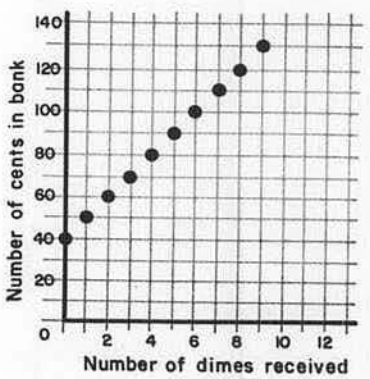
1. Tom is twice as old as his brother Bob. Bob is either 7, 8, or 9 years old.
2. I have 15¢ made up of an odd number of coins. What coins do I have?
3. I gave the clerk a quarter and he gave me three coins in change. What can you say?

Graphs

What Can You Say? problems lend themselves easily to graphing. (See Notes on Pupil Page 15.)

Here are graphs for the problems on pupil page 19:





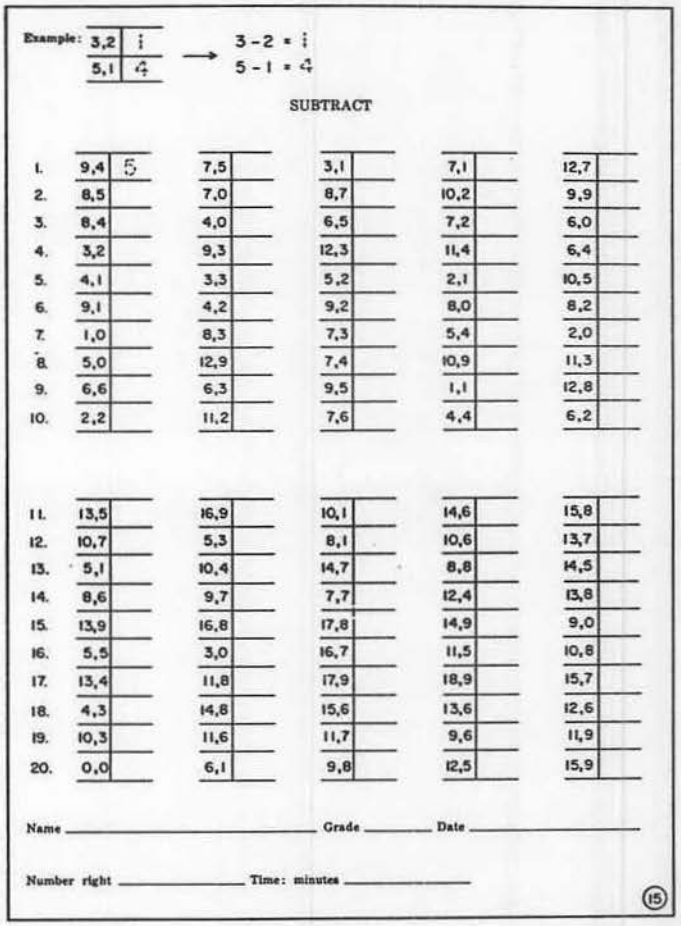
Make these graphs on a graph board and let pupils make them on graph paper. They should understand that each point of the graph represents a possible situation which meets the special conditions of the problem. To test this understanding, put your finger on various points and ask what the points tell. For example, put your finger on the point for the pair (6,100)—if he received 6 dimes for his birthday then he had 100 cents in his bank.

Drill

Here are some stories for headline practice:

1. Caleb decided to buy five apples. He paid thirty cents for all of them.
2. Polly had a birthday party at her house for Louise. Seven boys and eight girls (including Louise) came to the party.
3. Seventeen boys from Collinwood School play on Little League baseball teams. Twenty-four boys from the South Elementary School are also members of the same Little League baseball teams.

Consider using as a drill sheet or as a test on subtraction facts page 15 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.



Example: 3,2 | 1 ; 5,1 | 4 → 3 - 2 = 1 ; 5 - 1 = 4

SUBTRACT

1.	9,4	5	7,5		3,1		7,1		12,7	
2.	8,5		7,0		8,7		10,2		9,9	
3.	8,4		4,0		6,5		7,2		6,0	
4.	3,2		9,3		12,3		11,4		6,4	
5.	4,1		3,3		5,2		2,1		10,5	
6.	9,1		4,2		9,2		8,0		8,2	
7.	1,0		8,3		7,3		5,4		2,0	
8.	5,0		12,9		7,4		10,9		11,3	
9.	6,6		6,3		9,5		1,1		12,8	
10.	2,2		11,2		7,6		4,4		6,2	
11.	13,5		16,9		10,1		14,6		15,8	
12.	10,7		5,3		8,1		10,6		13,7	
13.	5,1		10,4		14,7		8,8		14,5	
14.	8,6		9,7		7,7		12,4		13,8	
15.	13,9		16,8		17,8		14,9		9,0	
16.	5,5		3,0		16,7		11,5		10,8	
17.	13,4		11,8		17,9		18,9		15,7	
18.	4,3		14,8		15,6		13,6		12,6	
19.	10,3		11,6		11,7		9,6		11,9	
20.	0,0		6,1		9,8		12,5		15,9	

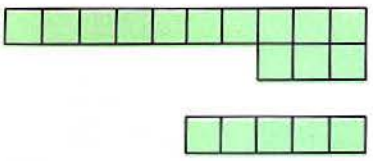
Name _____ Grade _____ Date _____

Number right _____ Time: minutes _____

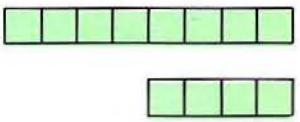
(15)

Count the tiles and record your results.

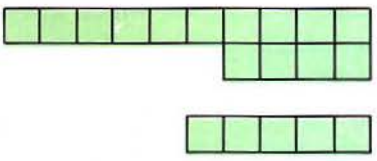
A. 13 + 5 = 18



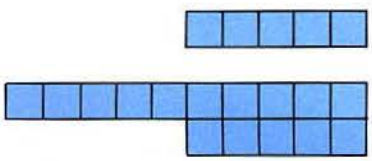
B. 8 + 4 = 12



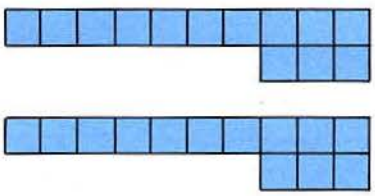
C. 14 + 5 = 19



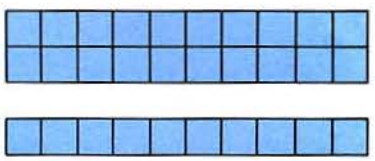
D. 5 + 15 = 20



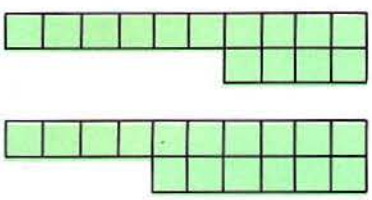
E. 13 + 13 = 26



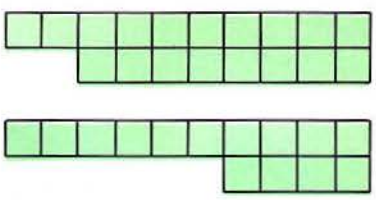
F. 20 + 10 = 30



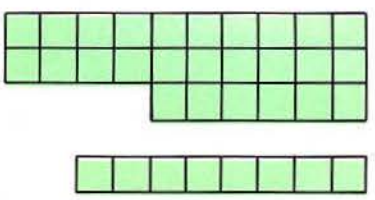
G. 14 + 16 = 30



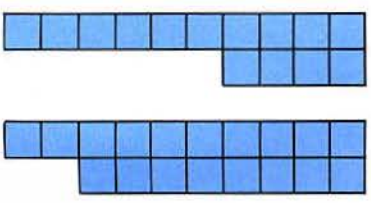
H. 18 + 14 = 32



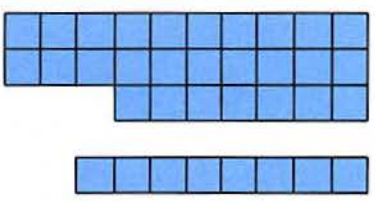
I. 26 + 8 = 34



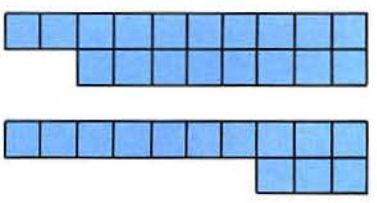
J. 14 + 18 = 32



K. 27 + 8 = 35



L. 18 + 13 = 31



M. What's My Rule?

5	7	13	9	18	27	38	55	79	99	C
8	10	16	12	21	30	41	58	82	102	C + 3

20-21 Counting squares - encouraging shortcuts - using decimal numeration - addition and subtraction equations - What's My Rule?

Tasks: *In exercises A-L pupils record in the tinted areas number sentences which are suggested by each of the representations. In the game in exercise M pupils use the clues provided in the first three columns or the pattern indicators at the extreme right of the table to continue the pattern by making appropriate entries in the remainder of the table.*

Purpose: To perfect skills in interpreting representations of addition and in recording observations; to prepare for an algorithm for column addition.

Unifying Ideas: Addition and subtraction; functions and relations.

Materials for Teacher: Tagboard strips like those on the pupil page.

Materials for Pupils: Cuisenaire rods.

The Lesson:

This page and pupil page 21 are designed to remind pupils of an algorithm for column addition. You may wish to have pupils work through each problem with Cuisenaire rods. For example, in exercise H a pupil would use an orange rod and a brown rod together with an orange rod and a purple rod. The brown rod and the purple rod would then be exchanged for an orange rod and a red rod. Thus, the sum is represented by three orange rods and a red rod.

Along with teaching algorithms, we must give pupils an understanding of the process. By "understanding" we mean the ability to get the right answer when one forgets an algorithm. When memory fails —when a pupil forgets the rules—what has he to fall back on? Only on understanding. The test for understanding is just that—what can a pupil do without an algorithm?

We should see an algorithm as a final outcome of a learning process which involves five steps:

Step 1. *Counting—The World of Real Things*

If you understand what is meant by adding, subtracting, multiplying, and dividing, you can get to a correct answer to any problem by arranging objects and counting them. What is 17 + 15? Have 17 pupils line up on one side of the room and 15 on the

more

Headline for the Day: Tell a story for any of the sentences you wrote for exercises A through L on page 20.

other. Count them to find that 17 + 15 is 32. Tongue depressors or Popsicle®* sticks are more easily manipulated—so get to them as soon as possible.

Step 2. *Making Pictures and Sketches*

The next skill we develop is that of making pictures of things and treating the pictures as things. Abstraction has crept in. Use marks on the board instead of pupils or Popsicle sticks:

X X X X X X X X X X X X X X X X X (17)

O O O O O O O O O O O O O O O (15)

Perhaps we should notice how helpful it is to remember that 10 + 10 is 20 and start from there to count the strays on the end. Perhaps we should group the strays in 5's and remember that 5 + 5 is 10, that 10 + 10 is 20, and that 20 + 10 is 30 . . . and 2 more strays. This kind of activity leads to the next step.

Step 3. *Making Simplified Sketches*

It's tiring to draw so many X's and O's. So we use a shorthand notation:

(10) X X X X X X X (17)

(10) O O O O O (15)

And here is an even more abbreviated shorthand notation:

(10) (5) X X (17)

(10) (5) (15)

Step 4. *Giving Arguments*

Simplified sketches and tricks to avoid tedious counting lead one to use reasoning or arguments. For example, from what one knows of place value, it is clear that 17 + 15 = (10 + 7) + (10 + 5). In view of the fact that the order and grouping in an addition problem can be changed without affecting the sum it follows that

$$(10 + 7) + (10 + 5) = (10 + 10) + (7 + 5).$$

Also

$$7 + 5 = 12 = 10 + 2.$$

So

$$\begin{aligned} 17 + 15 &= (10 + 10) + (10 + 2) \\ &= (10 + 10 + 10) + 2 \end{aligned}$$

which, by place value, is 32.

Step 5. *Using an Algorithm*

Now we're ready for an algorithm. But if we have carefully worked through each step in the development:

Things → Sketches → Brief Sketches → Arguments → Algorithm

how much is left to do? If pupils have not in the past been able to master algorithms, it is probably because we have been skipping part of the development—we've been hurrying . . . and the only way to hurry to an algorithm is by using rote memory. Then, when a pupil's memory fails, he has nothing to fall back on.

An algorithm is just a special line of argument, or a written record of manipulations with objects. If you forget that particular line of argument, use the best line of argument you can devise. If arguments are elusive, draw a brief sketch which will, of course, suggest a line of argument. If you can't think of a brief sketch, draw a full one . . . if you can't draw any sketch then go get the Popsicle sticks!

When pupils falter at the algorithm step, they should be led back down toward more familiar ground—and they should keep going back until their feet find something firm and reassuring.

If you have trouble teaching algorithms, it's probably because you are getting there too fast—leaping over the steps we've outlined. If you let the program outlined above unfold, most pupils will have an algorithm before you teach it. It's just a record of reasonable arguments, arguments that refer to a brief sketch, an abbreviation of a full sketch, a suggestion of real objects.

On pupil pages 20, 21, and 23-27, most of the activities aim toward understanding algorithms as indicated in the above discussion. Return to this series of steps frequently in analyzing the situation in your class if pupils are not making adequate progress.

*The name POPSICLE is a registered trademark of the Joe Lowe Corporation, Englewood, New Jersey.

You may wish to begin preparing for pupil page 20 by using strips of tagboard marked as suggested in the sketches on each page. Hold up various combinations of pairs of such cards and have pupils tell you how many there are in all. Be sure that they notice how helpful it is to remember to count by tens and to group ones into so many tens and so many more.

(See Notes on Pupil Pages 54-65 in the Teacher's Guide, LEVEL B (Revised) for a careful development and discussion of algorithms for column addition and column subtraction.)

Drill

Here are some stories for headline practice:

1. Reno spent 90 cents for 9 boxes of popcorn.
2. Eighteen children were divided into teams with 6 children on each team.
3. A jet pilot landed after flying 433 miles. After taking on some more passengers he took off to complete his flight plan. His plan called for a total trip of 914 miles.

A suggestion for a drill sheet to give pupils practice in hunting for tens is shown in the next column.

1. $6 + 5 + 4 = \square$ **2.** $8 + 7 + 2 = \square$

3. $7 + 2 + 3 = \square$ **4.** $6 + 2 + 8 = \square$

5. $5 + 9 + 5 = \square$ **6.** $4 + 9 + 1 = \square$

7.
$$\begin{array}{r} 7 \\ 3 \\ +\,5 \\ \hline \end{array}$$

8.
$$\begin{array}{r} 3 \\ 2 \\ +\,7 \\ \hline \end{array}$$

9.
$$\begin{array}{r} 6 \\ 7 \\ +\,3 \\ \hline \end{array}$$

10.
$$\begin{array}{r} 8 \\ 5 \\ +\,2 \\ \hline \end{array}$$

11.
$$\begin{array}{r} 4 \\ 6 \\ +\,8 \\ \hline \end{array}$$

12. $4+6+3+5 = \square$ **13.** $7+5+2+3 = \square$

14.
$$\begin{array}{r} 8 \\ 7 \\ 2 \\ +\,1 \\ \hline \end{array}$$

15.
$$\begin{array}{r} 9 \\ 3 \\ 2 \\ +\,1 \\ \hline \end{array}$$

16.
$$\begin{array}{r} 6 \\ 7 \\ 3 \\ +\,2 \\ \hline \end{array}$$

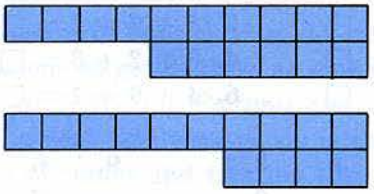
17.
$$\begin{array}{r} 7 \\ 5 \\ 3 \\ +\,5 \\ \hline \end{array}$$

18.
$$\begin{array}{r} 4 \\ 7 \\ 6 \\ +\,3 \\ \hline \end{array}$$

19.
$$\begin{array}{r} 1 \\ 9 \\ 9 \\ +\,1 \\ \hline \end{array}$$

N.

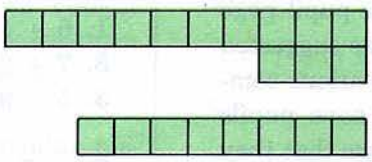
16 + 14 = 30



(10 + 6)	16
+ (10 + 4)	+ 14
(20 + 10)	30

O.

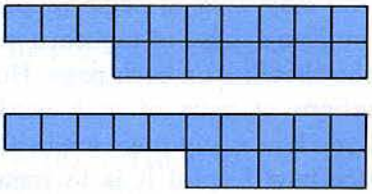
13 + 8 = 21



(10 + 3)	13
+ (0 + 8)	+ 8
(10 + 11)	21

P.

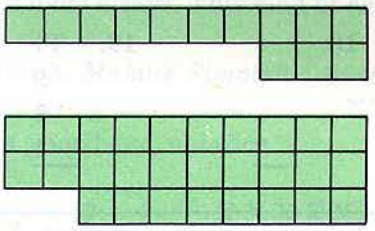
17 + 15 = 32



(7 + 10)	17
+ (5 + 10)	+ 15
(12 + 20)	32

Q.

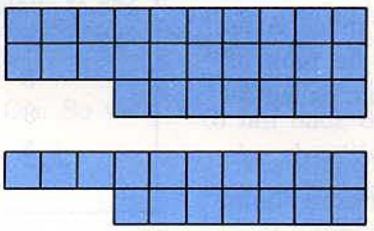
13 + 28 = 41



(10 + 3)	13
+ (20 + 8)	+ 28
(30 + 11)	41

R.

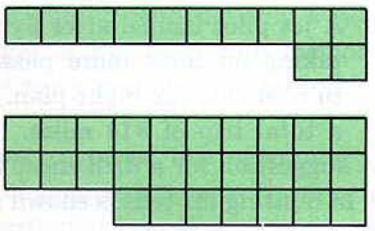
27 + 17 = 44



(20 + 7)	27
+ (10 + 7)	+ 17
(30 + 14)	44

S.

12 + 27 = 39



(10 + 2)	12
+ (20 + 7)	+ 27
(30 + 9)	39

What Are My Rules?

T.

3	7	10
9	1	10
12	8	20

U.

12	4	16
7	3	10
19	7	26

V.

10	9	19
10	7	17
20	16	36

W.

4	20	0	10	12	49	50	13	85	100	S
11	27	7	17	19	56	57	20	92	107	S + 7

Notes on Pupil Page 21

Tasks: *In exercises N-S pupils use the clues provided in exercise N to construct complete number sentences and partitions in each of the other exercises. In exercises T-V pupils complete each* Cross Number Puzzle. *In exercise W pupils use the clues provided in the first two columns, or the pattern indicators at the extreme right, to complete the table.*

Purpose: To review and extend understanding of addition and of an algorithm for column addition.

Unifying Ideas: Number and counting; numeration; addition and subtraction; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

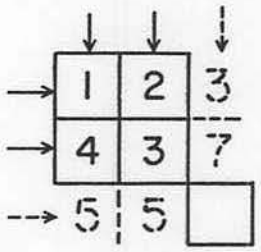
On this page pupils are exploring simplified sketches and arguments which lead to an algorithm for column addition.

Examples like those on the page should be placed on the board for discussion. Pupils should be asked why a grouping of counters in rows of 10 is better than one in which there is 8 or 11 or some number other than 10 in a row.

Cross Number Puzzles

Pupils who have used MATH WORKSHOP, LEVELS A and B, will know the secrets of the *Cross Number Puzzle.* Ask these pupils to keep the secrets and to look for new discoveries as examples are explored.

Put the following sketch, including the arrows, on the chalkboard.



The arrows, you will explain, tell us to add. As results of the four examples of addition indicated by the solid arrows are recorded, we are led to wonder where to record the additions indicated by the broken-line arrows.

Of course, in the one example shown above, we could write a 10 in the empty box and it would serve jointly as the sum of 3 and 7 and the sum of 5 and 5. Is this a coincidence? Most children and adults think it is, and that this outcome is dependent upon the choice of the four numbers originally written in the four cells.

Many of your pupils would be less surprised by the outcome if they imagined objects in the four cells—1 dot, 2 squares, 3 circles, and 4 stars. We can sweep the objects across into piles of 3 and 7; then sweep them down into a single pile of 10. Or, we could sweep them down into piles of 5 and 5; then sweep them across into a single pile of 10.

If we started with 10 objects and swept them together, we would not expect any to appear out of thin air or to miraculously disappear. If we start with 10 objects, we will end up with 10 objects.

We might keep a record of the alternatives above in the following way:

Sweeping across and then down

$$(1 + 2) + (4 + 3) = 3 + 7 = 10$$

Sweeping down and then across

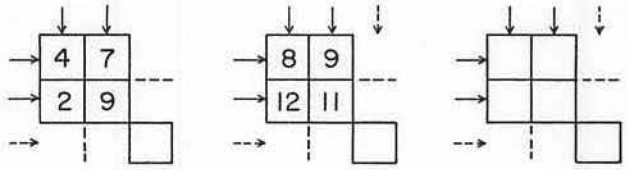
$$(1 + 4) + (2 + 3) = 5 + 5 = 10$$

Headline for the Day: Anita spent 7¢ more than Jonah. What can you say about this situation? Make a table.

There is a fundamental principle in arithmetic that the grouping and the order in which numbers are added does not affect the result. In columns of addition, we often "add up" the columns and then, to check our computations, we "add down" the columns. We are confident that this is a check because we are sure that the order and grouping used when we add will not change the result.

Yet, when most children (and many adults) are first confronted with a *Cross Number Puzzle*, they are amazed that the box in the lower right-hand corner will suffice for recording the result of the addition in the right-hand column and the result of the addition in the bottom row.

Try the next two examples below and then write any four numbers that come to mind for a further example:



You will probably have several pupils in your class who can give this puzzle a real workout while trying to discover four numbers that won't work. Even after they have swept objects on the floor and visually swept designs on paper—first one way and then the other—they may not fully understand that the numbers in the cells are shorthand notations for these more concrete situations, that is, that the numbers are answers to the *How many?* questions we could ask if we were to construct a physical representation.

The Rearrangement Rule for Addition

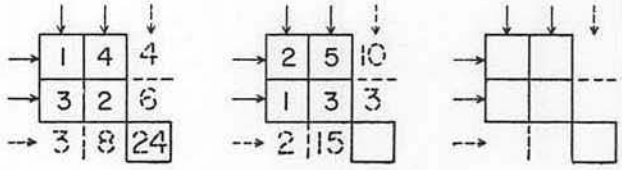
In this work with *Cross Number Puzzles* we are trying to help pupils see examples of a basic principle in mathematics: A string of numbers to be added can be rearranged (regrouped or reordered) in any way without changing the sum. This rearrangement rule depends, of course, on the fact that a display of objects can be partitioned in any manner without affecting the total number of objects in the display.

Any application of the rearrangement rule for addition can be justified mathematically by appealing to the commutative and associative principles for addition. For example, in the case of the *Cross Number Puzzle* given at the beginning of this discussion, the rearrangement rule tells us that $(1 + 2) + (4 + 3)$ is the same number as $(1 + 4) + (2 + 3)$. Let's see how the commutative (order) principle and the associative (grouping) principle come into the act:

$$
\begin{aligned}
&(1 + 2) + (4 + 3) \\
&\qquad \} \text{associativity} \\
&= [(1 + 2) + 4] + 3 \\
&\qquad \} \text{associativity} \\
&= [1 + (2 + 4)] + 3 \\
&\qquad \} \text{commutativity} \\
&= [1 + (4 + 2)] + 3 \\
&\qquad \} \text{associativity} \\
&= [(1 + 4) + 2] + 3 \\
&\qquad \} \text{associativity} \\
&= (1 + 4) + (2 + 3)
\end{aligned}
$$

The Rearrangement Rule for Multiplication

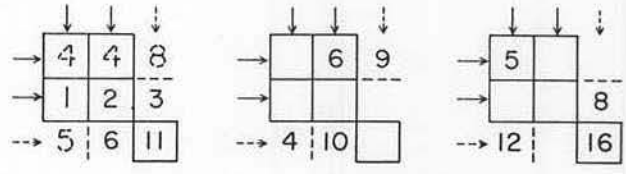
The format of the *Cross Number Puzzle* helps reveal that there is a rearrangement rule for multiplication, too. In the examples below, the arrows tell us to multiply. We leave you the job of making up a third example; try any four numbers that come to mind.



Subtracting a Number Is the Inverse of Adding It

The *Cross Number Puzzle* may also be used to introduce the pupil to subtraction.

The agreement for the puzzle below is, as earlier, that the arrows indicate addition and that the number for the lower right-hand cell is the result of adding the four numbers in the basic four cells. Subtraction enters the picture as soon as we are faced with a problem of reconstruction, or building backwards. Here are some examples:



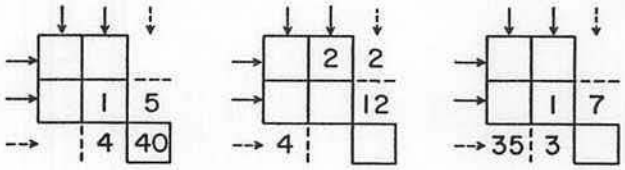
To rebuild in the first example you may argue as follows: I'm told that by adding 3 to some num-

ber, I get the result 11. To find this number I should "undo" the result of adding 3. In doing so, I must perform the inverse operation on 11. This inverse operation is called *subtracting 3*. Since $8 + 3 = 11$ and there is no other number whose sum with 3 is 11, the result of subtracting 3 from 11 must be 8.

Step by step, the number that belongs in each cell in each example can be discovered by similar reasoning. And most of the steps we take call for subtraction.

Dividing by a Number Is the Inverse of Multiplying by It

Similarly, we can reconstruct puzzles when the rule is that the arrows indicate multiplication. Examples:

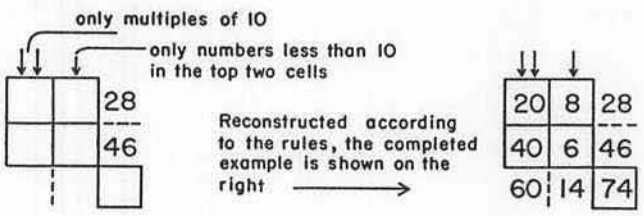


In the first example the question is asked: Multiplying what number by 5 results in 40? Or: 40 divided by 5 is what number? Etc.

The problem of reconstruction now involves division. Here we have examples of the fact that dividing by a number undoes the result of multiplying by that number.

Help with Algorithms

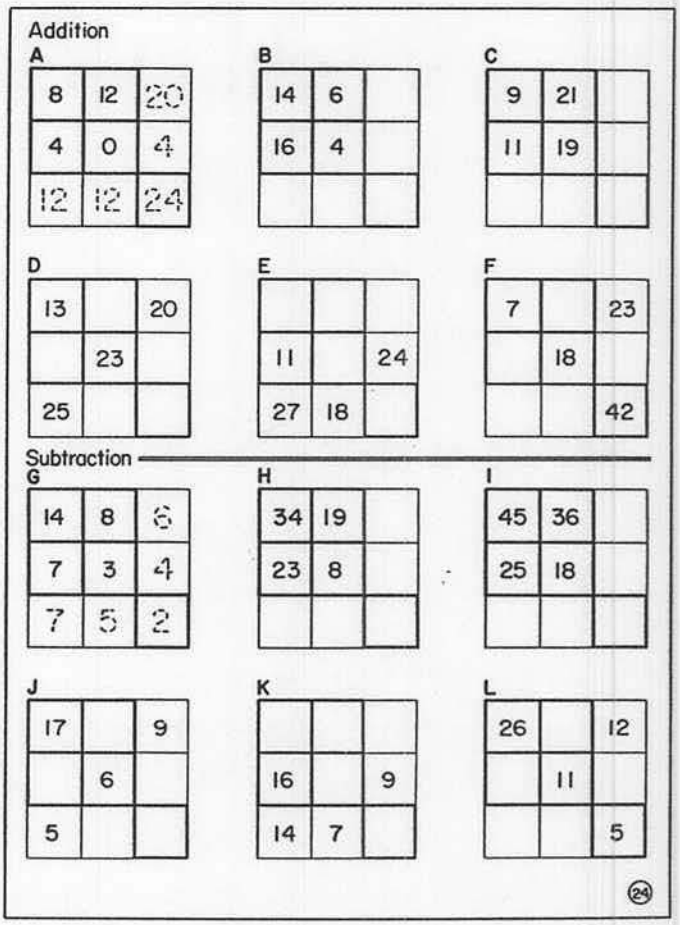
The addition *Cross Number Puzzle* will help pupils understand the steps in the common algorithm for column addition. The examples below call for reconstruction according to the agreement that only multiples of 10 can be used in the column under the double arrow, and only numbers less than 10 can be used in the top two cells of the column under the single arrow. Otherwise, the normal rules of addition are used.


only multiples of 10
only numbers less than 10 in the top two cells
28
46
Reconstructed according to the rules, the completed example is shown on the right
20 8 28
40 6 46
60 14 74

In following the rules, 28 and 46 are partitioned or "regrouped." This series of steps will be condensed or abbreviated and lead into the common algorithm for column addition.

Drill

Consider using as a drill sheet on *Cross Number Puzzles* page 24 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Here are some stories for headline practice:

1. Bridget paid 24 cents for 3 oranges.
2. A farmer had 6 pigs in his pen and he drove in 3 more pigs.
3. Fourteen campers were joined by 18 more. All of the campers made camp by the lake.

Mark correct answer. Mark the furthest from the correct answer.

Difference between the answers you marked.

Example

$$\begin{array}{r} 20 \\ -\ 7 \\ \hline 13 \end{array}$$

(1) 23 (2) 13 (3) 27 (4) 3

1 2 3 4 → 14

1

29 + [3] = 32

(1) [13] (2) [4] (3) [3] (4) [61]

1 2 3 4 → 58

2

△31 − 8 = 23

(1) △31 (2) △40 (3) △35 (4) △15

1 2 3 4 → 16

3

$$\begin{array}{r} 12 \\ 6 \\ +17 \\ \hline 35 \end{array}$$

(1) 25
(2) 47
(3) 8
(4) 35

1 2 3 4 → 27

4

$$\begin{array}{r} 125 \\ -18 \\ \hline 107 \end{array}$$

(1) 107
(2) 143
(3) 117
(4) 133

1 2 3 4 → 36

5

$$\begin{array}{r} 9 \\ 9 \\ 9 \\ 9 \\ +9 \\ \hline 45 \end{array}$$

(1) 36
(2) 59
(3) 45
(4) 70

1 2 3 4 → 25

6

50 − 18 = [32]

(1) [68] (2) [42] (3) [38] (4) [32]

1 2 3 4 → 36

Cheese Sandwiches	12¢ ea.	Corned Beef	30¢
Hot Dogs	15¢ ea.	Milk	8¢ Carton
Fish	20¢ ea.		
Hamburgers	17¢ ea.		

7 Bill bought 2 hamburgers. What did they cost him?

(1) 24¢ (3) 43¢
(2) 34¢ (4) 30¢

1 2 3 4 → 10¢

8 Alice bought a hot dog, a hamburger, and a carton of milk. How much did she spend?

(1) 32¢ (2) 30¢
(2) 18¢ (4) 40¢

1 2 3 4 → 22¢

9 Mary bought two cheese sandwiches and a carton of milk.

(1) 32¢ (3) 28¢
(2) 20¢ (4) 18¢

1 2 3 4 → 14¢

10 Mark had one dollar. He bought as many corned beef sandwiches as he could. How much change did he get?

(1) 20¢ (3) 1¢
(2) 10¢ (4) 29¢

1 2 3 4 → 19¢

11 Harry bought 2 fish sandwiches and a carton of milk. How much more did Sam spend for 2 corned beef sandwiches?

(1) 48¢ (3) 60¢
(2) 12¢ (4) 8¢

1 2 3 4 → 48¢

12 Together Al and Ed bought 5 sandwiches, one of each kind. They bought 2 cartons of milk. How much did they spend?

(1) $1.02 (3) $1.10
(2) 98¢ (4) $1.21

1 2 3 4 → 12¢

22 Test page - a variety of situations and applications involving addition, subtraction, and multiplication.

Tasks: *Pupils use the clues provided in the green region at the top of the page to guide them in completing the 12 test exercises.*

Purpose: To diagnose strength and weakness in addition and subtraction.

Unifying Ideas: Addition and subtraction; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupil page 22 is the first of seven test pages. The others are pupil pages 39, 52, 65, 72, 88, and 99. Of course, every pupil page is a test page in a very important sense. If pupils have difficulty with any work in this book, such difficulty should be regarded as symptomatic of a need for individual help.

The particular form of the test on pupil page 22 is not designed as a comprehensive test on all work studied in pupil pages 1 through 21. Rather, it gives us a sense of the general achievement of pupils. Those who fail to get at least 80% (29 or 30 points out of 36—3 points per item) correct on their own need special attention.

Prepare pupils for this page by discussing the method for marking and recording answers, using one or more examples like that at the top of the page.

Drill

Here are some stories for headline practice:

1. Gene has 50 marbles and 5 sacks. He put the same number of marbles in each of the 5 sacks.
2. Bonnie's cousin caught 8 fish. Bonnie and her dad caught 14 fish.
3. Wednesday 545 children bought milk with their lunches. Thursday 485 children had milk with their lunches.

Here is a suggestion for a drill sheet:

Rules:

(a) Add 2... 3, 5, 7, , , , , , ,

(b) Add 3... 3, 6, 9, , , , , , ,

(c) Add 5... 5, 10, , , , , , , ,

(d) Add 3... 10, 13, , , , , , , ,

(e) Add 4... 4, 8, , , , , , , ,

(f) Add 5... 13, , , , , , , , ,

(g) Add 4... 15, , , , , , , , ,

(h) Add 6... 6, , , , , , , , ,

(i) Add 7... 7, , , , , , , , ,

(j) Add 5... 21, , , , , , , , ,

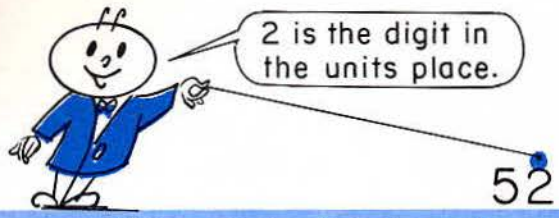
(k) Add 8... 8, , , , , , , , ,

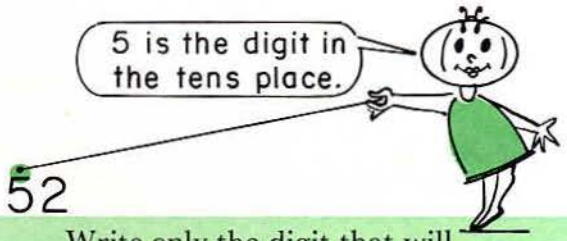
(l) Add 9... 9, , , , , , , , ,

(m) Add 8... 11, , , , , , , , ,

+	2	7	0	4	9	5	1	6	3	8
6										
0										
9										
1										
8										
5										
2										
4										
7										
3										

Headline for the Day: Tell a story to fit this headline: $50 - 18 = \square$





Write only the digit that will be in the units place of the sum.

	A	B	C	D	E
I	11 + 4 = 5	18 + 2 = 0	17 + 5 = 2	18 + 1 = 9	16 + 14 = 0
II	13 + 7 = 0	16 + 6 = 2	12 + 7 = 9	18 + 7 = 5	13 + 19 = 2
III	17 + 29 = 6	33 + 24 = 7	29 + 39 = 8	50 + 39 = 9	48 + 49 = 7

Write only the digit that will be in the tens place of the sum.

	F	G	H	I	J
I	11 + 4 = 1	18 + 2 = 2	17 + 5 = 2	18 + 1 = 1	16 + 14 = 3
II	13 + 7 = 2	16 + 6 = 2	12 + 7 = 1	18 + 7 = 2	13 + 19 = 3
III	17 + 29 = 4	33 + 24 = 5	29 + 39 = 6	50 + 39 = 8	48 + 49 = 9

Find the sums.

	K	L	M	N	O
IV	11 + 4 = 15	18 + 2 = 20	17 + 5 = 22	18 + 1 = 19	16 + 14 = 30
V	13 + 7 = 20	16 + 6 = 22	12 + 7 = 19	18 + 7 = 25	13 + 19 = 32
VI	17 + 29 = 46	33 + 24 = 57	29 + 39 = 68	50 + 39 = 89	48 + 49 = 97

What Are My Rules?

I.

5	5	10
1	6	7
14	0	14
6	3	9
7	8	15
9	11	20
10	20	30
13	14	27
$\frac{1}{2}$	$\frac{1}{2}$	1

II.

5	9	14	12	20	31	19	30	3	21	46	100	203	y
2	6	11	9	17	28	16	27	0	18	43	97	200	y − 3

23 Exercises to give insight into necessary place value concepts for an addition algorithm - What Are My Rules?

Tasks: *Pupils use the clues and directions provided to record in the white boxes of the blue region the units digit of each sum. In the white boxes of the green region they record the tens digit of each sum. In the blue and green boxes of the white region they record the required sums. In the games at the lower right and bottom of the page pupils use the clues to complete the table.*

Purpose: To focus attention on the steps in an algorithm for column addition.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: Counting Board and washers.

Materials for Pupils: None.

The Lesson:

On this page, pupils examine 18 problems in three ways:

First, they focus their attention on the units place of the sum.

Second, they focus their attention on the tens place of the sum.

Finally, they work the problem completely.

This attention to the algorithm one step at a time increases a pupil's insight into the arguments behind the algorithm.

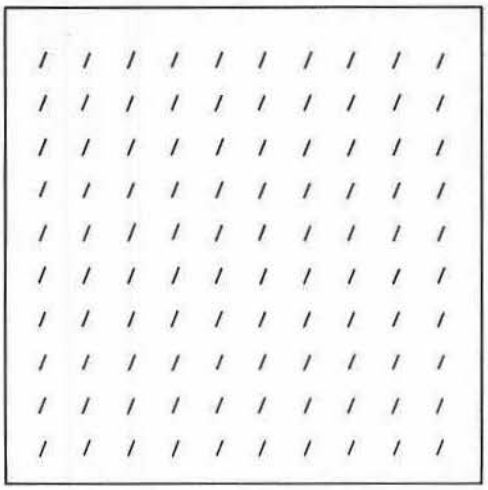
The Counting Board

Now is a good time to bring in the Counting Board. This versatile device can be made from a square piece

more

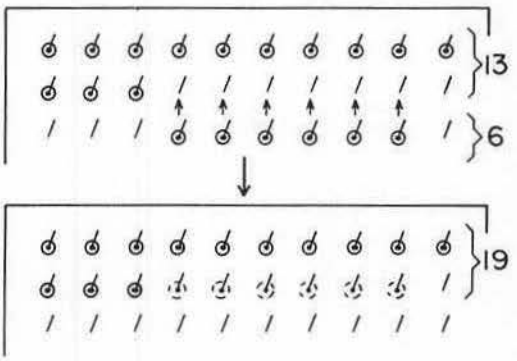
Headline for the Day: Kerby spent 3¢ less than Ingrid. Make a table to keep a record of what you can say about this situation.

of wood with a 10-by-10 arrangement of 100 nails driven in part way, and a box of washers which can be obtained at any hardware store. Use finishing nails —the kind without heads—so that the washers will slip over them easily.

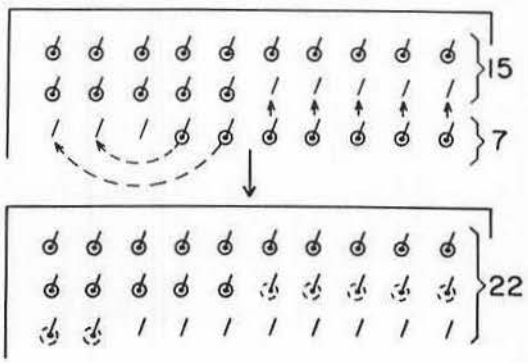


A few of the many uses of the Counting Board are shown in the following examples:

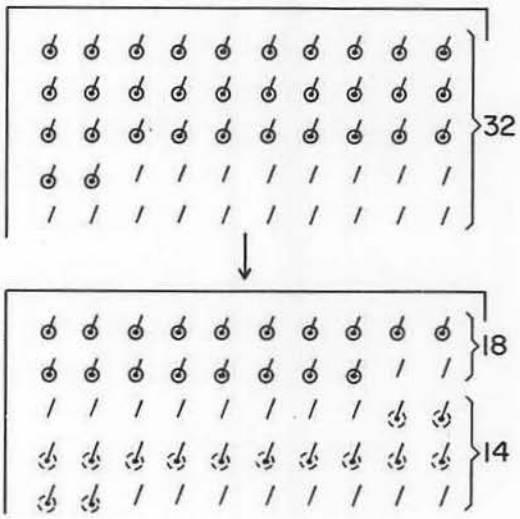
$13 + 6 = 19$ $\quad 15 + 7 = 22$ $\quad 18 + 14 = 32$



$13 + 6 = 19 = 6 + 13$



$15 + 7 = 22 = 7 + 15$



$18 + 14 = 32 = 14 + 18$

Any of the demonstrations can be regarded as interpreting any of the members of a fact team, such as:

$15 + 7 = 22$ $\quad 7 + 15 = 22$
$22 - 7 = 15$ $\quad 22 - 15 = 7$

You will also find the Counting Board to be very helpful when you are working with multiplication and division.

Drill

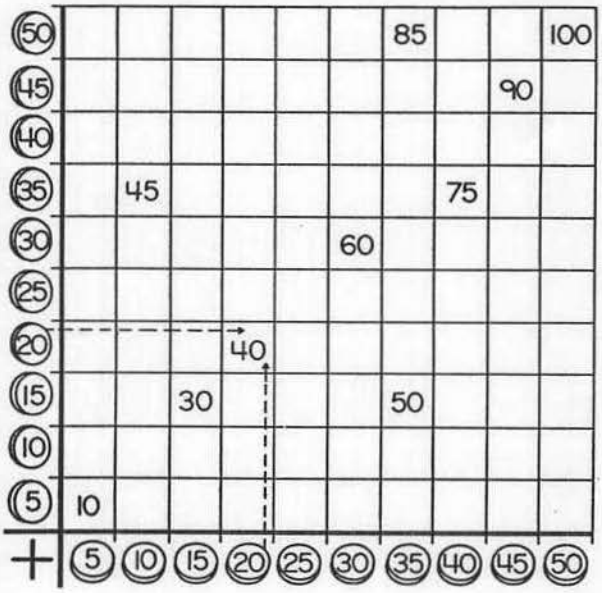
Here are some stories for headline practice:

1. Roy had 16 pieces of candy. He gave some to Pete, who didn't have any. Each boy now has the same number of pieces of candy.

$16 - \square = \square$ $\quad 16 \div 2 = 8'$

2. Mr. Gilgoyle's third-grade class had 99 cookies for their class party. After the party 17 cookies were left.
3. Petulla guessed there were 200 people at the boys' basketball game. There were actually 188 people at the game.

Consider using drill sheets pages 17 and 25 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



+	5	10	15	20	25	30	35	40	45	50
50							85			100
45									90	
40										
35		45						75		
30						60				
25										
20				40						
15			30				50			
10										
5	10									

$$\begin{array}{r} 20 \\ +20 \\ \hline 40 \end{array} \qquad \begin{array}{r} 25 \\ +30 \\ \hline \end{array} \qquad \begin{array}{r} 40 \\ +45 \\ \hline \end{array} \qquad \begin{array}{r} 15 \\ +30 \\ \hline \end{array}$$

$$\begin{array}{r} 35 \\ + \\ \hline 75 \end{array} \qquad \begin{array}{r} \\ +25 \\ \hline 40 \end{array} \qquad \begin{array}{r} 50 \\ + \\ \hline 85 \end{array} \qquad \begin{array}{r} 5 \\ + \\ \hline 45 \end{array}$$

20 + 20 = 40

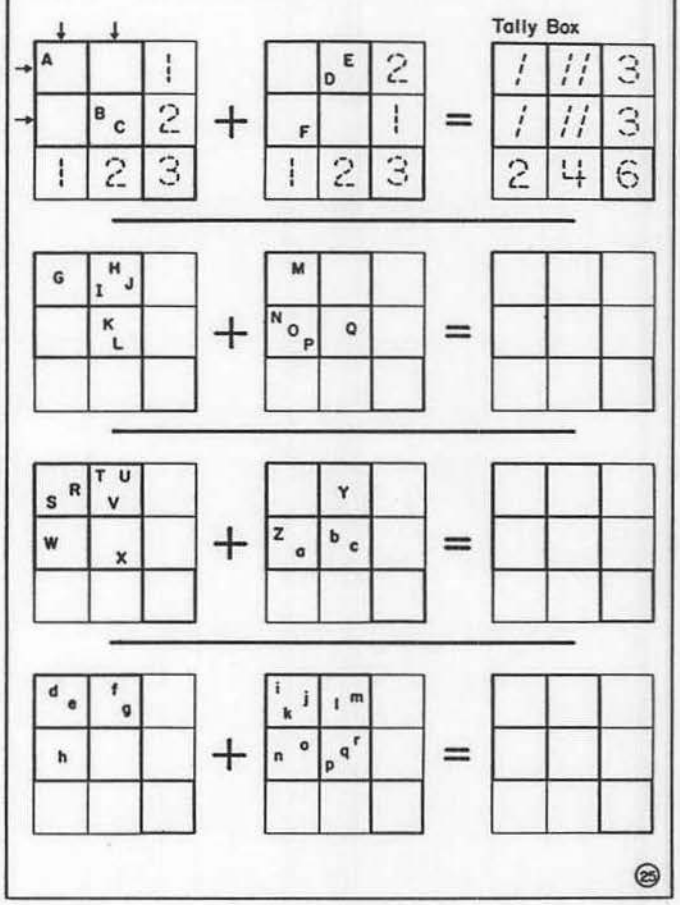
30 + 40 = ___

45 + ___ = 70

35 + 35 = ___

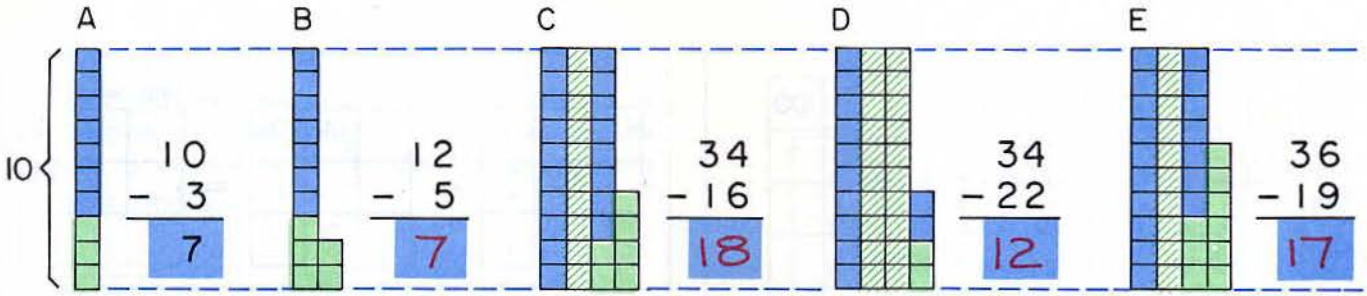
___ + 20 = 60

17

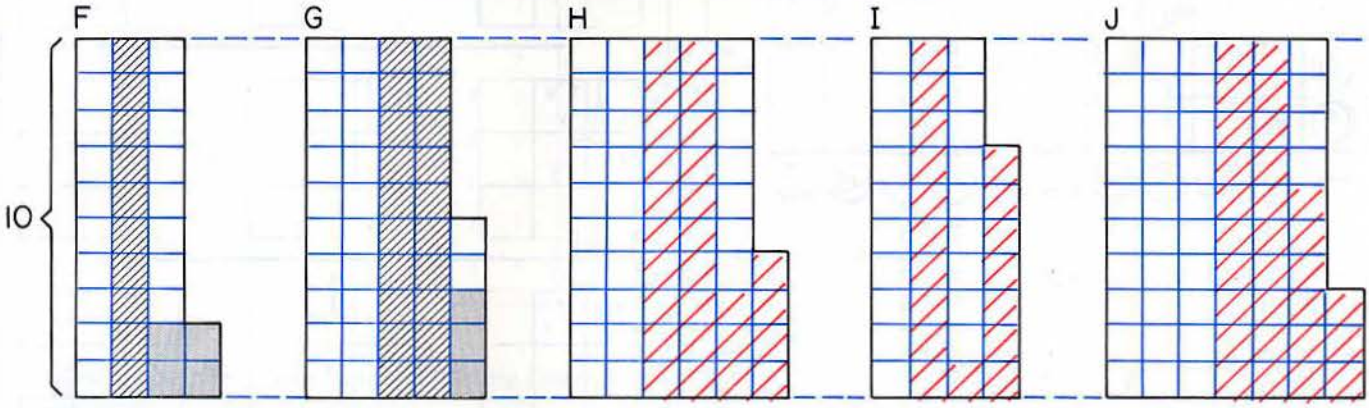


25

These pictures suggest the idea of subtraction. Please study them.



Complete the sketches below.
Are enough clues given?



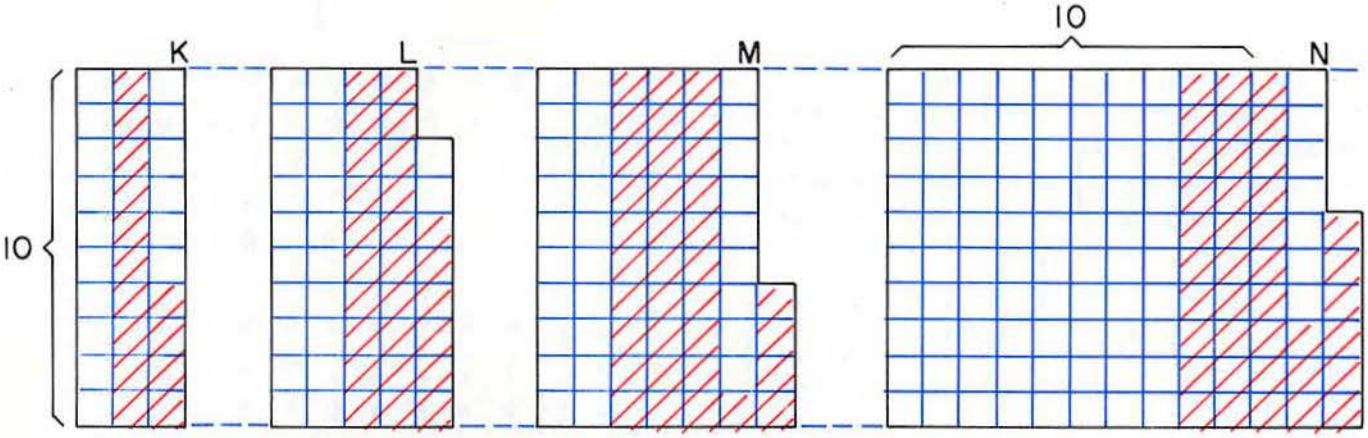
32 − 14 = 18

45 − 23 = 22

54 − 27 = 27

37 − 18 = 19

63 − 29 = 34



30 − 14 = 16

48 − 26 = 22

64 − 35 = 29

126 − 39 = 87

24 Using rectangular arrays of counters and decimal partitioning to gain in- sights into an algorithm for subtraction (with "exchanging").

Tasks: *Pupils use the clues provided to record in the tinted boxes the required differences and, in F-N, to reconstruct the subtraction problems.*

Purpose: To give practice with representations which lead to an algorithm for column subtraction.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Graph board; colored chalk.

Materials for Pupils: Cuisenaire rods; graph paper.

The Lesson:

An understanding of the common algorithm for column subtraction is easily developed by using graph paper as illustrated on pupil pages 24 and 25. It may also be developed with Cuisenaire rods. For example, consider the problem of subtracting 24 from 52. We represent 52 by 5 orange rods and 1 red rod. Our job is to remove 2 orange rods and a purple rod from this representation of 52. In order to do this, one of the orange rods must be exchanged for, say, a brown rod and a red rod. Now, 52 is represented by 4 orange rods, 1 brown rod, and 2 red rods. Remove 2 of the orange rods and the 2 red rods. We are left with 2 orange rods and 1 brown rod. Thus, the result of subtracting 24 from 52 is 28.

Cross Number Puzzles

The subtraction algorithm can also be developed through *Cross Number Puzzles* as we did in LEVEL B.

Problem

52		
24		
?		

Step 1

52	50	2
24	20	4
?		

Step 2

52	40	12
24	20	4
?		

Step 3

52	40	12
24	20	4
?	20	8

The puzzle is oriented so that the total-box is in the upper left-hand corner. The job is now one of reconstructing the puzzle in order to find the number for the box in the lower left-hand corner. We partition 52 and 24 as shown in Step 1. This is not a useful partition because there is no whole number whose sum with 4 is 2. Step 2 shows a more useful partition of 52. The number for the lower left-hand cell is the sum of 20 and 8.

Drill

Here is a suggestion for a drill sheet:

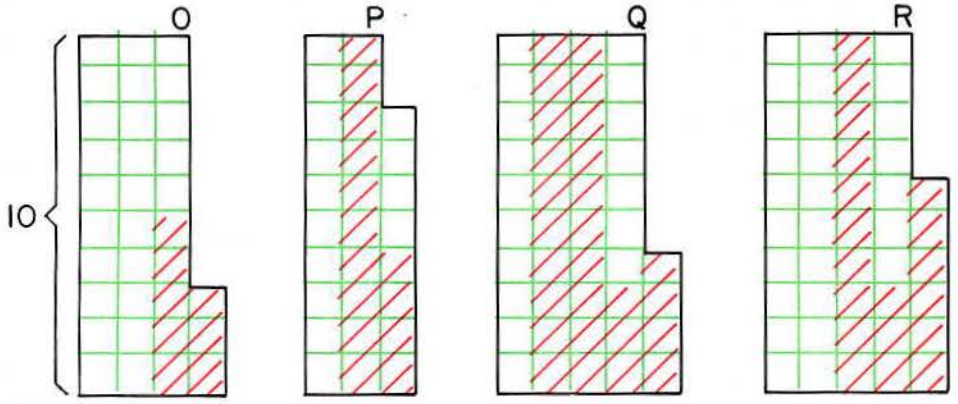
1. 50 + 70	2. 80 + 30	3. 20 + 30
4. 60 + 70	5. 90 + 80	6. 40 + 70
7. 16 + 32	8. 36 + 41	9. 32 + 55
10. 43 + 31	11. 83 + 15	12. 50 + 40
13. 76 + 52	14. 54 + 82	15. 96 + 42
16. 83 + 83	17. 50 + 68	18. 42 + 53

Here are some stories for headline practice:

1. Lisa paid 40 cents for 8 apples.
2. Manfred had 20 cents. His dad gave him 70 cents for his weekly allowance.
3. Melissa counted 51 boxcars, 23 flatcars, and 14 gondolas on the freight train.

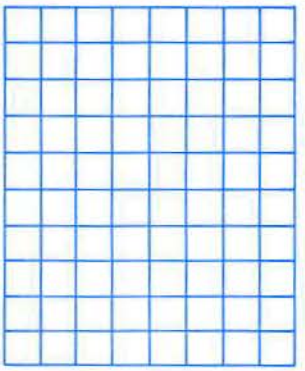
Headline for the Day: Tell a story to fit this headline: $126 - 39 = \square$

This page is very much like the last one. Please complete the sketches.



$$\begin{array}{r} 33 \\ -\ 8 \\ \hline 25 \end{array} \qquad \begin{array}{r} 28 \\ -\ 14 \\ \hline 14 \end{array} \qquad \begin{array}{r} 44 \\ -\ 27 \\ \hline 17 \end{array} \qquad \begin{array}{r} 46 \\ -\ 19 \\ \hline 27 \end{array}$$

(Draw sketches if you need to.)

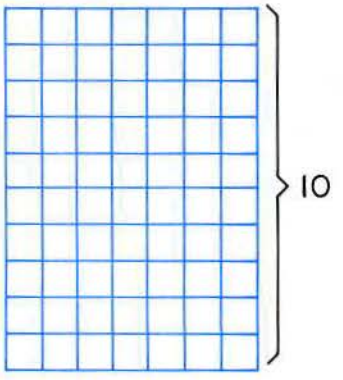


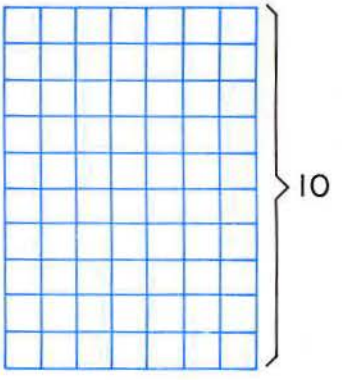
Complete the following subtraction problems. Draw sketches if you need to.

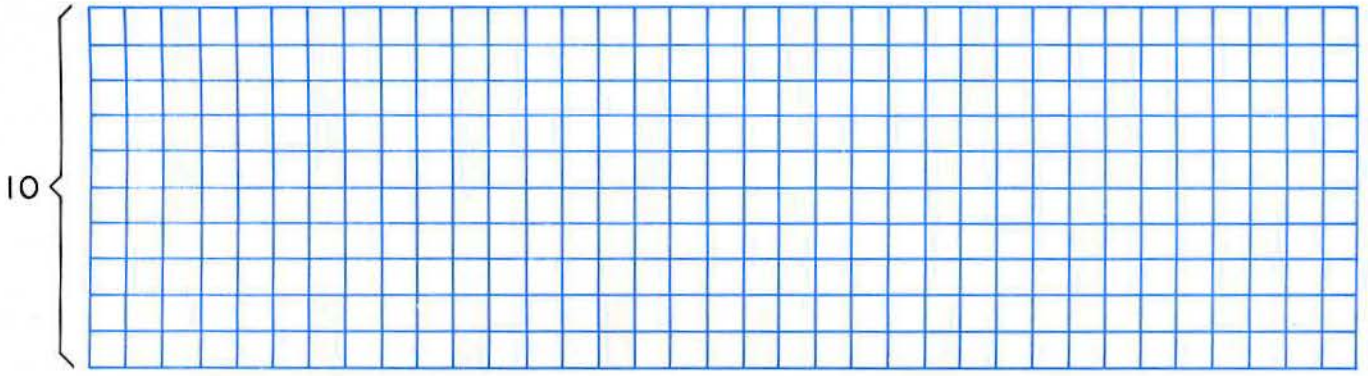
(a) $\begin{array}{r} 23 \\ -\ 8 \\ \hline 15 \end{array}$ (b) $\begin{array}{r} 30 \\ -\ 14 \\ \hline 16 \end{array}$ (c) $\begin{array}{r} 57 \\ -\ 30 \\ \hline 27 \end{array}$ (d) $\begin{array}{r} 70 \\ -\ 9 \\ \hline 61 \end{array}$ (e) $\begin{array}{r} 90 \\ -\ 20 \\ \hline 70 \end{array}$

(f) $\begin{array}{r} 35 \\ -\ 15 \\ \hline 20 \end{array}$ (g) $\begin{array}{r} 26 \\ -\ 12 \\ \hline 14 \end{array}$ (h) $\begin{array}{r} 40 \\ -\ 26 \\ \hline 14 \end{array}$ (i) $\begin{array}{r} 33 \\ -\ 18 \\ \hline 15 \end{array}$ (j) $\begin{array}{r} 78 \\ -\ 25 \\ \hline 53 \end{array}$

(k) $\begin{array}{r} 50 \\ -\ 25 \\ \hline 25 \end{array}$ (l) $\begin{array}{r} 88 \\ -\ 44 \\ \hline 44 \end{array}$ (m) $\begin{array}{r} 32 \\ -\ 19 \\ \hline 13 \end{array}$ (n) $\begin{array}{r} 27 \\ -\ 27 \\ \hline 0 \end{array}$ (o) $\begin{array}{r} 31 \\ -\ 13 \\ \hline 18 \end{array}$







25 Using rectangular arrays of counters and decimal partitioning to solve subtraction problems.

Tasks: *Pupils use the clues provided to record in the tinted boxes the required differences.*

Purpose: To further explore representations which lead to an algorithm for column subtraction, and to provide practice in using the algorithm.

Unifying Ideas: Number and counting; addition and subtraction.

Materials for Teacher: Graph board.

Materials for Pupils: Cuisenaire rods; graph paper.

The Lesson:

Although there should be little need to do so, you may wish to run through the steps used in column subtraction.

Example 1

```
  59
- 17
  2
```
First, subtract 7 from 9. The answer is 2. Write a 2 in the units place.

```
  59
- 17
  42
```
Next, subtract 10 from 50. The answer is 40. Write a 4 in the tens place.

Example 2

```
  53
- 17
```
First, subtract 7 from 3. This cannot be done. So, think of 53 as 40 + 13. Show this by crossing out the 5, writing a 4, and a 1. (Or, cross out the 3 and write a 13.)

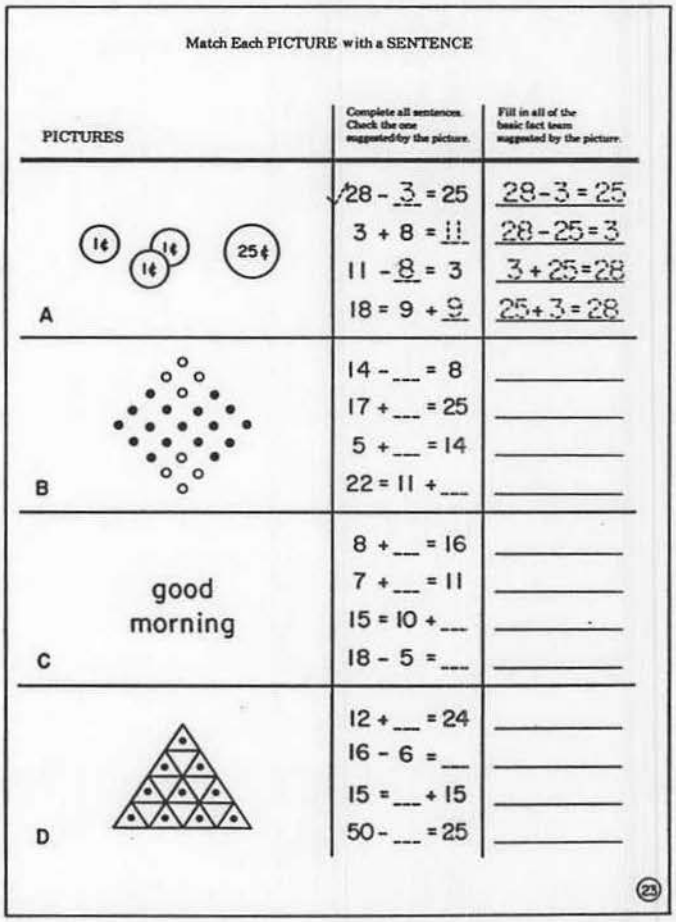
```
 4 13
- 17
  6
```
Now, subtract 7 from 13. The answer is 6. Write a 6 in the units place.

```
 4 13
- 17
  36
```
Next, subtract 10 from 40. The answer is 30. Write a 3 in the tens place.

Some pupils may feel more comfortable if they put in the "scratch marks." Others will not need to do this at all. We prefer not to establish a uniform policy in such matters.

Drill

Consider using as a drill sheet page 23 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:


Match Each PICTURE with a SENTENCE
PICTURES
Complete all sentences. Check the one suggested by the picture.
Fill in all of the basic fact team suggested by the picture.
1¢ 1¢ 1¢ 25¢
A
✓28 - 3 = 25
3 + 8 = 11
11 - 8 = 3
18 = 9 + 9
28-3=25
28-25=3
3+25=28
25+3=28
B
14 - ___ = 8
17 + ___ = 25
5 + ___ = 14
22 = 11 + ___
C
good morning
8 + ___ = 16
7 + ___ = 11
15 = 10 + ___
18 - 5 = ___
D
12 + ___ = 24
16 - 6 = ___
15 = ___ + 15
50 - ___ = 25
23

Here are some stories for headline practice:

1. Claire put 7 pencils in each of 9 boxes.
2. Nina's mother made 12 sandwiches for Nina and her 3 friends. They divided the 12 sandwiches equally among themselves.

Headline for the Day: Tell a story to fit this headline: $27 - 27 = \square$

Write only the digit that will be in the units place in the result.

	A	B	C	D	E
I	35 − 3 = 2	30 − 16 = 4	29 − 23 = 6	40 − 17 = 3	32 − 23 = 9
II	55 − 23 = 2	55 − 32 = 3	66 − 33 = 3	67 − 47 = 0	51 − 15 = 6
III	75 − 43 = 2	67 − 48 = 9	75 − 65 = 0	95 − 84 = 1	98 − 89 = 9

Write only the digit that will be in the tens place in the result.

	F	G	H	I	J
I	35 − 3 = 3	30 − 16 = 1	29 − 23 = 0	40 − 17 = 2	32 − 23 = 0
II	55 − 23 = 3	55 − 32 = 2	66 − 33 = 3	67 − 47 = 2	51 − 15 = 3
III	75 − 43 = 3	67 − 48 = 1	75 − 65 = 1	95 − 84 = 1	98 − 89 = 0

Find the results.

	K	L	M	N	O
IV	35 − 3 = 32	30 − 16 = 14	29 − 23 = 6	40 − 17 = 23	32 − 23 = 9
V	55 − 23 = 32	55 − 32 = 23	66 − 33 = 33	67 − 47 = 20	51 − 15 = 36
VI	75 − 43 = 32	67 − 48 = 19	75 − 65 = 10	95 − 84 = 11	98 − 89 = 9

What Are My Rules?

I.

4	2	2
8	6	2
10	8	2
6	4	2
9	7	2
12	10	2
15	13	2
31	29	2
2	0	2
100	98	2

II.

5	1	14	6	7	13	19	20	0	29	39	57	98	x
8	4	17	9	10	16	22	23	3	32	42	60	101	x + 3

26 Subtraction exercises focusing on place value aspects of a subtraction algorithm - What Are My Rules?

Tasks: *Pupils use the clues and directions provided to record in the white boxes of the green region the units digit of each result. In the white boxes of the blue region they record the tens digit of the result. In the blue and green boxes of the white region pupils record the required results. For the* What's My Rule? *games pupils use the clues provided to complete the tables.*

Purpose: To focus attention on the steps in an algorithm for column subtraction.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: None.

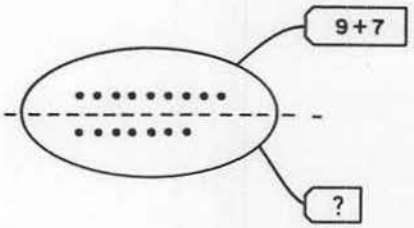
Materials for Pupils: None.

The Lesson:

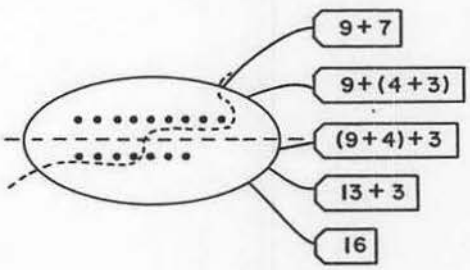
The exercises on this page are designed to allow pupils to focus their attention on the steps involved in finding the units digit and the tens digit in the answer to a problem involving the subtraction of a 1-digit or a 2-digit number from a 2-digit number.

Partitioning—The Basis for Algorithms

Let's pretend that we have forgotten the basic fact that 9 + 7 is 16. How could we rediscover it? Our basic procedure is to take a display of 9 objects and a display of 7 objects none of which is in the display of 9 objects and combine the two displays. The sum of 9 and 7 is the number of objects in the combined display.



We could determine the sum by straightforward one-by-one counting. We might avoid the counting by using some basic addition facts already known to us. For example, if we had memorized the facts that 7 is 4 + 3, 9 + 4 is 13, and 13 + 3 is 16, our problem would be solved. For in this case we would partition the combined display as follows:



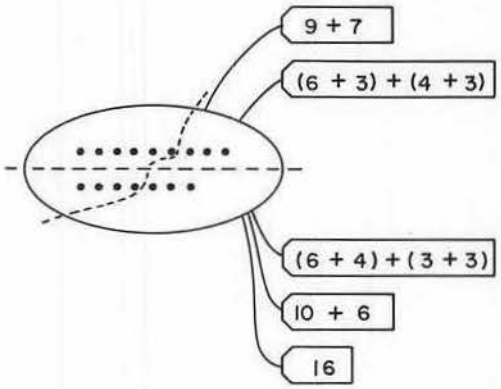
Our acquaintance with the world of physical objects assures us that this partitioning will not change the sum.

Of course there are many partitionings we might use. Our selection depends on which addition facts

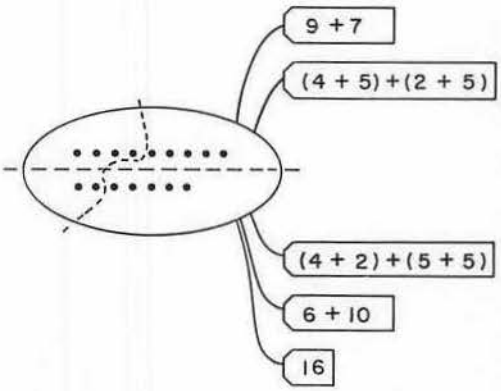
more

Headline for the Day: Nevin spent a penny more than Clara. Make a table to keep a record of what you can say about this situation.

we know. For example, suppose that we know that 9 is 6 + 3, 7 is 4 + 3, 6 + 4 is 10, 3 + 3 is 6, and 10 + 6 is 16. With this knowledge at our disposal, we would partition the combined display as shown in the diagram below:



Another way of discovering that 9 + 7 is 16 depends on the knowledge that 9 is 4 + 5, 7 is 2 + 5, 4 + 2 is 6, 5 + 5 is 10, and 6 + 10 is 16:



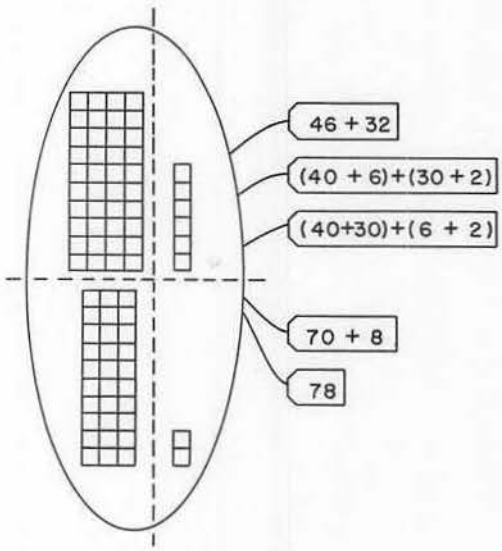
Partitioning and Positional Notation

Now, of course, it is highly unlikely that someone who had memorized all the addition facts mentioned above would have failed to memorize the fact that 9 + 7 is 16. The purpose of the foregoing discussion is to illustrate the general procedure that partitioning does not affect the total. The general procedure becomes useful when you can *choose a partitioning which allows you to make use of memorized facts.*

Consider how we might find the sum of 46 and 32. One could not be expected to have memorized this fact and it would be quite tedious to try to count, one by one, the combination of a display of 46 objects and a display of 32 objects. Instead, one starts calling upon certain basic addition facts and his knowledge of positional notation. Here are the facts:

1)	$46 = 40 + 6$	positional notation
2)	$32 = 30 + 2$	positional notation
3)	$40 + 30 = 70$	positional notation and basic fact
4)	$6 + 2 = 8$	basic fact
5)	$70 + 8 = 78$	positional notation

With these five facts at hand, one partitions the combined display as follows:



We can keep track of these steps in more abstract form as follows:

Problem: $46 + 32 = ?$

		46
		32
		?

Step 1: Get ready for a partitioning.

Step 2: Partition in accordance with positional notation (facts 1 and 2).

40	6	46
30	2	32
		?

Step 3: Avoid the counting by using basic memorized facts and positional notation (facts 3, 4, and 5).

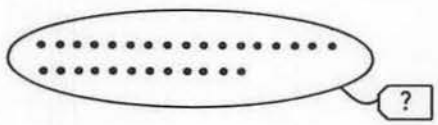
40	6	46
30	2	32
70	8	78

Our knowledge that no rearrangement of a physical display will change the count is the basis for asserting that the sum of 70 and 8 is the sum of 46 and 32.

Finally, positional notation is such a powerful device that one need not be conscious of choosing a partitioning. The choice is made as soon as you decide to line up the places and add in columns.

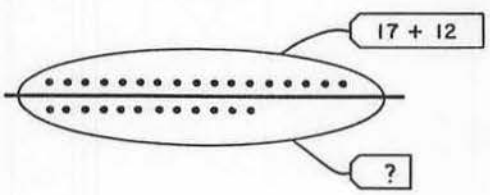
An Analysis of Addition without Exchanging

Let's analyze the job of finding the sum of 17 and 12. In physical terms this amounts to finding the number of objects in a combined display of two nonoverlapping displays—a display of 17 objects and a display of 12 objects. Here is a picture of such a combined display:



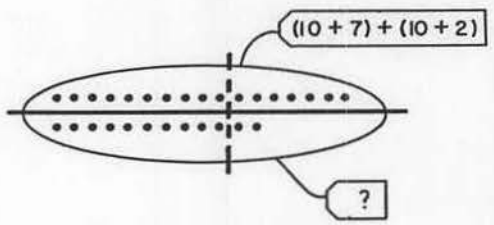
The job of finding the total really amounts to *reporting* the number in such a way that a person receiving the report will get an accurate picture of the numerosity of the combined display. We ordinarily assume that such a person has an excellent acquaintance with the numbers from 0 through 10. So, we ordinarily give our report in terms of a partition each of whose parts consists of no more than 10 entities. In other words, we ordinarily give our report in terms of a *decimal* partitioning. (If, however, our respondent were limited in his acquaintance to just the numbers from 0 through 5, we would give our report in terms of a *quinary* partitioning. Or, if he were limited to the numbers 0 through 8, we would give our report in terms of an *octal* partitioning.)

One of the things we know immediately about the combined display is that it can be partitioned into a

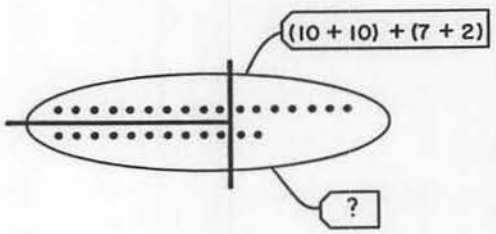


17-group and a 12-group. If our respondent were well acquainted with the numbers 17 and 12, all we would need to report to him is that the combined display could be partitioned into one 17-group and one 12-group. This report would give him a clear picture of the numerosity of the combined display. He would feel perfectly comfortable with the symbol "17 + 12" as a name for the sum. But, we are assuming that our respondent feels comfortable only with reports which refer to groups which consist of at most ten objects. So he would not feel comfortable with "17 + 12" as a report, since both the 17 and 12 refer to groups which consist of more than 10 objects in each.

Now how do we go about obtaining a decimal partitioning of the combined display? We start with the knowledge that one partitioning is 17 + 12. Because of our understanding of positional notation and because we know that a decimal partitioning was used in reporting the numbers for the given displays, we know that one of the given displays can be partitioned into a 10-group and seven 1-groups, and the other into a 10-group and two 1-groups. We have discovered that the combined display can be



partitioned into two 10-groups and one group consisting of 7 + 2 objects.

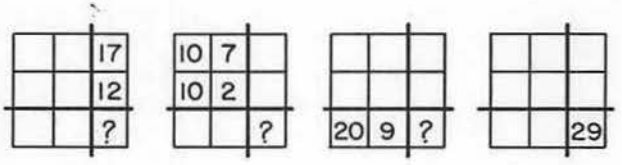


Since 7 + 2 is less than 10, the partitioning job is finished.

We are now ready to give the report. Because we have memorized certain basic facts we know that 7 + 2 = 9. We can tell our decimal-minded respondent that the combined display can be partitioned into two 10-groups and nine 1-groups. For short, we simply tell him that the combined display consists

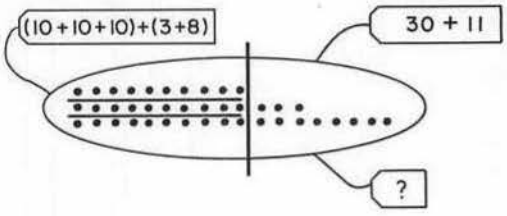
of 29 members. (The symbol "29" means nothing more than two 10-groups and nine 1-groups.)

The *Cross Number Puzzle* form shows the steps very neatly:

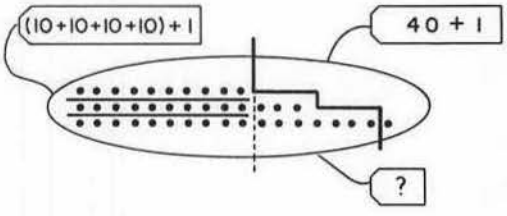


Addition with Exchanging

Now let's consider the job of finding the sum of 30 and 11. We shall have to do more work to achieve the partition we are seeking. For example, here might be one of the steps:



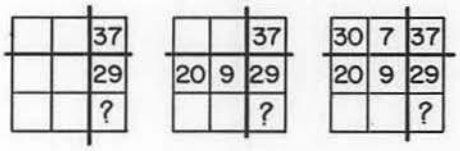
Since 3 + 8 is greater than 10, our job is not finished. Here is the finished job:



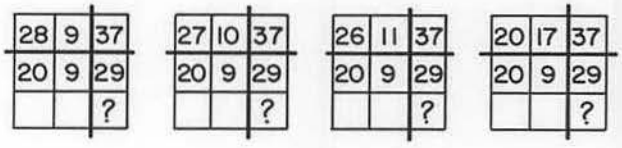
The regrouping from the partition 30 + 11 to the partition 40 + 1 is commonly called *carrying*. But this term is certainly not suggested by any of the work we have done. It might be better to use the word *exchanging* because it is more suggestive of what is done.

An Analysis of Subtraction

Let's consider the problem of subtracting 29 from 37. A pupil can solve this problem by reconstructing an addition *Cross Number Puzzle*. A pupil might first proceed as shown at the top of the following column.



At this stage, he realizes that there is no whole number he can add 9 to in order to get the sum 7. Given complete freedom, he might decide to change his partition of 29 from 20 + 9 to 22 + 7. But if he wishes to stick with a decimal partitioning of 29, he would not do this. So he must change the partition of 37. Any of the following could help:



However, only the last partition capitalizes fully on positional notation. (Otherwise, one would be hard-pressed to defend it over the first or second.)

The regrouping from the partition 30 + 7 to the partition 20 + 17 is commonly called *borrowing*. Clearly, this term is not suggested by any of the work we have done. As with carrying, you may wish to use the word *exchanging* instead.

Drill

Here are some stories for headline practice:

1. Aldo bought ten 5-cent stamps.
2. Simeon's third-grade class had an art show. The girls displayed 37 paintings. The boys displayed 25 paintings.
3. The three third-grade classes in Bunny's school put on a play together. There were 35 children in one class, 29 in another class, and 31 in the other class.
4. Pablo spent 25 cents on bananas and 31 cents on grapes.
5. Randy's apartment house is 6 stories high. There are 14 rooms on each floor.
6. On the first night 275 people came to our concert. On the second night 288 came to the concert.
7. Mitzi's dog Sadsac just had 10 puppies. Last year Sadsac had 8 puppies.

At the top of the following page is a suggestion for a drill sheet.

	I	II	III	IV	V	VI
	30	12	31	25	14	27
A	+ 14	+ 18	+ 9	+ 25	+ 16	+ 23
	8	7	4	6	15	8
B	2	8	9	4	5	26
	+ 9	+ 3	+ 1	+ 7	+ 5	+ 2
	32	47	6	12	17	15
C	3	8	9	18	1	18
	+ 5	+ 3	+ 14	+ 6	+ 19	+ 15
	3	9	6	12	18	16
	7	5	4	8	2	25
D	2	1	9	9	13	4
	+ 8	+ 5	+ 9	+ 3	+ 7	+ 5
	4	13	19	3	25	25
	6	7	4	2	13	5
E	1	2	6	17	4	15
	9	4	6	8	7	15
	+ 8	+ 8	+ 4	+ 16	+ 16	+ 35
	18	11	25	27	19	32
	2	9	13	14	15	22
	4	2	5	8	11	14
F	6	7	5	9	15	6
	7	3	14	16	17	8
	+ 3	+ 18	+ 6	+ 13	+ 17	+ 18

Here is another suggestion for a drill sheet:

	I	II	III	IV	V	VI
	10	10	9	20	4	37
A	+ 0	+ 7	+10	+ 6	+30	+20
	2	4	6	2	1	5
B	3	2	3	0	7	5
	+ 5	+ 4	+ 1	+ 8	+ 2	+ 2
	8	3	1	6	3	2
C	2	5	9	4	9	5
	+ 7	+ 5	+ 4	+ 6	+ 7	+ 8
	6	12	7	5	1	6
D	10	8	3	20	40	4
	+ 4	+ 0	+20	+ 5	+ 9	+15
	1	8	17	24	7	6
E	19	5	3	1	8	34
	+ 6	+15	+ 6	+ 6	+13	+ 9
	3	6	9	3	9	5
F	7	3	6	7	4	5
	4	2	2	2	1	7
	+ 4	+ 8	+ 1	+ 8	+ 6	+ 4
	17	12	15	13	15	21
G	3	9	5	17	2	19
	+10	+ 8	+17	+ 6	+15	+ 5

Write only the digit that will be in the units place in the result.

	A	B	C	D	E
I	13 + 18 = 1	49 − 12 = 7	52 − 11 = 1	24 + 19 = 3	53 − 6 = 7
II	19 + 28 = 7	65 − 12 = 3	92 − 33 = 9	27 + 34 = 1	99 − 32 = 7
III	56 + 15 = 1	89 − 16 = 3	91 − 12 = 9	14 + 69 = 3	100 − 13 = 7

Write only the digit that will be in the tens place in the result.

	F	G	H	I	J
	13 + 18 = 3	49 − 12 = 3	52 − 11 = 4	24 + 19 = 4	53 − 6 = 4
	19 + 28 = 4	65 − 12 = 5	92 − 33 = 5	27 + 34 = 6	99 − 32 = 6
	56 + 15 = 7	89 − 16 = 7	91 − 12 = 7	14 + 69 = 8	100 − 13 = 8

Find the results.

	K	L	M	N	O
IV	13 + 18 = 31	49 − 12 = 37	52 − 11 = 41	24 + 19 = 43	53 − 6 = 47
V	19 + 28 = 47	65 − 12 = 53	92 − 33 = 59	27 + 34 = 61	99 − 32 = 67
VI	56 + 15 = 71	89 − 16 = 73	91 − 12 = 79	14 + 69 = 83	100 − 13 = 87

What Are My Rules?

I.

9	1	10
5	5	10
10	0	10
6	4	10
1	9	10
2	8	10
7	3	10
4	6	10
10	0	10
3	7	10

II.

2	10	5	4	6	3	7	11	13	17	0	57	99	x
12	20	15	14	16	13	17	21	23	27	10	67	109	x + 10

27 Addition and subtraction exercises focusing on place value aspects of addition and subtraction algorithms - What Are My Rules?

Tasks: *Pupils use the clues and directions provided to record in the white boxes of the blue region the units digit of the required sum or difference. In the white boxes of the green region pupils record the tens digit of the required sum or difference. In the blue and green boxes of the white region they record the required sums or differences. For the games at the lower right and bottom of the page, pupils use the clues provided to complete the tables.*

Purpose: To further develop efficient procedures for adding and subtracting 2-digit numbers.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

Addition and subtraction problems are mixed on this page but again the pupils are asked to focus their attention on the units place and on the tens place before tackling the problem as a whole.

Partitioning

As we have seen in Notes on Pupil Page 26, the common algorithm for column addition of 2-digit numbers is nothing more than a written record of a decimal partitioning of a display of objects.

Let's look at the situation a little more abstractly. Each whole number between 9 and 100 is the sum of some multiple of 10 and some whole number less than 10. The decimal notation for the given number tells you precisely which multiple of 10 and which whole number less than 10. Consider the problem of finding the sum of 73 and 28.

$$
\begin{aligned}
& 73 + 28 \\
&= (70 + 3) + (20 + 8) && \text{positional notation} \\
&= (70 + 20) + (3 + 8) && \text{rearrangement rule} \\
&= 90 + 11 && \text{basic facts} \\
&= 90 + (10 + 1) && \text{positional notation} \\
&= (90 + 10) + 1 && \text{rearrangement rule} \\
&= 100 + 1 && \text{basic fact} \\
&= 101 && \text{positional notation}
\end{aligned}
$$

Now, when we use the common algorithm for column addition, we certainly do not write down all of this information. We use shortcuts, and the shortcuts can be justified in the same way as are the steps above.

Consider:

$$
\begin{array}{r}
73 \\
+\ 28 \\
\hline
\end{array}
$$

If we start our "figuring" by saying to ourselves: 3 + 8 is ___, we certainly have already partitioned 73 and 28 in our minds, for how else did we pick 3 and 8?

Next, we say to ourselves: 3 + 8 is 11; write a 1 in the units place and remember 10. We have partitioned 11 into 10 + 1. We also know that the rest

more

Headline for the Day: Keith spent a dime more than Stella. Make a table to keep a record of what you can say about this situation.

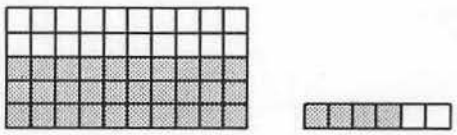
of the problem will not involve groups smaller than 10; so, we are sure of the units digit in the sum. (It is this certainty that leads us to use a right-to-left algorithm. If we were merely estimating the sum then we would just work with the tens digits alone.)

Then, we say to ourselves: 7 + 2 is 9 (this is short for 70 + 20 is 90). Then: 9 + 1 is 10 (or: 90 + 10 is 100). We record the result by writing a 0 in the tens place and a 1 in the hundreds place. All of this can be done with understanding only by someone who has an excellent grasp of the notion of decimal notation.

Finally, the fact that we can construct a name for the sum by adding in columns depends not only on decimal notation but also upon the rearrangement rule for addition which, in turn, depends upon the fact that a physical rearrangement of the objects in a display does not change the number of objects in the display.

This last remark may appear to deal with obvious matters. But, to see that there is some substance to it, consider the problem of justifying the common algorithm for column subtraction. In terms of displays of objects, subtracting in columns makes good sense:

$$\begin{array}{r} 56 \\ -\ 34 \\ \hline \end{array}$$



But, from an abstract point of view, it isn't at all obvious that (50 + 6) − (30 + 4) = (50 − 30) + (6 − 4). To justify this, we make use of the fact that adding a number and subtracting that number are inverses of each other and of the rearrangement rule for addition. We start by adding 30 + 4 to (50 − 30) + (6 − 4).

$$\begin{aligned} &[(50-30)+(6-4)]+(30+4) \\ &=[(50-30)+30]+[(6-4)+4] \qquad \text{rearrangement} \\ &= \quad 50 \quad + \quad 6 \qquad \text{adding a number is the inverse of subtracting it} \end{aligned}$$

So, 50 + 6 = [(50 − 30) + (6 − 4)] + (30 + 4). Since subtracting a number is the inverse of adding it, the last equation implies that (50 + 6) − (30 + 4) = (50 − 30) + (6 − 4). And this is what we wished to establish.

It seems clear from the sophistication of the preceding demonstration that any attempt to justify the common algorithm for column subtraction by referring to properties of numbers would be out of place in a primary grade classroom. The most practical thing to do is to refer to manipulations with physical objects or to a reconstructed *Cross Number Puzzle*, which is merely a record of such manipulations:

56	50	6
34	30	4
?		

	50	6
	30	4
?	20	2

22	20	2

Partitioning Games

The idea of choosing a partitioning according to specified rules can be introduced to pupils by the following kind of board work.

Game 1

Less than 10
More than 10, but less than 20
Multiples of 10
Less than 10
Multiples of 10

$$\begin{aligned} 35 &= \underline{\quad + \quad} = \underline{\quad + \quad + \quad} \\ 78 &= \underline{\quad + \quad} = \underline{\quad + \quad + \quad} \\ 47 &= \underline{\quad + \quad} = \underline{\quad + \quad + \quad} \\ 53 &= \underline{\quad + \quad} = \underline{\quad + \quad + \quad} \end{aligned}$$

Game 2

Odd or even number less than 10
Even number less than 10
Odd number less than 10
Multiples of 10

$$\begin{aligned} 18 &= \underline{10 + 5 + 2 + 1} \\ 91 &= \underline{80 + 7 + 4 + 0} \end{aligned}$$

Game 3

Using *Cross Number Puzzles*, the rules are that the top two cells in column A are reserved for multiples of 10 while the top two cells in column B are reserved for numbers less than 10.

A	B	
		12
10	0	10

A	B	
	0	10
		16

A	B	
	0	20
		12

A	B	
	0	80
	0	10

A	B	
		19
0		3

A	B	
0		7
		19

A	B	
		24
		8

A	B	
		85
		5

A	B	
		17
		48

A	B	
		8
		63

A	B	
		19
		43

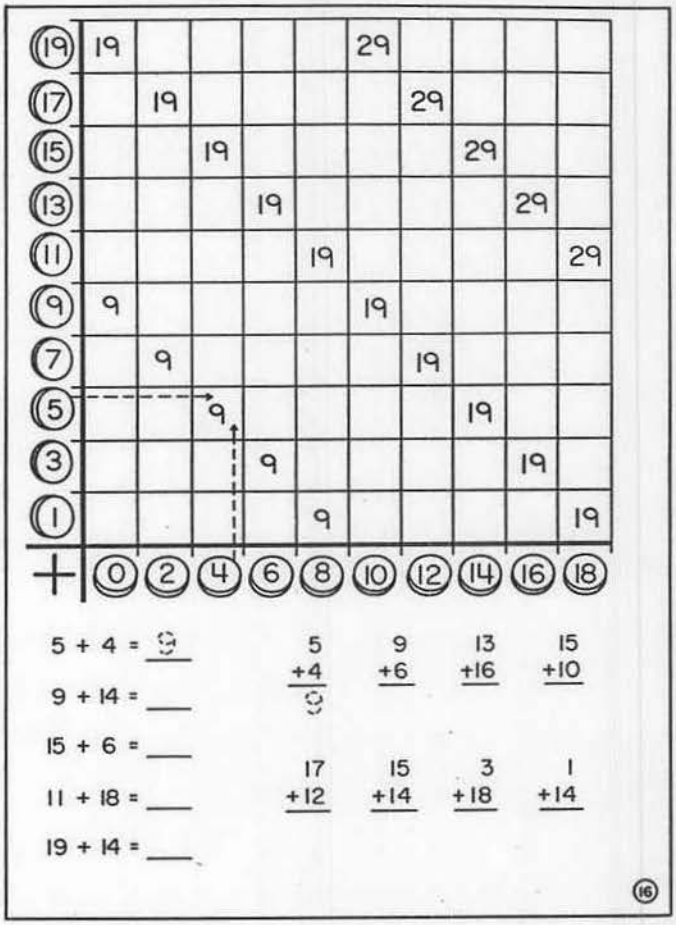
A	B	
		47
		60

Drill

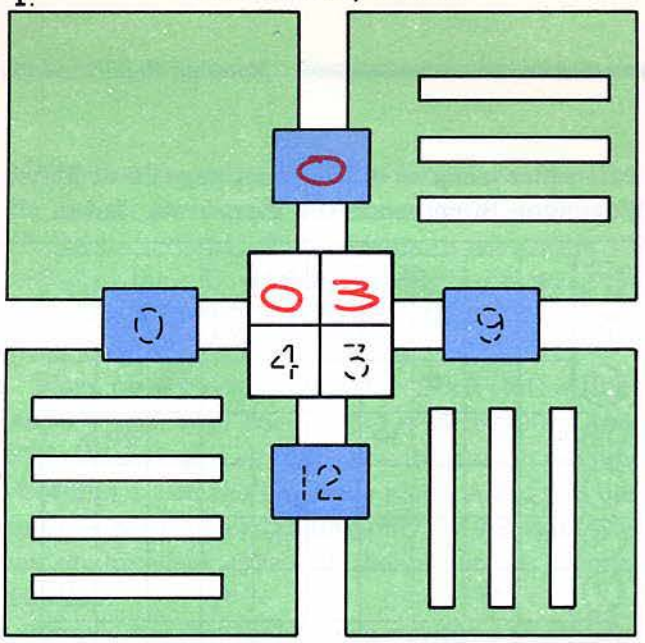
Here are some stories for headline practice:

1. Carlos bought seven 5-cent stamps.
2. Judy spent 23 days of her summer vacation at Granny's. She had 37 days left in the summer holidays.
3. Danny lives near the railroad station. One week he counted thirty-five northbound trains and twenty-seven southbound trains stopping at the station.

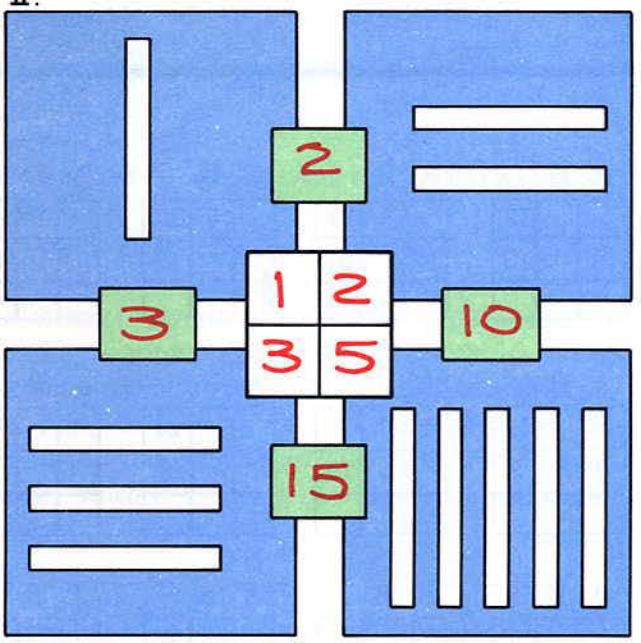
Consider using as a drill sheet page 16 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



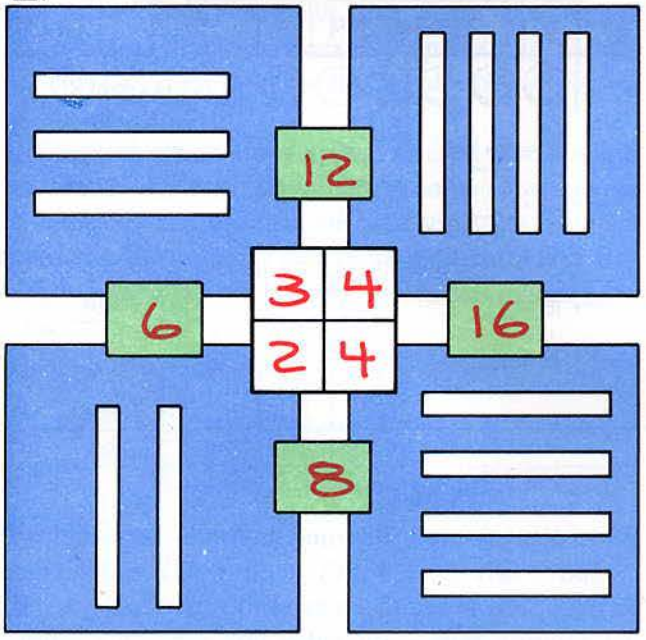
I. What Are My Rules?



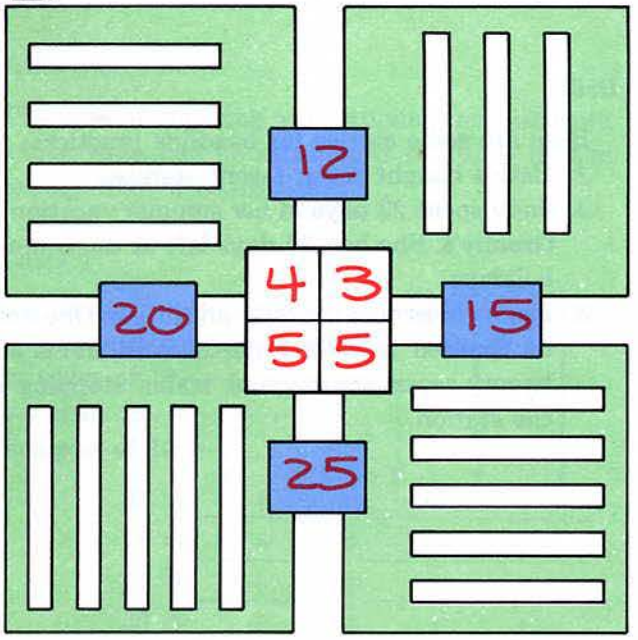
II.



III.



IV.

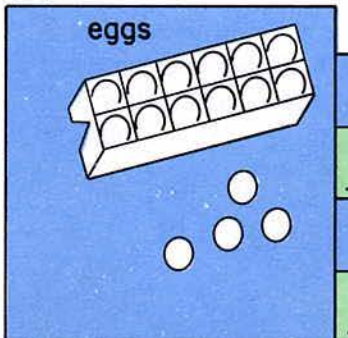


V.

Fewest Coins — dimes and pennies.

twice as much	
2 d., 8 p.	
half as much	
0 d., 7 p.	
3 times as much	4 times as much
4 d., 2 p.	5 d., 6 p.

VI.



Fewest Units — dozens and eggs.

twice as many	
2 doz., 8 eggs	
half as many	
0 doz., 8 eggs	
3 times as many	4 times as many
4 doz., 0 eggs	5 doz., 4 eggs

28 Using horizontal and vertical strips to explore sums and products of the numbers of such strips and their intersections - What Are My Rules?

Tasks: *For the first four exercises pupils record in the white boxes the number of slots in each tinted card; and in the tinted boxes they record the product of the numbers of slots in cards of adjacent pairs. In exercises V and VI pupils use the information provided by the pictures and the described operations to compute the missing information for each table. Note the "fewest units" restrictions.*

Purpose: To review and extend knowledge of multiplication and to provide a use for this knowledge in several applications.

Unifying Ideas: Multiplication and division.

Materials for Teacher: Demonstration-size slotted cards as described in Notes on Pupil Page 8; 4 dimes and 16 pennies; 4 egg cartons and 64 marbles or other objects to represent eggs.

Materials for Pupils: None.

The Lesson:

Note that all of the computation problems on pupil page 28 are presented in a concrete setting so that pupils must decide which data are relevant data and which operations to use in order to do the exercises. This is particularly true of problems in V and VI; these exercises are in a format which is used heavily throughout LEVEL C.

In some elementary mathematics programs the "4 times as much" problem in V would be given as a straightforward multiplication problem

$$\begin{array}{r} 14 \\ \times\ 4 \\ \hline \end{array}$$

and such a problem would be delayed until an algorithm for multiplying a 2-digit number by a 1-digit number had been taught.

In other programs the problem might be posed in the form of a story:

> Bill had 14¢. John had four times as much. How much did John have?

The format we have adopted allows a pupil to derive the same benefit he would get from the story problem but it doesn't burden him with reading. Also, in a very small space we present four such story problems, and we make it possible for pupils to observe certain relationships among the problems. For example, when he has solved the "twice as much" problem, he can use the answer as a datum for solving the "4 times as much" problem (4 times as much is twice as much as twice as much).

Finally, the format we have used in V makes it possible for the pupil to find answers without knowing the standard algorithms. Clearly, an amount 4 times as much as 1 dime and 4 pennies can be made with 4 dimes and 16 pennies. (This involves two of the steps in the standard algorithm for multiplication.) But, the pupil is also asked to use the smallest possible number of dimes and pennies to make up this same amount of money. (To find this out, he goes through another step in the standard algorithm.)

Fewest Units

In case pupils need help with V and VI, use the actual objects as a source of explanation. You may also wish to prepare for V and VI by working with the following exercises at the chalkboard:

With only nickels and pennies available, how can I make the listed amounts, using the fewest possible coins?

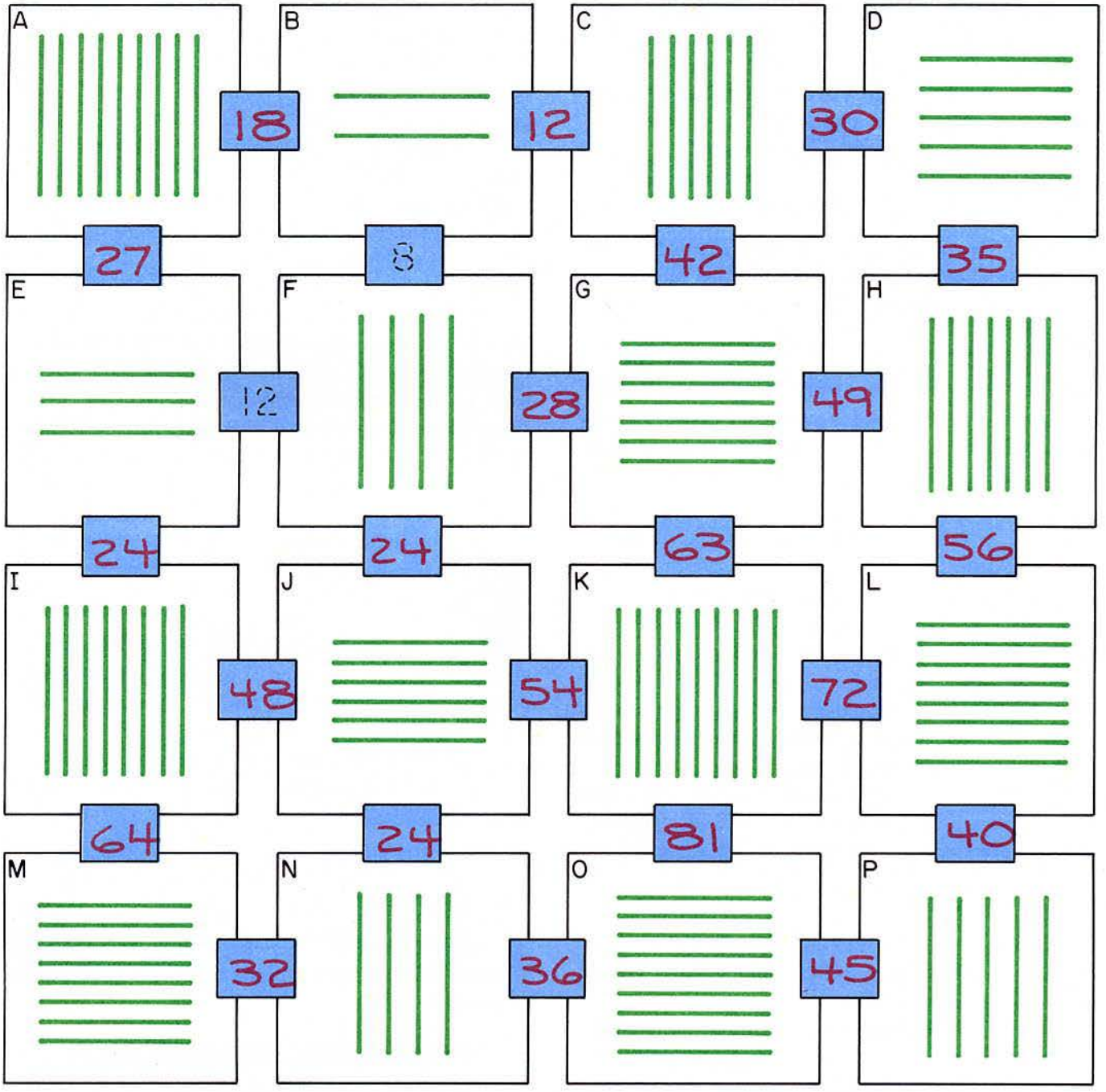
Amount	4¢	5¢	6¢	10¢	15¢	18¢
Nickels	0	1				
Pennies	4	0				

Represent the time periods using the fewest possible units.

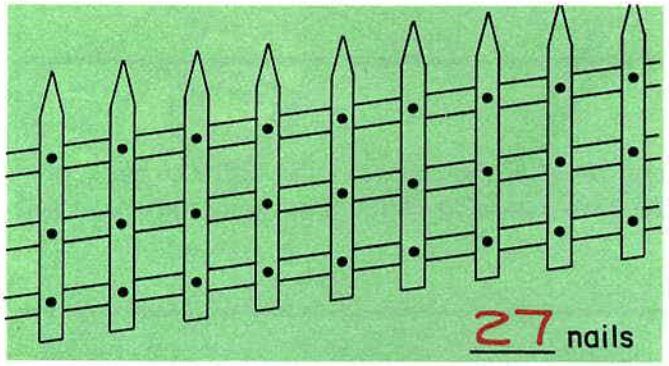
Period	3 days	7 days	13 days	14 days	23 days	
Weeks	0	1				3
Days	3	0				4

Headline for the Day: Cecil spent 43¢ at the candy store. He gave the clerk a dollar bill. The clerk gave Cecil the correct change using the fewest coins he could. What were the coins Cecil received?

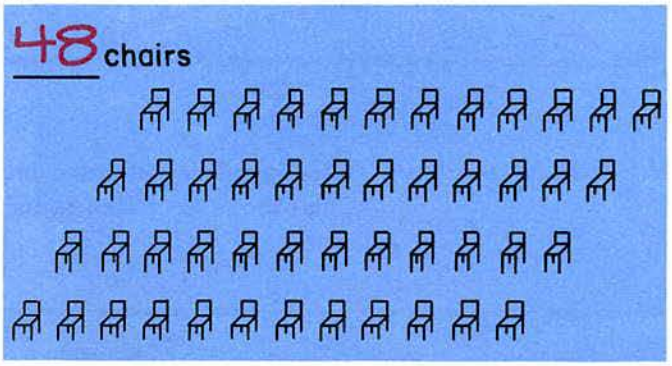
You may need a piece of scratch paper to see what happens when the lines cross in some of the examples.



I



II



29 Using horizontal and vertical lines to explore sums and products of the numbers of such lines and their intersections - counting (or shortcuts to counting) rectangular arrays of nails and chairs to determine the total number of nails and chairs.

Tasks: *Pupils record in the blue boxes the product of the numbers of lines pictured on pairs of adjacent cards. At the bottom of the page pupils count or use shortcuts to counting to record in the blanks the number of nails and chairs pictured.*

Purpose: To further extend knowledge and understanding of multiplication.

Unifying Ideas: Structure; multiplication.

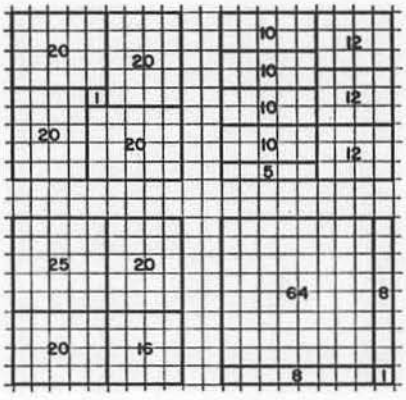
Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

Encourage pupils to use their own diagrams to discover multiplication facts. Show them shortcuts to counting. For example, in finding the product of 9 by 9, a pupil might sensibly draw a rectangle on graph paper and then count the number of unit squares, one by one. This approach through counting should not be tolerated at this level. You should take an active part in encouraging shortcuts by praising those pupils who use them and by demonstrating their use to pupils who may be insecure.

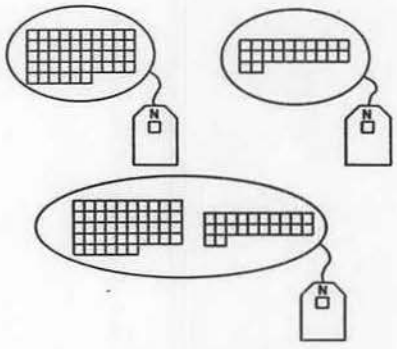
The basic idea in avoiding the counting of rectangular arrays is to partition them into smaller rectangular arrays whose counts have been memorized. The total is then obtained by addition. Here are some examples of such partitionings:



The partitioning indicated by drawing lines such as those in the bottom pair of pictures provides a basis for explaining the fact that multiplication distributes over addition. It is also the basic idea used in explaining the algorithm for long multiplication treated later in LEVEL C.

Drill

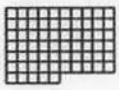
Here are some developmental exercises for a Ditto sheet on addition and subtraction:

1. 

2.

35	58	17	27
+ 24	+ 21	+ 42	+ 41

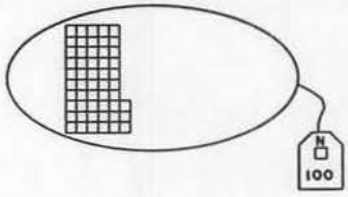
3. Shade 32 little squares and tell how many are left unshaded.



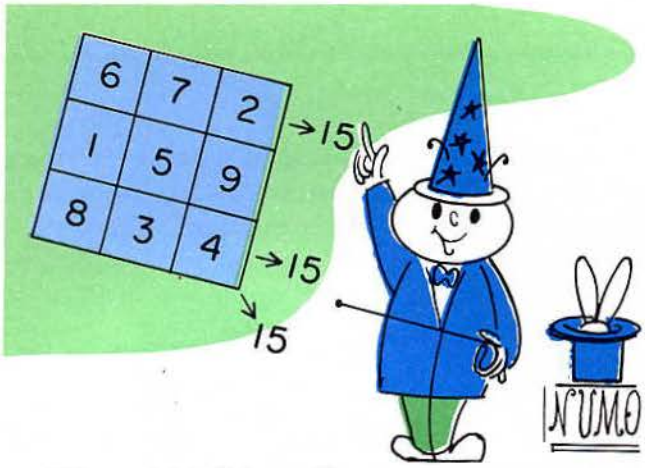
4.

59	38	27	85
− 21	− 22	− 21	− 35

5. Draw more little squares to make 100 little squares in all. How many do you need to draw?



Headline for the Day: Lance spent 4¢ less than Tania. Make a table to keep a record of what you can say about this situation.



"Gee whiz! It's really a Magic Square. Pick any row or column or diagonal: add the 3 numbers — and the sum is 15.

"That's magic! That gives me a score of 8 points — one for each row or column or diagonal with the sum 15.

"Here's another square with the same numbers — 1 through 9. Is it magic? Do I get 8 points for this one? You find out."

A

2	9	4	15
7	5	3	15
6	1	8	15
15	15	15	15

Score: 8

"Here is another. What score do I get for it?"

B

4	3	8	15
9	5	1	15
2	7	6	15
15	15	15	15

Score: 8

"Some squares are not magic all the way; so, I wouldn't get 8 points for them. How many points would you give me for this one?"

C

1	8	6	15
3	5	7	15
4	9	2	15
8	22		8

Score: 5

"Here's one that gave me a very low score."

D

1	2	3	6
4	5	6	15
7	8	9	24
12	15	18	15

Score: 4

"Can you get a zero score?"

E *

3	7	1	11
4	9	8	21
2	6	5	13
9	22	14	17

Score: 0

* OTHER POSSIBLE ANSWERS

30-31 A brief excursion into Magic Squares and their properties.

Tasks: *Pupils supply the missing sums and scores in accordance with the stated conditions. They may have to do a bit of exploring for exercise E. The objective here is to arrange the numbers so that no two of the eight rows, columns, and diagonals have the same sum. There are many such arrangements.*

Purpose: To explore Magic Squares while practicing addition and subtraction skills.

Unifying Ideas: Addition and subtraction.

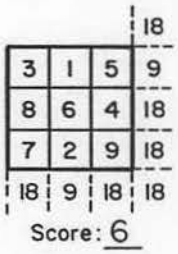
Materials for Teacher: Ditto master.

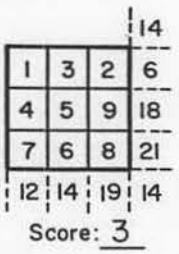
Materials for Pupils: Ditto sheets.

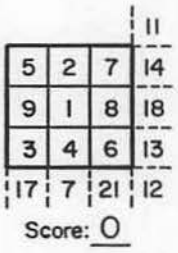
The Lesson:

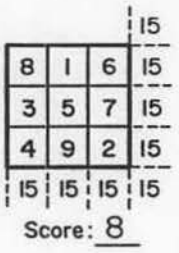
In Magic Squares the sum for each row, column, and diagonal is the same number. Magic Squares are interesting to pupils, offer opportunities for drill, and their structure invites further exploration. On pupil page 30 (and in F, G, and H on pupil page 31) Numo makes up a game in which the score for a completed square is the total number of rows, columns, and diagonals having the sum 15.

An interesting alternative approach to the subject would be to propose a game in which the score for a completed square is the largest number of rows, columns, and diagonals having the same sum. Here are four completed squares and their scores:









Pupils might then be challenged to find a square with a score of 8.

Pupils should be provided with Ditto sheets of scratch paper bearing blank Magic Square work forms with places to write the sums.

Another variation on the game of arranging the numbers 1-9 in a 3-by-3 square is to make the score

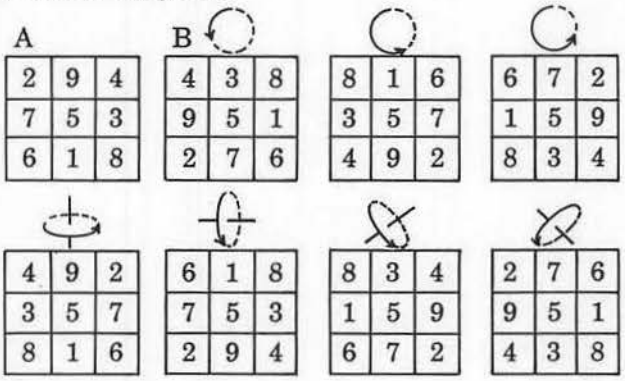
more

Headline for the Day: Tell a story to fit this fact team: $72 + 28 = \square$ $\square - 28 = 72$ $28 + 72 = \square$ $\square - 72 = 28$

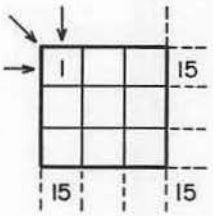
the total of the sums for the rows, columns, and diagonals. For example, under this scoring rule, the scores for the four completed squares shown above are 126, 118, 113, and 120, respectively. Challenge pupils to discover the lowest and highest possible scores. This should be easy to do after they have discovered that for each completed square, the total of the row-sums and column-sums must be 90 (this is, of course, twice the sum of the numbers from 1 through 9). This means that the score for a completed square is 90 plus the total of the diagonal-sums. So, to find the lowest possible score . . .

Exploring 3-by-3 Magic Squares

Notice that the completed squares in exercises A and B are magic "all the way." The square in B can be obtained by rotating the square in A about its center a quarter-turn to the left. Six other Magic Squares can be obtained from the one in A by various rotations about the center or the center column or either diagonal.

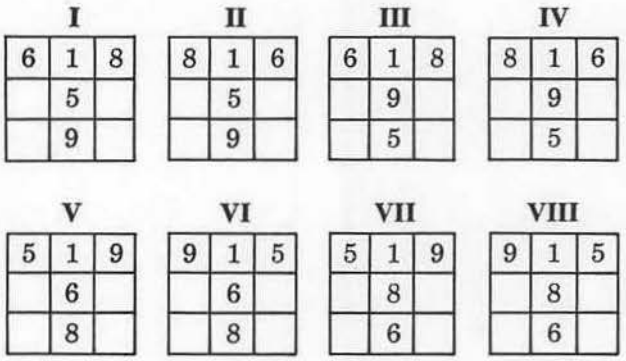


It is natural to ask if these 8 completed squares are the only ones which are Magic Squares for the numbers 1-9. One could continue exploring but it is doubtful from a practical point of view that the answer would be obtained by exhaustion since there are 362,880 different completed squares for the nine numbers from 1 through 9. (See Notes on Pupil Page 31.) In view of the very large number of possibilities it seems unlikely that merely 8 of them are Magic Squares. Yet, that is the case. As a way of seeing this, consider the possible positions for 1 in a completed square. Is there a Magic Square with 1 in the upper left-hand cell? If there were, there would have to be three pairs of different numbers from { 2, 3, 4, 5, 6, 7, 8, 9 } such that each couple together with 1 would have the sum 15. It turns out



that there are only two such couples, { 5, 9 } and { 6, 8 }. So, there is no Magic Square with 1 in the upper left-hand cell. For the same reason, there is no Magic Square with 1 in any of the other corner cells. Nor is there a Magic Square with 1 in the center cell for that would require four triples of different numbers each with sum 15 and each having 1 as a component.

Therefore, the only position for 1 in a Magic Square is in the middle cell of an edge row or of an edge column. Let's examine all the possibilities for 1 in the middle cell of the top row:



As you can readily see, only two of these (I and II) can be Magic Squares and these are the sixth and third of the Magic Squares shown in the earlier discussion. All other Magic Squares can be obtained from I (or II) by rotations.

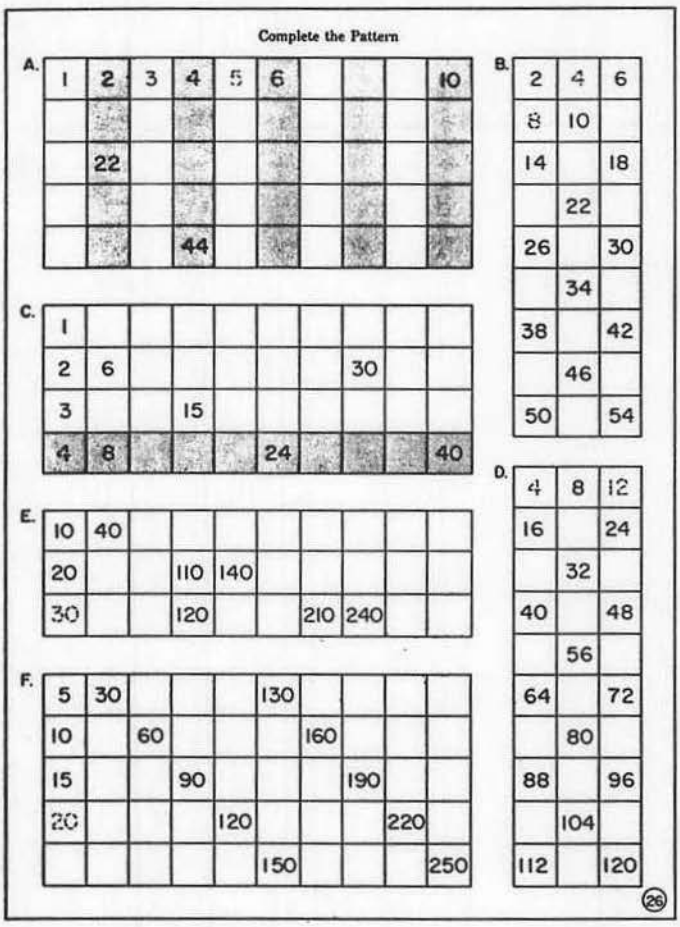
Drill

Here are some stories for headline practice:

1. Al has 14 bottle tops. Rudy has 7.
2. Jane figured she would weigh 18 pounds on the moon. People weigh 6 times as much on earth as they would on the moon.
3. Chloe's teacher had 25 pieces of chalk. Each of eight children received a piece.
4. Pat paid 4 dimes for a toy. Her cousin paid 8 nickels for a toy just like the one Pat bought.
5. Sophia's class put on a puppet show. The names of 96 boys and 79 girls were on the visitors list.

6. Allison had a collection of sea shells. Vince brought his collection of 185 shells over to Allison's house. Together they had 335 sea shells.

Consider using as a drill sheet page 26 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Here is a suggestion for a drill sheet on subtraction:

	I	II	III		IV					
	8	12	13		10	19	3	20	31	72
A	− 3	− 3	− 3	−3						
	10	19	32		28	40	23	51	34	100
B	− 5	− 5	− 5	−5						
	12	15	20		29	32	13	41	83	124
C	− 4	− 4	− 4	−4						
	15	21	89		47	30	54	72	36	103
D	− 7	− 7	− 7	−7						
	14	22	30		43	26	51	35	99	117
E	− 8	− 8	− 8	−8						
	15	20	34		23	87	41	59	136	92
F	− 6	− 6	− 6	−6						
	14	18	30		21	45	33	96	62	107
G	− 9	− 9	− 9	−9						
	19	20	27		18	24	30	22	70	51
H	−12	−12	−12	−12						

Rule: Subtract 7

I

89, 82, , , 61, , , , , ,

"What score do I get for these squares?"

F

8	1	5	15
3	6	7	15
4	9	2	15
15	16	14	15

Score: 3

G

2	9	4	15
3	5	7	15
6	1	8	15
11	15	19	15

Score: 6

H

2	7	6	15
5	9	1	15
4	3	8	15
11	19	15	19

Score: 4

When Numo gets tired of using only the numbers 1 through 9, he tries other groups of numbers.

"Once I tried odd numbers. Here is my work. What's my score?"

I

9	1	11	21
5	15	13	33
7	17	3	27
21	33	27	27

Score: 3

"That's a low score! I counted the number of times I got 27. That was low. I counted the times I got 33. That was no better. I tried again:

J

3	13	11	27
17	9	1	27
7	5	15	27
27	27	27	27

Score: 8

"That's a better score."

"Next, I tried even numbers and got a perfect score of 8. Can you finish the square as I did?"

K

8	6	16	30
18	10	2	30
4	14	12	30
30	30	30	30

Score: 8

"Next, I jumped around in picking my numbers. What score would you give me?"

L

11	7	4	22
1	8	15	24
12	9	5	26
24	24	24	24

Score: 6

"Can you use these same 9 numbers to get a perfect score?" YES, INTERCHANGE THE 7 AND THE 9.

Notes on Pupil Page 31

Tasks: *Pupils supply the missing sums and scores in accordance with the stated conditions on pages 30 and 31.*

Purpose: To further explore Magic Squares while perfecting addition and subtraction skills.

Unifying Ideas: Addition and subtraction.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets of Magic Square forms.

The Lesson:

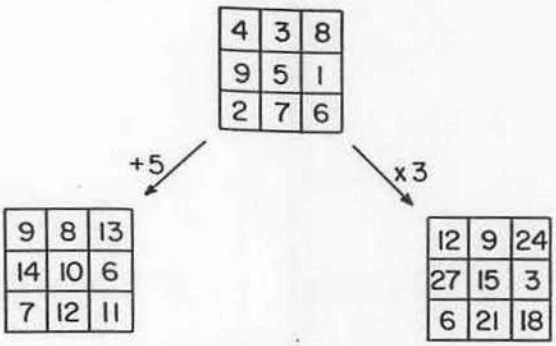
Numo might indeed get tired of arranging the numbers 1 through 9 before he had tried all possible arrangements. How many possible arrangements are there? There are 9 numbers which could be used to fill the first cell (top row, left). When the first cell has been filled, there are 8 numbers which could be used to fill the second cell (top row, middle). So, there are 9×8 pairs of numbers to fill the first two cells. (See Notes on Pupil Page 12.) When the first two cells have been filled, there are 7 numbers which could be used to fill the third cell. So, there are $9 \times 8 \times 7$ ways (that is, triples of numbers) to fill the first three cells. Continuing with this argument, it is easy to see that there are $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways to complete a 3-by-3 square with the numbers from 1 through 9.

Exercises I and J explore the completed squares formed by using the first nine consecutive odd numbers. The "magic sum" for these nine numbers is 27. It is easy to find what the magic sum must be for any given numbers. Here's how: The total of the row-sums must be the sum of the given nine numbers. So, if the row-sums are to be equal, each row-sum must be one-third of the sum of the given numbers. In the case of the first nine consecutive odd numbers, their sum is 81; so, the magic sum is 27.

In similar fashion, we can compute the magic sum for the nine consecutive even numbers beginning with 2 (exercise K). The magic sum is 30.

More Exploring of 3-by-3 Magic Squares

As we mentioned in Notes on Pupil Page 30, there are exactly 8 Magic Squares with the numbers 1-9, and each of these can be obtained from any of the others by a rotation. One can also obtain Magic Squares from any of these by making certain kinds of systematic changes in all nine numbers. For example, consider the two completed squares obtained by adding 5 to each of the nine numbers in the square in B (on pupil page 30) and by tripling each of the numbers:



Both are Magic Squares. If you double each of the numbers in the square in B, you get the square in K. If you subtract 1 from each number in the square in K and rotate the resulting square a half-turn about its center row, you get the square in J.

Drill

Here are some stories for headline practice:

1. Wally had 14 pencils. His brother gave him 12 more.
2. Gene is 8 years old. There are 12 months in 1 year.
3. Mila's class bought 6 yards of wood to make puppets for their show. They found they needed 2 feet of wood to make each puppet.

Headline for the Day: Fill in the blanks to complete the pattern.

600, 585, 570, ___, ___, ___, ___, ___, 480

All About Coins

Find the 6 ways to make 18¢ with coins. In keeping track, you quickly learn the shorthand we use.

How many coins are there in each of the ways to make 18¢?

18¢

0 d, 0 n, 18 p. (or 18 coins)

1 d, 0 n, 8 p. (or 9 coins)

0 d, 3 n, 3 p. (or 6 coins)

0 d, 2 n, 8 p. (or 10 coins)

0 d, 1 n, 13 p. (or 14 coins)

1 d, 1 n, 3 p. (or 5 coins)

Notice that only 5 coins were required in the last combination. We say that 5 coins are the "fewest coins" needed to make 18¢.

FEWEST COINS

Show how the amounts given could be made with the fewest coins.

14¢ ..... 1 d, 0 n, 4 p.

7¢ ..... 0 d, 1 n, 2 p.

16¢ ..... 1 d, 1 n, 1 p.

20¢ ..... 2 d, 0 n, 0 p.

24¢ ..... 2 d, 0 n, 4 p.

More About Fewest Coins

1. Show how the amounts in the blue blocks can be made with the fewest coins.
2. Find the sum of the amounts shown in the blue blocks.
3. Then show the fewest coins needed to make this amount.

A.

8¢ ..... 0 d, 1 n, 3 p.

\+ 7¢ ..... 0 d, 1 n, 2 p.

15¢ ..... 1 d, 1 n, 0 p.

B.

9¢ ..... 0 d, 1 n, 4 p.

\+ 12¢ ..... 1 d, 0 n, 2 p.

21¢ ..... 2 d, 0 n, 1 p.

C.

6¢ ..... 0 d, 1 n, 1 p.

\+ 16¢ ..... 1 d, 1 n, 1 p.

22¢ ..... 2 d, 0 n, 2 p.

D.

15¢ ..... 1 d, 1 n, 0 p.

\+ 9¢ ..... 0 d, 1 n, 4 p.

24¢ ..... 2 d, 0 n, 4 p.

Problem: Can you find any amount less than 25¢ which can be made by the same number of coins in two different ways? No!

32-33 Numeration systems exercises in terms of Fewest Coins problems - addition and subtraction exercises in terms of Fewest Coins problems.

Tasks: *For the exercises on the left half of the page, pupils record in the blanks either the number of different coins needed or the number of each kind of coin needed. In exercises A-D pupils record the fewest coins for each amount of money specified in the tinted boxes and the fewest coins for the sum of the amounts specified in the tinted boxes.*

Purpose: To explore numeration systems via the vehicle of the *Fewest Coins* game while simultaneously practicing computational skills.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Coins.

Materials for Pupils: Cuisenaire rods.

The Lesson:

Fewest Coins is a game we have played frequently in MATH WORKSHOP. On pupil pages 32 and 33 pupils are given practice in making amounts of money with the fewest possible coins. "Fewest coins" are then added to "fewest coins." The purpose of the activity is to provide experience with the idea of place value (or positional notation) in a new context—one in which the symbol 1 2 4 by temporary agreement stands not for one hundred twenty-four, but rather for 1 dime, 2 nickels, 4 pennies, or 24 cents.

A way of developing readiness for this page is to play the following game: I have coins worth 18¢ in my hand. What can you say for sure? Continue for several minutes. Then say: Now, I am going to give you some more information about the coins in my hand. The combination uses the fewest coins to make 18¢. Now, what can you say for sure? This new information abruptly halts the speculation, because it has fully determined that the combination is 1 dime, 1 nickel, and 3 pennies. (Pupils can use Cuisenaire rods as they work toward the answer—the orange rods represent dimes, the yellow rods represent nickels, and the white rods represent pennies.)

Try the game again with another combination: What can you say for sure? Many things. And, finally, introduce the condition that it is a "fewest coins" combination. Again, the speculation is halted.

Ask: Is there any amount less than 50 cents that can be made up of the fewest coins in more than one way? Try several examples. (There are no such amounts!) The time spent in a fruitless search for such combinations is time well spent.

What Can You Say?

Try another kind of game: I gave the clerk a quarter. He gave me two coins as change. What can you say for sure about this situation? Answers may be:

If he gave you 2 dimes then you spent 5¢.

If he gave you 1 dime and 1 penny then you spent 14¢.

Now ask the question: How many different amounts might I have spent? Make a chart of the numbers of coins he gave me:

D	N	P	Value	Spent
2			20	5
	2		10	15
		2	2	23
1	1		15	10
	1	1	6	19
1		1	11	14

Are all of these possibilities examples of making change with the "fewest coins"? (No—2 nickels is not; 1 dime would suffice.)

Make a chart for the fewest coins we need to make all amounts of 1¢ through 15¢.

Headline for the Day: Tell a story to fit this fact team: $185 + 15 = \square$ $\quad$ $\square - 15 = 185$
$15 + 185 = \square$ $\quad$ $\square - 185 = 15$

More About Coins

Show each amount in the fewest dimes and pennies.

8¢	0 d, 8 p	25¢	2 d, 5 p
13¢	1 d, 3 p	10¢	1 d, 0 p
21¢	2 d, 1 p	36¢	3 d, 6 p
59¢	5 d, 9 p	82¢	8 d, 2 p

Find the Sums.

Use the fewest dimes and pennies for each amount.

A

14¢	1 d, 4 p	
+ 8¢	0 d, 8 p	
22¢	2 d, 2 p	

B

20¢	2 d, 0 p	
+ 13¢	1 d, 3 p	
33¢	3 d, 3 p	

C

25¢	2 d, 5 p	
+ 6¢	0 d, 6 p	
31¢	3 d, 1 p	

D

48¢	4 d, 8 p	
+ 12¢	1 d, 2 p	
60¢	6 d, 0 p	

E

18¢	1 d, 8 p	
+ 17¢	1 d, 7 p	
35¢	3 d, 5 p	

F

37¢	3 d, 7 p	
+ 26¢	2 d, 6 p	
63¢	6 d, 3 p	

Find the Differences.

G

30¢	3 d, 0 p	
− 12¢	1 d, 2 p	
18¢	1 d, 8 p	

H

25¢	2 d, 5 p	
− 7¢	0 d, 7 p	
18¢	1 d, 8 p	

I

47¢	4 d, 7 p	
− 23¢	2 d, 3 p	
24¢	2 d, 4 p	

J

29¢	2 d, 9 p	
− 9¢	0 d, 9 p	
20¢	2 d, 0 p	

K

37¢	3 d, 7 p	
− 18¢	1 d, 8 p	
19¢	1 d, 9 p	

L

56¢	5 d, 6 p	
− 29¢	2 d, 9 p	
27¢	2 d, 7 p	

M

81¢	8 d, 1 p	
− 34¢	3 d, 4 p	
47¢	4 d, 7 p	

N

70¢	7 d, 0 p	
− 47¢	4 d, 7 p	
23¢	2 d, 3 p	

Find the Sums.

O

18¢	1 d, 8 p	
36¢	3 d, 6 p	
+ 25¢	2 d, 5 p	
79¢	7 d, 9 p	

P

47¢	4 d, 7 p	
19¢	1 d, 9 p	
+ 5¢	0 d, 5 p	
71¢	7 d, 1 p	

Notes on Pupil Page 33

Tasks: *In the exercise in the upper left quarter of the page pupils record in the blanks the fewest coins for each amount shown. In the other exercises pupils record in the blanks the fewest coins for each amount specified and the fewest coins for the sum of the amounts specified. In the tinted boxes they record the values of the coins, or the sums or differences of the amounts, as instructed.*

Purpose: To explore numeration systems via the vehicle of the *Fewest Coins* game while simultaneously perfecting computational skills.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets (see below).

The Lesson:

Since this page is limited to the condition of fewest dimes and pennies, the patterns of addition and subtraction using fewest coins are identical with the patterns for the decimal system of numeration.

Follow up this page by preparing a Ditto master of the exercise that follows:

	Fewest coins				Amounts of money			
	25¢	5¢	1¢	Total	Quarters	Nickels	Pennies	Total
a.	1	2	4	7	25¢	10¢	4¢	39¢
b.	2	1	4	7	50¢	5¢	4¢	59¢
c.	1	3	3	7	25¢	15¢	3¢	43¢
d.	0	3	4	7	0¢	15¢	4¢	19¢
e.	2	3	2	7	50¢	15¢	2¢	67¢
f.	2	2	3					
g.			1		50¢	30¢		
h.								77¢
i.								69¢
j				5	50¢	15¢		

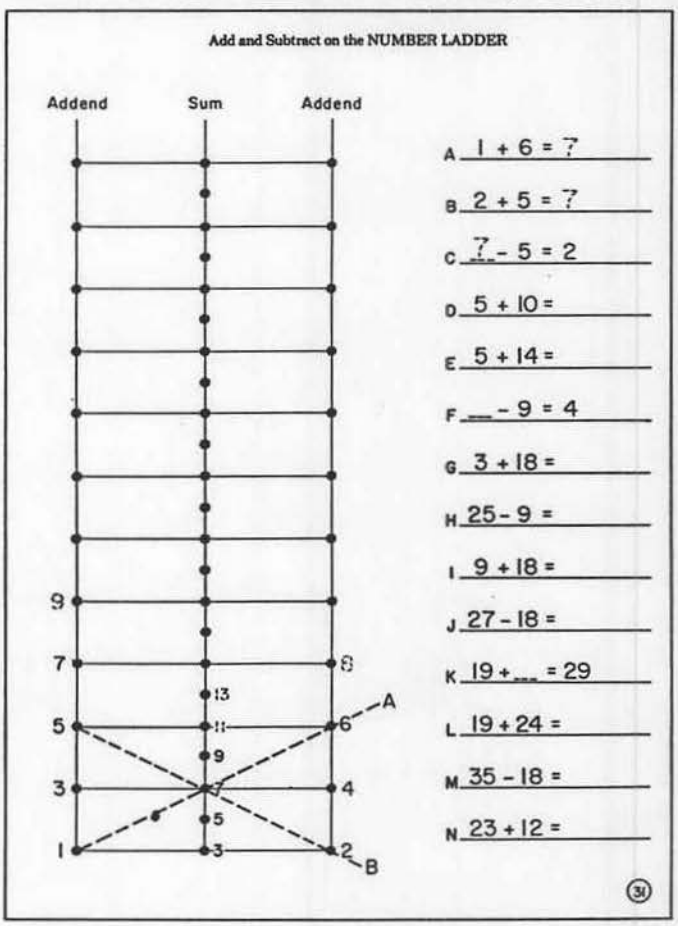
Example *a* tells you that, given quarters, nickels, and pennies to choose from, the least number of coins needed to make 39¢ is 7: 1 quarter, 2 nickels, and 4 pennies.

Drill

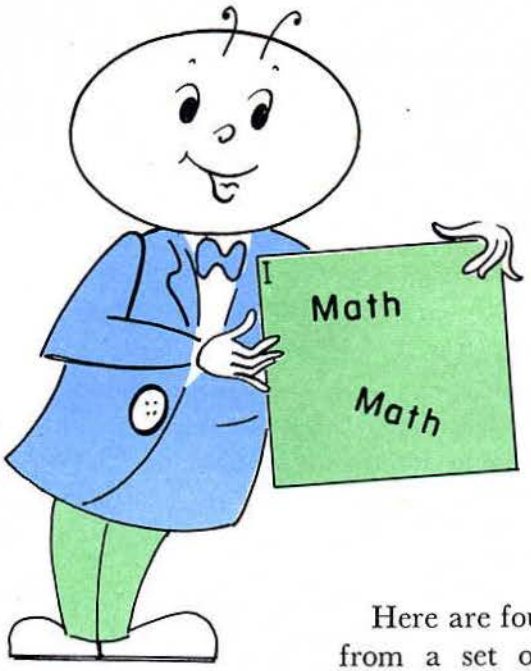
Here are some stories for headline practice:

1. Mary has 3 pears. Joan has 12 pears.
2. Zeke, Sue, and Simon went fishing. They each caught a trout. Zeke's fish was 14 inches long, Sue's fish was 17 inches long, and Simon's fish was 16 inches long.

Consider using as a drill sheet page 31 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Headline for the Day: Grant exchanged 1 half-dollar, 1 quarter, and 4 nickels for dimes and pennies. How many dimes and pennies should he have received?



Here are four cards from a set of cards made by Numo, the number-explainer.

He calls them his W-A-M-R cards. That's his shorthand for "What-Are-My-Rules?"

Here is what you will find on the backs of those cards.

BACK

Rule A	Rule B
1	2
Rule C	Rule D
3	4

BACK

Rule A	Rule B
3	6
Rule C	Rule D
9	12

BACK

Rule A	Rule B
4	8
Rule C	Rule D
12	16

BACK

Rule A	Rule B
2	4
Rule C	Rule D
6	8

Can you find the back which goes with each card above? Can you fill in the missing numbers?

FRONT

II
Math

FRONT

III
Math
Math
Math
Math

FRONT

IV
Math
Math
Math

Here are some more cards from this set. You fill in the boxes.

FRONT

V
Math
Math
Math
Math

BACK

Rule A	Rule B
5	10
Rule C	Rule D
15	20

FRONT

VI

BACK

Rule A	Rule B
0	0
Rule C	Rule D
0	0

FRONT

VII
Math
Math
Math
Math
Math
Math
Math
Math
Math
Math

BACK

Rule A	Rule B
10	20
Rule C	Rule D
30	40

FRONT

VIII
Math
Math
Math
Math

BACK

Rule A	Rule B
6	12
Rule C	Rule D
18	24

34 Introduction of WAMR cards - practice in multiplication and division via What Are My Rules? games in a new format.

Tasks: *Pupils make the appropriate matching of the front and back of each card in the WAMR set. They then apply the rules to record on the cards the missing entries.*

Purpose: To introduce *What Are My Rules?* games in a new setting, i.e., with WAMR cards; to continue to practice computational skills.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations.

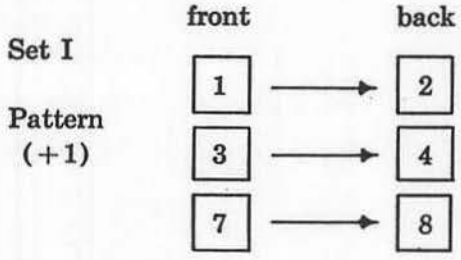
Materials for Teacher: Demonstration-size WAMR cards (green front, blue back) just like those shown on pupil page 34 or those described below.

Materials for Pupils: Cuisenaire rods.

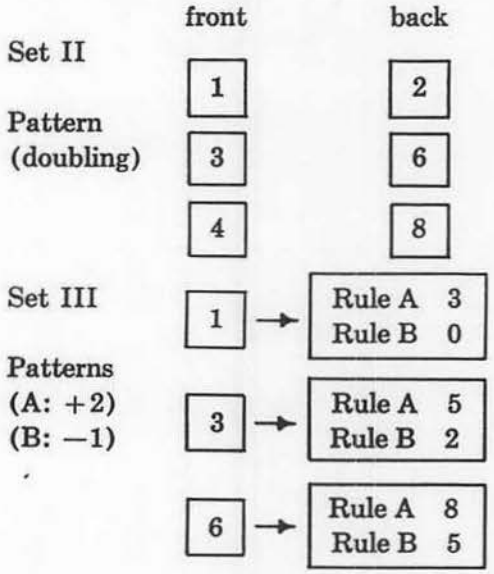
The Lesson:

As explained on this page, WAMR (pronounced wammer) sets are decks of cards on which there are items pictured on the front and back. From here through pupil page 80, new sets of WAMR cards are introduced by Numo. The game is played this way. Look at the front of the cards. Then study the numbers on the back to see how they are derived by applying some rule to the numbers or items on the front.

Begin this game by actually preparing sets of cards, gradually leading from very easy examples to those which are more challenging.



After showing the backs of two or three cards as indicated, show the fronts and have the pupils guess the rule. The rule, of course, in the examples given above is that the number on the back is 1 greater than the number on the front.



Pupils are to regard the display of four cards at the top of pupil page 34 as showing the fronts of four cards from a new set of cards. For this set of cards, there are four rules or ways of thinking about the front of the card. A study of the clues given on the backs of these cards should lead to the discovery that Rule A is $\times 1$, Rule B is $\times 2$, Rule C is $\times 3$, and Rule D is $\times 4$.

To help pupils discover the rules and as a basis for extending the activity, the following chart can be developed on the board as another way of recording the relationships noted:

Number of times "Math" is used	Rule A	Rule B	Rule C	Rule D
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Headline for the Day: Circle the number which does not fit the pattern: 4, 7, 13, 25, 50

WAMR... Set A

A

FRONT	BACK
4	7
3	1

B

FRONT	BACK
7	9
2	5

C

10	14
4	6

D

20	25
5	15

E

17	20
3	14

F

FRONT	BACK
16	22
6	10

G

12	20
8	4

H

16	23
7	9

I

30	40
10	20

J

20	28
8	12

K

32	42
10	22

L

5	7
2	3

M

12	21
9	3

N

8	16
8	0

O

10	15
5	5

WAMR... Set B

What Are My Rules?

U

FRONT	BACK	
	Rule A: 4¢	Rule B: 7¢
4¢ each or 2 for 7¢	Rule C: 11¢	Rule D: 14¢

V

	Rule A: 2¢	Rule B: 4¢
2¢ each or 3 for 5¢	Rule C: 5¢	Rule D: 7¢

W

FRONT	BACK	
	Rule A: 3¢	Rule B: 5¢
3¢ each or 2 for 5¢	Rule C: 8¢	Rule D: 10¢

X

	Rule A: 4¢	Rule B: 8¢
4¢ each or 3 for 10¢	Rule C: 10¢	Rule D: 14¢

Y

	Rule A: 6¢	Rule B: 12¢
6¢ each or 3 for 15¢	Rule C: 15¢	Rule D: 21¢

Z

	Rule A: 5¢	Rule B: 10¢
5¢ each or 3 for 13¢	Rule C: 13¢	Rule D: 18¢

Tasks: *In exercises A-O pupils discover and apply the two rules for WAMR Set A to complete each of the given cards. In exercises W-Z pupils use the clues provided to discover the four rules for WAMR Set B, and apply these rules to complete each card.*

Purpose: To discover rules and apply them; to work with common merchandising rates.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

WAMR Set A has a pair of rules. The first rule is to add the pair of numbers on the front and write the sum in the green region on the back. The second rule is to find the difference of the numbers on the front and write it in the blue region on the back.

Prepare pupils for WAMR Set A by playing *What Are My Rules?* games, making a record on the chalkboard as follows:

	A	B
3, 1 →	4	2
7, 2 →	9	5
10, 1 →		

You might ask pupils to invent new rules:

	A	B	Jill's rule	Ed's rule
3, 1	4	2	3	14
7, 2	9	5	14	19
10, 1	11	9	10	21

Jill's rule, of course, is that you find the product of the two numbers while Ed's rule tells you to add 10 to the sum of the given numbers.

A pupil should not be asked to reveal his rule until most of the pupils have had a chance to discover it. Instead, he should be made a monitor of his column of the chart and should decide whether a response of a classmate is correct and may be entered.

WAMR Set B has four rules. Rule A asks you to find the cost of 1 item at the rate stated on the front of the card. Rule B asks for the cost of 2 items. Rule C asks for the cost of 3 items, and Rule D asks for the cost of 4 items.

Before pupils attempt this page you may wish to have them work up a chart like the following:

	Cost for				
Candy	1	2	3	4	5
3¢ each 2 for 5¢	3¢	5¢	8¢	10¢	

Pupils should realize that amounts are computed from the sales prices for one and for more than one.

Following this page, consider work along the lines suggested by the following discussion.

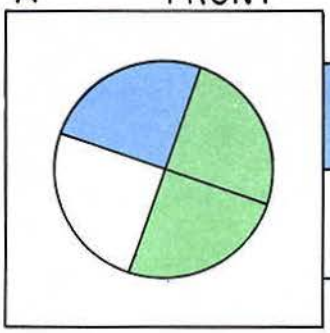
A problem in a traditional textbook reads: *Mary went to the store. She paid 33¢ for 3 cans of soup. How much did she pay for each can?* The answer of course is 11¢.

But let's change the problem slightly: *Mary paid 31¢ for 3 cans of beans. John bought only 1 can of beans. How much do you think John paid?* Now we have a problem that almost certainly will lead to a discussion involving differences of opinion. Certainly, John did not pay 10⅓¢ for one can. Some pupils might argue that 10⅓¢ is closer to 10¢ than 11¢, so he must have paid 10¢. Does that sound reasonable? No, because Mary would have bought her 3 cans of beans one at a time and paid 10¢ each, or 30¢ in all, in order to save 1¢. Other pupils might argue that John must have paid 11¢, and that Mary got a bargain by buying 3 at a time. Still others might apply the same argument to support the conclusion that John could have paid 12¢—and Mary's bargain would be more impressive. Why do stores sometimes offer bargains? How much do you think 2 cans would cost if Mary paid 32¢ for 3 cans?

Headline for the Day: Della had a dollar to spend on some candy. The candy sold for 7¢ each or 2 for 13¢. Make a table to keep a record of some of her possible purchases and their costs.

WAMR... Set C

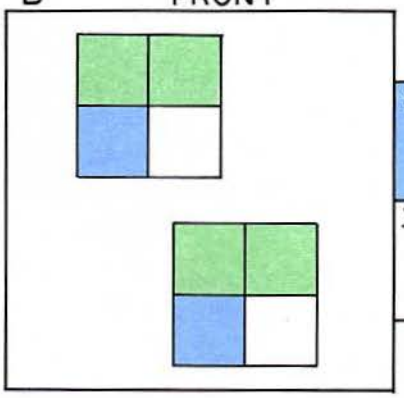
A FRONT



BACK

Rule I	Rule II
1	2
Rule III	Rule IV
3	4

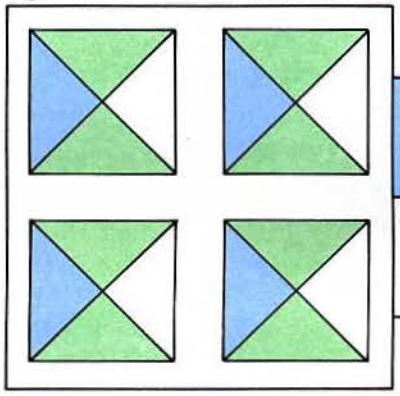
B FRONT



BACK

I	II
2	4
III	IV
6	8

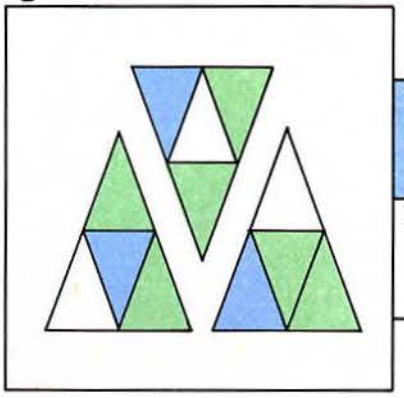
C FRONT



BACK

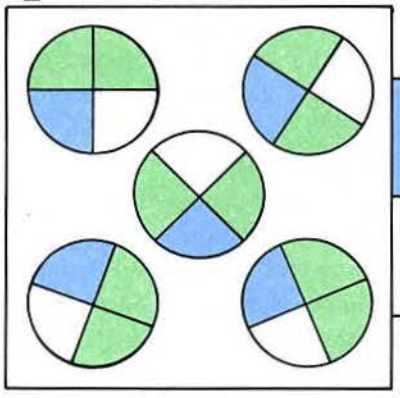
I	II
4	8
III	IV
12	16

D



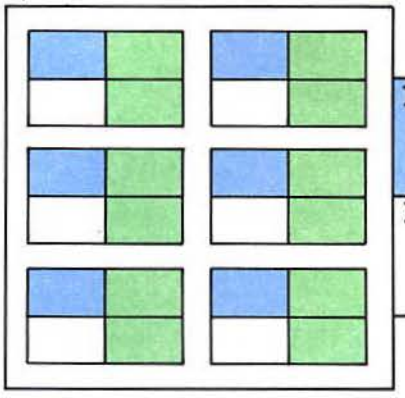
I	II
3	6
III	IV
9	12

E



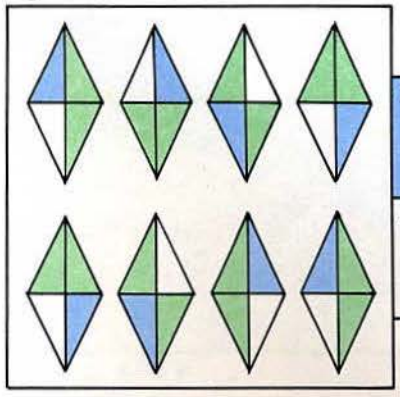
I	II
5	10
III	IV
15	20

F



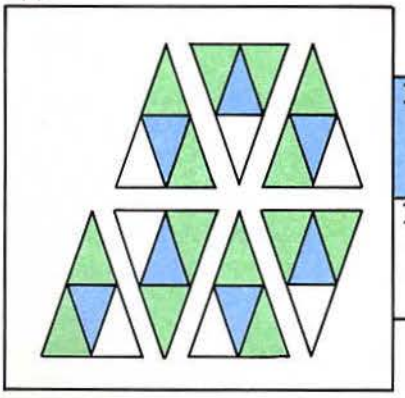
I	II
6	12
III	IV
18	24

G



I	II
8	16
III	IV
24	32

H



I	II
7	14
III	IV
21	28

36 Using WAMR cards depicting partitioning of geometric regions to practice multiplication and division.

Tasks: *Pupils use the clues provided in A to discover the four rules for WAMR Set C.*

Purpose: To continue exploring functions using WAMR cards.

Unifying Ideas: Multiplication and division; functions and relations.

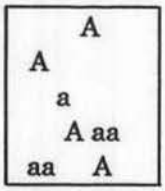
Materials for Teacher: A number line chart as described below.

Materials for Pupils: None.

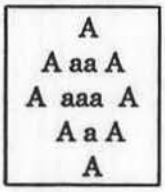
The Lesson:

An activity requiring similar thinking to that involved in discovering the rules in WAMR Set C might be used to prepare pupils for this page:

1.



2.



	Rule I	Rule II	Rule III
1.	4	5	9
2.	8	6	14

Rule I is to find the number of capital A's. Rule II is to find the number of small a's, and Rule III is to find the total number of small letters and capital letters.

Here is another similar activity:

1.

	Rule I	Rule II	Rule III
1.	9	10	1
2.			

2.

Rule I is to find the number of letters above the line. Rule II is to find the number of letters below the line, and Rule III is to find the difference between these numbers.

For WAMR Set C on pupil page 36 Rule I is to find the number of regions colored blue. Rule II is to find the number colored green. Rule III is to find the number of regions colored green or blue.

more

Headline for the Day: Add the numbers from card A on page 36 to those on card B. The numbers on which card are the sums of those on A and B?

Rule IV is to find the number of regions colored or not colored.

Number Line Chunks

Chop a number line from 0-149 into fifteen horizontal segments, or chunks, of ten numbers each.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109
110	111	112	113	114	115	116	117	118	119
120	121	122	123	124	125	126	127	128	129
130	131	132	133	134	135	136	137	138	139
140	141	142	143	144	145	146	147	148	149

A large copy of such a number line chart can be put to good use in the classroom. It can be used to demonstrate that it is possible to jump from 2 to 12, or 5 to 15, or 7 to 37 with great ease. Pupils won't need to count in order to do this, either. Every jump down (↓) is equivalent to adding 10. A jump ↓ followed by a jump ↷2 is equivalent to adding 12. A jump ↓ followed by a jump ↶2 is equivalent to adding 8. To find the sum of 10 and 17, you start at 10, jump ↓ and ↷7, and you're on 27. Or, start at 10, jump ↷7, and then jump ↓.

Before going on, allow your pupils to spend a little while in learning how to move about the number line grid. They will enjoy exercises such as these: Where will you land if you start at 0 and take three jumps down? Four jumps down? Two jumps down? Six jumps down? Ten jumps down?

Describe ways to make the trip from 11 to 22. From 5 to 12. From 3 to 24. From 7 to 18. From 25 to 37. From 111 to 123.

Where are we if we start at 15, jump down two rows and then make four jumps of 1 to the right? $(15 + 20 + 4 = 39)$ If we start at 15, make four jumps of 1 to the right, and jump down two rows? $(15 + 4 + 20 = 39)$ If we make two jumps of 1 to the right, jump down two rows, and jump 2 to the right? $(15 + 2 + 20 + 2 = 39)$

Have pupils jump back, or up, to subtract on the number line. Just as there are many ways to get from 15 to 39, there are also many ways to get from 39 to 15.

When pupils understand how to use the number line grid, it is possible to play games with it. For instance, one game is started by saying: I'm thinking of a number. Start at 15, jump three rows down and four to the left. What is the number? The first pupil to tell you what the number is becomes the leader and has to think of a number for which he must give appropriate clues. Some pupils may become so proficient that they won't need to use the number line chart. Encourage such pupils to "do it in their heads."

Other games could proceed like this: Start at 0, add 11. Add 11 more. Add 11 more. Where are you?

Or: Start at 3, add 6. Add 3 more. Add 6 more. Where are you?

Use repeated additions and subtractions of 3, 4, 5, 6, 7, etc., and ask pupils to look for patterns as they progress.

Jumping on the number line can be used to help emphasize the fact that in subtraction we simply undo what we do in addition. If a jump down one row is equivalent to adding 10 then a jump up one row must be equivalent to subtracting 10. If a jump of two to the right is equivalent to adding 2 then a jump of two to the left is equivalent to subtracting 2.

The structure of the decimal system is emphasized in the way the number line grid is designed. Bases for understanding algorithms for addition and subtraction can be firmly established by developing the ability to jump freely on the chart. Enterprising children will be able to discover many shortcuts for making jumps. The search for shortcuts should be encouraged because in discovering shortcuts pupils will also discover basic patterns in the decimal system.

You may wish to put some problems on the board, telling the children that they can use the number line chunks to help get the results:

27	42	38+24= ___	118+33= ___
+35	−17	55−19= ___	149−50= ___

Drill

Here is a suggestion for a drill sheet:

	I	II	III						
	19	22		20	13	33	97	100	102
A	−13	−13	−13						
	21	59		30	47	86	24	43	115
B	−16	−16	−16						
	38	20		47	21	44	30	59	103
C	−14	−14	−14						
	50	33		40	51	34	68	73	100
D	−25	−25	−25						
	30	25		27	31	53	118	122	104
E	−18	−18	−18						
	49	35		40	46	34	107	61	142
F	−27	−27	−27						
	67	50		41	52	89	93	61	164
G	−32	−32	−32						
	25	30		23	51	34	229	76	117
H	−19	−19	−19						

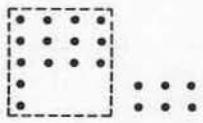
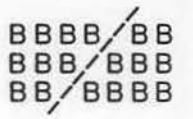
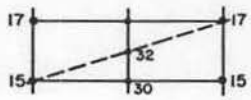
Rule: Subtract 8

I

100, , , , , 60, , , , ,

Also consider using as a drill sheet page 35 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:

Match Each PICTURE with a SENTENCE

PICTURES	Complete all sentences. Check the one suggested by the picture.	Fill in all of the basic fact team suggested by the picture.
A 	14 - 9 = 5 ✓20 - 6 = 14 16 - 8 = 8 10 - 0 = 10	20-6=14 20-14=6 6+14=20 14+6=20
B 	18 - ___ = 9 24 - 12 = ___ 13 - 8 = ___ ___ - 19 = 1	___ ___ ___ ___
C	50 - ___ = 5 33 - 11 = ___ 18 - ___ = 6 21 - 7 = ___	___ ___ ___ ___
D 	32 - ___ = 15 40 - 20 = ___ 12 - ___ = 6 19 - 17 = ___	___ ___ ___ ___

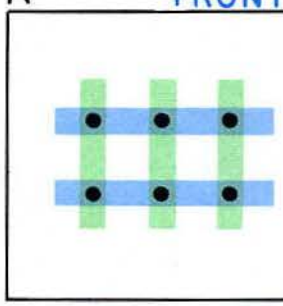
35

Here are some stories for headline practice:

1. Mr. Bacon is 40 years old. Mr. Bigsby is 29 years old.
2. Dad bought Steve a pair of shoes for $4.50, a tie for $0.98, and a sweater for $3.75.
3. Ramona lives 1 mile away from Sophia. Yesterday she walked to Sophia's house and back again. There are 5280 feet in 1 mile.
4. Daisy had 68 cents and earned 25 cents.
5. Emma bought 5 yards of ribbon. There are 36 inches in 1 yard.

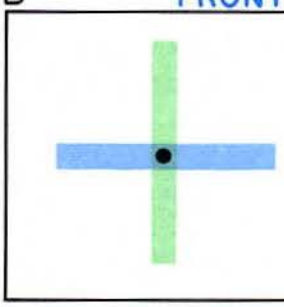
WAMR...Set D

A



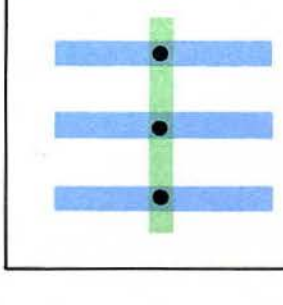
FRONT	BACK	
	Rule I: 5	Rule II: 6
	Rule III: 10	Rule IV: 12

B



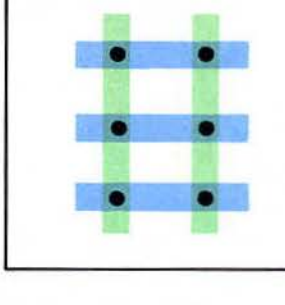
FRONT	BACK	
	Rule I: 2	Rule II: 1
	Rule III: 4	Rule IV: 2

C



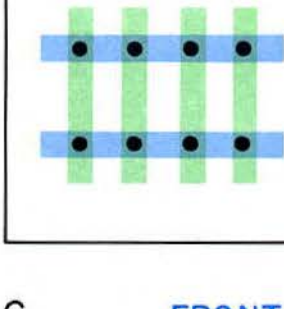
FRONT	BACK	
	Rule I: 4	Rule II: 3
	Rule III: 8	Rule IV: 6

D



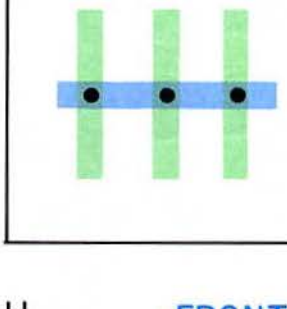
FRONT	BACK	
	I: 5	II: 6
	III: 10	IV: 12

E



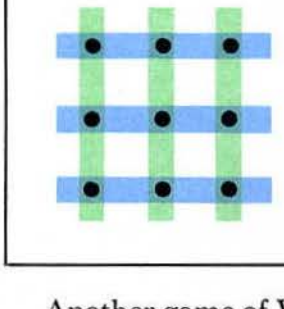
FRONT	BACK	
	I: 6	II: 8
	III: 12	IV: 16

F



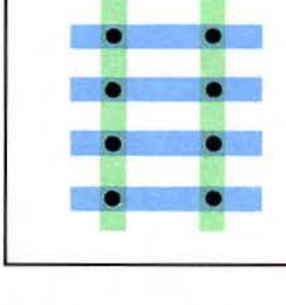
FRONT	BACK	
	I: 4	II: 3
	III: 8	IV: 6

G



FRONT	BACK	
	I: 6	II: 9
	III: 12	IV: 18

H



FRONT	BACK	
	I: 6	II: 8
	III: 12	IV: 16

Another game of What's My Rule?

a	b	c	d	e	f	g	h	i	j	k	l
8	13	11	20	15	12	20	50	50	50	50	m
5	13	4	10	11	3	9	10	20	15	24	n
3	0	7	10	4	9	11	40	30	35	26	m−n

37 Using WAMR cards depicting intersections of lines to practice multiplication and division - What's My Rule?

Tasks: *Pupils use the clues provided on the cards to determine the rules to complete exercises A-H. At the bottom of the page pupils use the clues provided by the first two columns of the table or the pattern indicators at the extreme right of the table to record the missing entries.*

Purpose: To continue exploring functions and relations via WAMR cards; to continue to perfect computational skills.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

WAMR Set D asks pupils to look at the pictures on the front of the cards in four different ways. They should study the clues and discover that Rule I is: find the number of "bars." Rule II is: find the number of "nails." Rule III is: double the number of bars. Rule IV is: double the number of nails.

Here is a good boardwork activity to introduce this page.

Picture	Complete the number sentences and find the one that fits the picture.	All members of the fact team.
	5 + ___ = 8 2 x 0 = ___ 6 = 3 x ___ ___ − 4 = 6	After pupils have recognized "3x2=6" as the proper sentence, have them write the other related sentences: 2 x 3 = 6 6 ÷ 3 = 2 6 ÷ 2 = 3
	1 x ___ = 5 10 ÷ 2 = ___ 6 + ___ = 7 15 = 3 x ___	
	___ = 1 + 5 8 − ___ = 6 2 + 2 + ___ = 8 ___ x 0 = 0	

Headline for the Day: Think of each column of the *What's My Rule?* game on page 37 as a headline for a story. Tell a story to fit each headline.

What's My Rule?

2 | 4	0 | 0	3 | 6	1 | 2	4 | 8		
6 | 12	5 | 10	10 | 20	8 | 16	7 | 14	15 | 30	50 | 100

Select cards so that the blue sides make true mathematical sentences.

WAMR... Set E

A

2	4
+ 3	6
5	10

B

6	12
+ 4	8
10	20

C

5	10
+ 0	0
5	10

D

6	12
+ 9	18
15	30

E

9	18
+ 6	12
15	30

F

13	26
+ 7	14
20	40

G $8 + 7 = 15$ (16, 14, 30)

H $15 + 5 = 20$ (30, 10, 40)

I $6 + 15 = 21$ (12, 30, 42)

J $15 + 9 = 24$ (30, 18, 48)

K $15 + 25 = 40$ (30, 50, 80)

L $18 + 32 = 50$ (36, 64, 100)

What do you notice about the green sides? THEY MAKE TRUE SENTENCES.

Let's try some subtraction sentences.

M $5 - 2 = 3$ (10, 4, 6)

N $12 - 8 = 4$ (24, 16, 8)

O $20 - 5 = 15$ (40, 10, 30)

P $21 - 7 = 14$ (42, 14, 28)

Q $20 - 15 = 5$ (40, 30, 10)

R $21 - 14 = 7$ (42, 28, 14)

S

23	46
− 17	34
6	12

T

14	28
− 9	18
5	10

U

40	80
− 30	60
10	20

V

44	88
− 18	36
26	52

W

37	74
− 37	74
0	0

X

36	72
− 14	28
22	44

Extra 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233

38 Using WAMR cards to oonstruct equations depicting addition and subtraction situations - What's My Rule?

Tasks: *Pupils use the clues provided at the top of the page to discover the rule for these WAMR cards. For the "Extra" at the bottom of the page pupils study the given numbers to discover the rule of formation of the sequence and record the appropriate entries.*

Purpose: To develop an awareness of the fact that the sum of the doubles of two numbers is the double of the sum of the numbers (a special case of the distributive principle for multiplication over addition); to further perfect computational skills.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

Materials for Teacher: Cards like those described on pages 23-27 of the booklet *Games and Enrichment Activities;* Counting Board; rubber bands.

Materials for Pupils: Geo-boards.

The Lesson:

WAMR Set E is based on one rule—the number on the back is the double of the number on the front. If you arrange three of the WAMR cards so that their fronts form a true addition (or subtraction) sentence then their backs will form a true addition (or subtraction) sentence. This is a simple consequence of the fact that multiplication distributes over addition. For example, since

$$3 + 7 = 10 \qquad \text{(fronts)}$$

it follows that

$$6 + 14 = 20 \qquad \text{(backs)}$$
$$(3 \times 2) + (7 \times 2) = 10 \times 2$$

because, by the distributive principle,

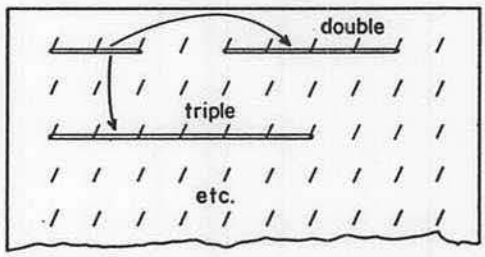
$$(3 \times 2) + (7 \times 2) = (3 + 7) \times 2$$

and $3+7 = 10$.

See pages 23-27 of the booklet *Games and Enrichment Activities* for an extended account of cards like the ones on pupil pages 38 and 40.

Another Use for the Counting Board

We suggest that you follow up these exercises on doubling with exercises involving the doubling and tripling of line segments. With the use of rubber bands, pupils can demonstrate doubling and tripling of line segments as well as halving and otherwise partitioning a line segment into segments of the same length:



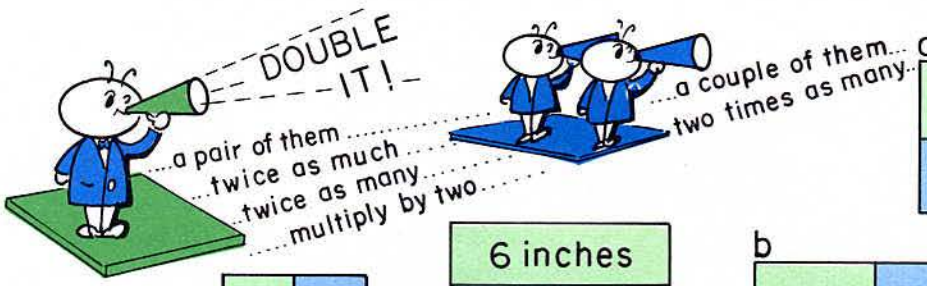
See, also, the section on the Geo-board on pages 80-86 of the booklet *Games and Enrichment Activities.*

Drill

Here are some stories for headline practice:

1. Loretta traveled to Mexico one summer. In a pottery shop she and her sister counted 239 pieces of pottery. Eighty-five pieces of pottery had "sold" tickets on them.
2. Ray had 56 marbles. He gave each of his 8 friends the same number of marbles.
3. Roger hit 60 home runs. Mickey hit 54 home runs.

Headline for the Day: Drew entered the *Play for Keeps Marble Contest.* He won 1 marble the first game and 1 marble the second game; the third game he won as much as he did the first two games. In the fourth game he won as many marbles as he did in the second and third games. In each new game he played he won as many as he did in the previous two games. How many marbles did he win in the 13th game?



8	16

6 inches
1 foot

b

15¢	30¢

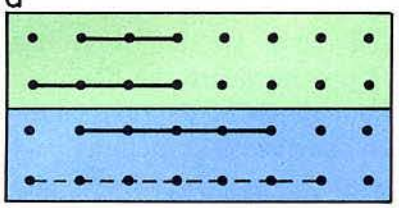
c

12¢	24¢

a

5 days
1 week and 3 days

d



e

5	10

f

9	18

g

25
50

h

8 in.
1 ft. and 4 in.

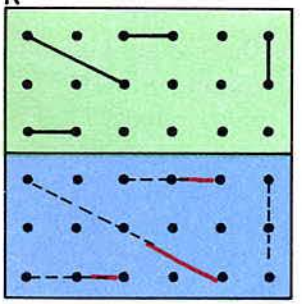
i

17 miles	34 miles

j

a rise from 15° to 21°	a rise from 15° to 27°

k



l

200	400

m

6 months
1 year (12 months)

n

𝍸 | | |
卌 卌 |

o

8 ounces
1 pound (16 ounces)

p

6 days	1 wk. and 5 da.

q

35 min.	1 hr. and 10 min.

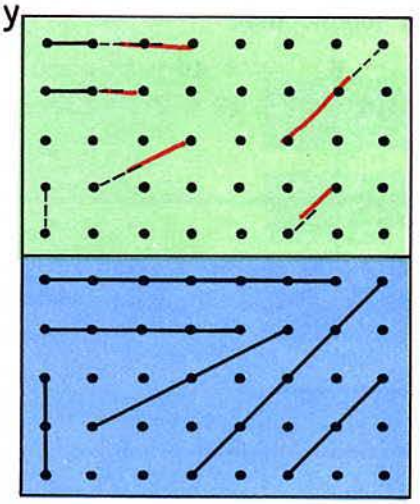
r

eleven	twenty-two

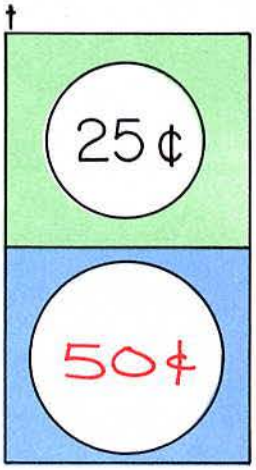
x

250	500

y



t



s

½ doz. eggs (6 eggs)	1 doz. eggs

u

A PRICE of 2 for 5¢
A PRICE of 1 for 5¢

v

11 inches
1 ft., 10 in.

w

75	150

39 Test page - using WAMR cards depicting a variety of situations to examine the relationship between doubling and halving.

Tasks: *Pupils use the clues provided at the top of the page to discover the rules for WAMR Set F.*

Purpose: To test the ability to apply the doubling (and the halving) operation in a variety of situations.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement; geometry.

Materials for Teacher: Ditto masters of *Wrapping Machine* tables as described below.

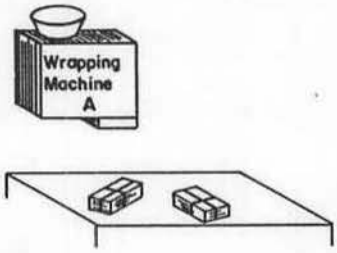
Materials for Pupils: Ditto sheets.

The Lesson:

We suggest that you use pupil page 39 as a test. Score 1 point per exercise for a total of 25 points. The test items cover a wide range of applications of arithmetic.

Wrapping Machines

Make a sketch of a machine on the board, labeling it *Wrapping Machine A*. Any sketch will do as long as it shows a "place to put things in" (input) and a "delivery station" (output).



We don't know what's inside this machine. We only know what we put in the machine and what comes out of it. If we put 6 marbles in through the funnel, the machine gives us 2 packages, neatly wrapped. When we open the packages, we find that each contains 3 marbles. What kind of machine is this? Let's try it again. We put in 18 cookies. What do you suppose the machine delivers?

It turns out that this machine always tries to give us 2 neatly wrapped packages with the same number of objects in each package. How many cookies do you suppose we'll find in each package? 9, of course.

Now, let's see if we know what the machine will do in the following situations:

Input	Output	
	Number of packages	Contents of each package
6 marbles	2	3 marbles
18 cookies	2	9 cookies
30 pennies		
a dozen eggs		
10 trucks and 4 cars		
a half-dozen apples		
2 dimes and 6 pennies		
3 nickels and 5 pennies		
		7 pencils
		25 stories
		11 letters of the alphabet

Headline for the Day: Elmer and Eddie are twin brothers. Whenever their father thinks of buying something for them, he checks the price of one item and then pays the cashier double that amount. Make a table to record some of the possible purchases and costs.

The fourth item (a dozen eggs) may cause a feeling of doubt. Some pupils may think that the machine couldn't handle such an input any better than it could handle 1 marble . . . or 1 nickel. But most pupils will agree that the machine would give us 2 packages with 6 eggs in a package.

With 10 trucks and 4 cars as the input, the machine might wear a puzzled look—if it had one to wear. The machine might put 2 cars and 5 trucks in each package. That would be consistent. Or, it might put 7 vehicles in each package: one containing 4 trucks and 3 cars, the other 6 trucks and 1 car.

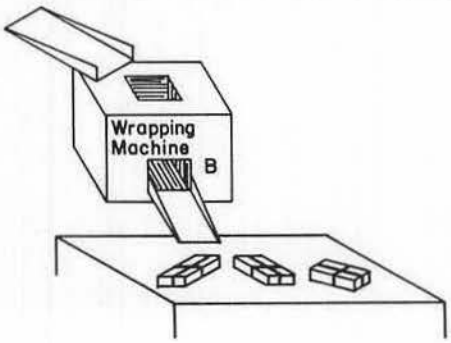
Next item—a half-dozen apples: this is easy.

Next item—2 dimes and 6 pennies: does the machine recognize that a dime is worth 10 pennies? Well, in one package there were 4 pennies. What do you suppose was in the other? 2 dimes and 2 pennies. There were 4 coins in each package. Evidently, the machine simply separates the objects into 2 groups with the same number in each group.

Suppose that we know the contents of one of the packages. What can we say about the input? If one package contains 7 pencils then the input must have been 14 objects, 7 of which were pencils. It is possible that there were 14 pencils, but it is also possible that there were 11 pencils and 3 erasers, etc.

What could we feed into the machine that would stump it completely? Certainly 1 quarter (25¢), 1 egg, 3 eggs, etc. This machine can't cut things in half.

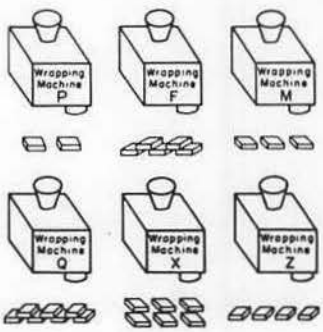
Draw another machine—call it *Wrapping Machine B*—which is similar in operation, only it always delivers 3 packages. If you were to feed it 6 marbles, you would get them wrapped in 3 packages, with 2 marbles in each package. Pupils record outputs and inputs. How many of the examples of input we have considered could both machines handle? (6 marbles, 30 pennies, a dozen eggs . . . but not 10 trucks and 4 cars, nor 2 dimes and 6 pennies.)



Input	Output	
	Number of packages	Number items in each package
6 marbles	3	2 marbles
30 pennies	3	10 pennies
1 dozen eggs	3	4 eggs
a set of tricycle wheels	3	1 wheel
1 penny, 4 nickles 2 dimes, 5 quarters		4 coins
2 dozen eggs		
		7 marbles
		4 nickels, 2 pennies
		3 trucks, 3 cars
		11 stars

This same activity can be extended to machines that deliver 4, 5, 6, etc., packages with the same number of items in each package.

After the idea has been sufficiently developed, draw a series of machines and a series of input-output records on the board. There has been a mix-up, and the tables (or records) have been separated from the machines they refer to. Can you match each machine with its record?



Machine M		
Input	No. pkg.	Items ea. pkg.
12 coins		
		8 eggs
	3	2 whistles
30 books		
		7 pencils

Machine ___		
Input	No. pkg.	Items ea. pkg.
14 marbles		
		10 beads
35 cookies		5 cookies
21 cars		

Machine ___		
Input	No. pkg.	Items ea. pkg.
10 wheels		
5 apples		
15 cents		
40 stamps		
		4 pears
		½ doz. eggs
		10 marbles

Machine ___		
Input	No. pkg.	Items
		7 books
		3 rulers
8 pears		
3 doz. bananas		18 bananas

Machine ___		
Input	No. pkg.	Items
1 doz. eggs		a pair of eggs
12 trucks		
		5 books
		10 erasers
		1 apple, 3 oranges

Machine ___		
Input	No. pkg.	Items
6 pr. shoes		3 shoes
2 doz. oranges		
		½ doz. eggs
		7 marbles

Of course, the first record must belong to *Wrapping Machine M* because the number of packages is shown.

The clue for the second record is given by the information that its wraps 35 cookies in a certain number of packages—there are 5 cookies in each package . . . so, there must be 7 packages—and that chart must belong with *Machine Q*.

Similar clues are given, with varying degrees of directness, for all others except the record that belongs with *Machine F*. This problem can be solved in at least two ways: (a) since all other records have been assigned to machines, the unassigned record and machine must belong together; or (b) assuming the machine can successfully handle all the inputs listed in the record, it must wrap each in 5 packages because the machines that wrap 2, 3, 4, 6, or 7 packages could not handle all the inputs sucessfully.

Now we have enough information to complete each line of each record. And, we could expand each record as far as we like.

To summarize these results, another chart can be drawn:

Number of items in input	Machine P		Machine M		Machine Z		Machine F		Machine X		Machine Q	
	Pkg.	In each	Pkg.	In each	Pkg.	In each	Pkg.	In each	Pkg.	In each	Pkg.	In each
2	2	1										
3			3	1								
4	2	2			4	1						
5							5	1				
6	2	3	3	2					6	1		
7											7	1
8	2	4			4	2						
9			3	3								

If the chart is extended through inputs of 20 or more, interesting patterns will be detectable. If the class enjoys the wrapping machine problems, continue with the idea by asking about the "can't handle" situations: *Machine W* wraps every input in 3 packages. If you fed 7 marbles into it, the machine would be stumped. Could a machine be designed to solve this problem in some way?

Drill

Here are some stories for headline practice:

1. Mr. Olson caught 14 fish. His wife Hilda caught 15 fish.

2. Kitty bought a skirt for $3.89. She paid 50 cents more when she bought the same kind of skirt last year.
3. There are 3 third-grade classes at Anabru Elementary School. There are 34 children in each class.
4. The cook at the lumber camp told Marsha that the lumberjacks ate 153 flapjacks yesterday and 188 this morning.
5. Jacques saved 3 dollars. He bought a hockey stick for $2.39 and a puck for 49 cents.
6. Tommy had 14 apples. Tami gave him 4 peaches.

Here are suggestions for two drill sheets:

	I	II	III	IV	V	VI
A	25 + 25	30 + 30	35 + 35	40 + 40	45 + 45	50 + 50
B	55 + 55	60 + 60	65 + 65	70 + 70	75 + 75	80 + 80
C	85 + 85	90 + 90	95 + 95	100 + 100	105 + 105	110 + 110
D	43 + 47	43 + 57	43 + 67	43 + 77	43 + 87	43 + 97
E	38 + 42	38 + 52	38 + 62	38 + 72	38 + 82	38 + 92
F	50 + 59	59 + 51	52 + 59	59 + 53	54 + 59	59 + 55
G	27 + 80	27 + 81	82 + 27	83 + 27	27 + 84	85 + 27
H	150 + 50	155 + 53	157 + 53	157 + 57	150 + 150	156 + 154

I 50, 47, 44, , , , , , , , 20

	I	II	III	IV	V	VI
A	20+ 8 − 7 = 20+ 1	28 − 7 = 21	20+8 −(10+7) = 10+1	28 − 17 =	38 − 7 =	39 − 17 =
B	30+ 5 − 2 =	35 − 2 =	30+5 −(10+2) =	35 − 12 =	45 − 12 =	45 − 32 =
C	10+15 − 6 =	25 − 6 =	10+15 −(10+6) =	25 − 16 =	45 − 16 =	45 − 36 =
D	20+16 − 8 =	36 − 8 =	20+16 −(10+8) =	36 − 18 =	36 − 28 =	56 − 28 =
E	40+ 9 − 4 =	49 − 4 =	40+9 −(20+4) =	49 − 24 =	79 − 24 =	89 − 54 =
F	50+12 − 8 =	62 − 8 =	50+12 −(30+8) =	62 − 38 =	72 − 38 =	72 − 58 =
G	70+14 − 9 =	84 − 9 =	70+14 −(20+9) =	84 − 29 =	84 − 49 =	86 − 39 =
H	90+11 − 5 =	101 − 5 =	90+11 −(30+5) =	101 − 35 =	101 − 75 =	103 − 79 =

I Rule: Subtract 3

37, 34, 31, , , , 19, , , , ,

Also consider using as drill sheets pages 38 and 39 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown on the next page.

Discover the Rules so you can Play the Games.

(Addition) ( ) ( )

A.

1	1	2
3	4	7
4	3	7
25	5	30

0	0	0
2	2	
4	4	
6	6	
8	8	
10	10	
12	12	
14		

B.

1	1	0
3	1	2
10	2	8
13	2	11

1	1	0
3	2	
5	3	
7	4	
9	5	
11	6	
13	7	

C.

8	2	10
10	3	13
15	5	20
20	5	25

18	10	26
17	10	
16	10	
15	10	
14	10	
13	10	
12	10	

a. N → Q

N	Q
3	6
5	8
11	14
20	23
1	4
6	
10	
17	

N+3=Q

b. A → Z

A	Z
6	0
8	2
12	6
24	18
10	
14	
26	
15	

c. P → O

P	O
5	10
12	17
19	24
25	30
0	
10	
23	
50	

d. M → N

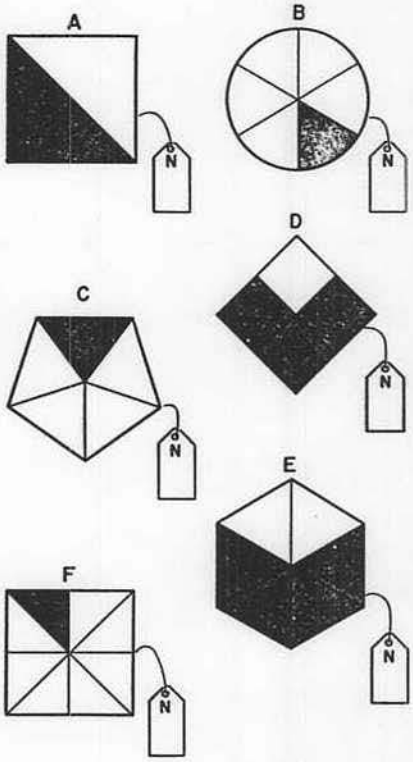
M	N
2	12
10	20
6	16
15	25
11	
9	
20	
90	

e. Y → X

Y	X
11	1
15	5
10	0
19	9
12	
16	
20	
25	

38

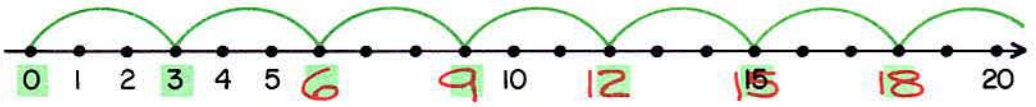
"N" on the tags means the number suggested by the whole shape.



	Shaded	Un-shaded	Tag
A	1	1	2
B	1		
C	1		
D	3	1	
E	4		
F		7	
A			2
B	2		
C	2		
D	6		
E		4	
F	10		
A		5	
D		3	
E	8		

39

What's My Rule?



WAMR...Set G

1	0	10	5	3	2
3	0	30	15	9	6

A

3	+ 3	= 6
9	9	18

B

10	− 6	= 4
30	18	12

C

3	+ 2	= 5
9	6	15

D

4	+ 0	= 4
12	0	12

E

10	+ 5	= 15
30	15	45

F

8	− 5	= 3
24	15	9

G

5	− 2	= 3
15	6	9

H

4	− 0	= 4
12	0	12

I

7	− 4	= 3
21	12	9

J

6	+ 4	= 10
18	12	30

K

5	+ 2	= 7
15	6	21

L

7	− 2	= 5
21	6	15

M

11	+ 9	= 20
33	27	60

N

14	− 7	= 7
42	21	21

Pencils 3¢ each

Erasers 2¢ each

Pads of paper 5¢ each

Notebooks 8¢ each

Number purchased

	0	1	2	3	4	5	6
I Erasers at 2 ¢ ea.	0 ¢	2 ¢	4 ¢	6 ¢	8 ¢	10 ¢	12 ¢
II Pencils at 3 ¢ ea.	0 ¢	3 ¢	6 ¢	9 ¢	12 ¢	15 ¢	18 ¢
III Pads at 5 ¢ ea.	0 ¢	5 ¢	10 ¢	15 ¢	20 ¢	25 ¢	30 ¢
IV Notebooks at 8 ¢ ea.	0 ¢	8 ¢	16 ¢	24 ¢	32 ¢	40 ¢	48 ¢

40 Using WAMR's to oonstruct equations depicting addition and subtraction situations - problems calling for tabulations of possible purchases.

Tasks: *Pupils use the clues provided by the number line jumps and the WAMR cards to construct true mathematical sentences using completed WAMR cards. For the bottom half of the page pupils use the information provided to complete each table.*

Purpose: To develop an awareness of the fact that the triple of the sum of two numbers is the sum of their triples, and to use this fact and its extensions (doubles, quadruples, quintuples, sextuples) in computing products in exercises III and IV.

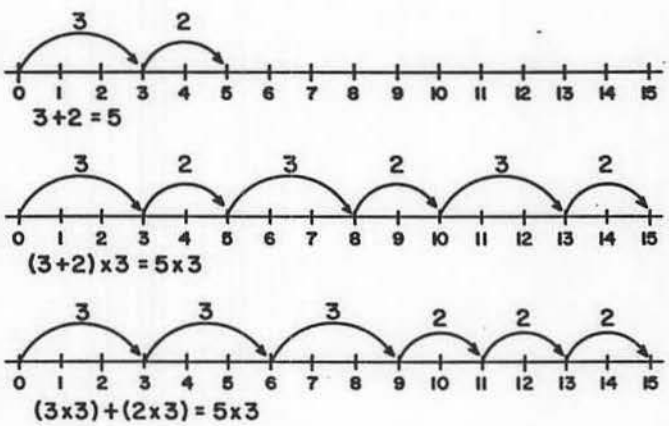
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

Materials for Teacher: None.

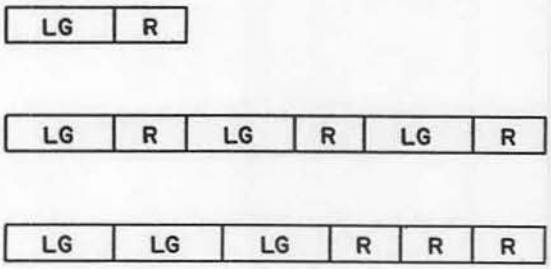
Materials for Pupils: Cuisenaire rods.

The Lesson:

The rule for WAMR Set G is that the number on the back of any card is the triple of that on the front. The fact that a triple of "front numbers" which make a true addition (or subtraction) sentence will be accompanied by a triple of "back numbers" which also make a true addition (or subtraction) sentence can be nicely illustrated by jumps on a number line. Consider exercise C:



The same points can be made with Cuisenaire rods. Here is exercise C again:



Exercises I-IV are designed to give pupils a chance to see the distributive principle at work. For example, by noticing that the cost of a pad is the sum of the cost of an eraser and the cost of a pencil, one can easily find the cost of 2 pads by adding the cost of 2 erasers and the cost of 2 pencils; the cost of 3 pads by adding the cost of 3 erasers and the cost of 3 pencils; etc. Similarly, the cost of 8 notebooks is the sum of the cost of 8 pencils and the cost of 8 pads because the cost of 1 notebook is the sum of the cost of 1 pencil and the cost of 1 pad.

All of this is merely an application of the well-known facts that the $\times$ 5 table can be derived from the $\times$ 2 and $\times$ 3 tables, and that the $\times$ 8 table can be derived from the $\times$ 3 and $\times$ 5 tables:

	$\times 2$	$\times 3$	$\times 5$	$\times 8$
0	0	0	$0 + 0 = 0$	$0 + 0 = 0$
1	2	3	$2 + 3 = 5$	$3 + 5 = 8$
2	4	6	$4 + 6 = 10$	$6 + 10 = 16$
3	6	9	$6 + 9 = 15$	$9 + 15 = 24$
4	8	12	$8 + 12 = 20$	$12 + 20 = 32$
5	10	15	$10 + 15 = 25$	$15 + 25 = 40$
6	12	18	$12 + 18 = 30$	$18 + 30 = 48$
7	14	21	$14 + 21 = 35$	$21 + 35 = 56$
8	16	24	$16 + 24 = 40$	$24 + 40 = 64$
9	18	27	$18 + 27 = 45$	$27 + 45 = 72$

Headline for the Day: Find the sum for each of the columns in tables I-IV on page 40.

WAMR... Set H

A

FRONT	BACK	
	Rule I: 3¢	Rule II: 6¢
	Rule III: 13¢	Rule IV: 1¢

B

FRONT	BACK	
	Rule I: 6¢	Rule II: 12¢
	Rule III: 16¢	Rule IV: 4¢

C

FRONT	BACK	
	I: 12¢	II: 24¢
	III: 22¢	IV: 10¢

D

FRONT	BACK	
	I: 25¢	II: 50¢
	III: 35¢	IV: 23¢

E

FRONT	BACK	
	I: 20¢	II: 40¢
	III: 30¢	IV: 18¢

F

FRONT	BACK	
	I: 30¢	II: 60¢
	III: 40¢	IV: 28¢

G

FRONT	BACK	
	I: 40¢	II: 80¢
	III: 50¢	IV: 38¢

H

FRONT	BACK	
	I: 50¢	II: $1.00
	III: 60¢	IV: 48¢

Another game of What's My Rule?

a	b	c	d	e	f	g	h	i	j	k	l
4	10	2	8	9	6	15	17	5	14	6	r
3	5	2	2	4	4	5	4	25	16	15	s
7	15	4	10	13	10	20	21	30	30	21	r+s

41 WAMR cards of coin problems dealing with the operations x2, +10, and -2 - What's My Rule?

Tasks: *Pupils use the clues provided by the WAMR cards A-D to complete each of exercises B-H. For the game at the bottom of the page pupils use the clues provided in the first 3 columns or the pattern indicators at the extreme right to complete the table.*

Purpose: To gain further experience in computing in application situations; to further perfect computational skills.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The rules for WAMR Set H are:

Rule I .....Find the value of the coins shown.

Rule II .....Double this amount.

Rule III.....Add 10¢ to the value of the coins shown.

Rule IV .....Subtract 2¢ from the value of the coins shown.

Tell pupils that they are to study the page on their own, that you will answer none of their questions

more

Headline for the Day: Suppose that each column in the table at the bottom of page 41 is a headline for a story. Tell a story to fit each headline.

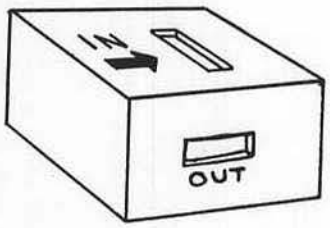
today. Circulate to see how pupils are working. Collect their books and study their papers to see how effectively pupils have worked. Use such an evaluation procedure from time to time to help you determine what special needs individual pupils (or small groups of pupils) have.

A Peculiar Machine

Have teams of pupils compete to discover how the machine works in the following story.

Once upon a time some very peculiar machines were discovered in a warehouse. On the top of each machine there was a slot. Pointing to it was an arrow; and above the arrow appeared the word IN. No one knew what possible use there was for the machines. Then one day, by accident, someone dropped into one of the machines a piece of paper with a 9 written on it. The machine made a purring noise and the piece of paper with an R written on it floated out of a slot at the side of the machine. Someone then put in a note that had a 4 written on it. Out from the machine came the piece of paper marked with an X. Someone else tried a 4 + 4, and out came the piece of paper marked with a Y. No matter what number was fed into this machine, out came a piece of paper marked with an R, or an X, or a Y—nothing else.

One of the men at the warehouse kept track of these strange happenings. His record is displayed below. Can you guess what pattern the machine followed?



In	Out
9	R
4	X
4 + 4	Y
14 − 7	X
4 + 10	R
8	Y
6	X
10 − 2	Y
2 × 6	
12 ÷ 3	
1 + 7	

If pupils have difficulty getting started, suggest that they set up a table:

X	Y	R

and have them begin to put the numbers in the table as they occur in the record:

X	Y	R
4	4 + 4	9
	(8)	

Those who have the "Aha!" experience will tell you that all pieces of paper marked with numbers less than 8 are marked with an X, those marked with 8 are marked with a Y, and those marked with numbers greater than 8 are marked with an R.

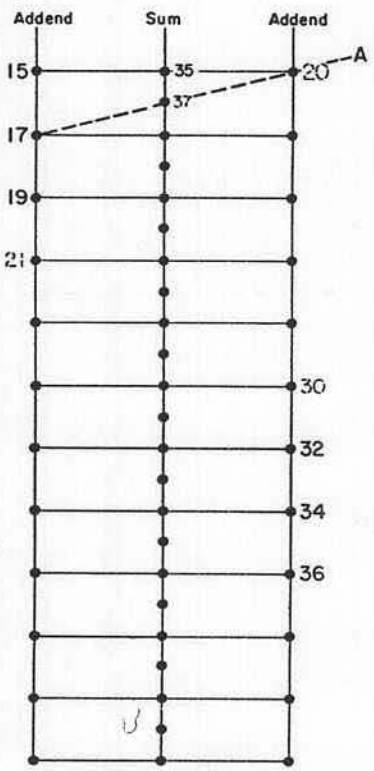
Drill

Here are some stories for headline practice:

1. Waldo had 18 rows of garden plants to care for during his summer holidays. Some of the plants were sweet corn, the rest were radishes. Corn made up the first 10 rows.
2. Larry saved 720 pennies. He put these pennies into 10 cups with the same number of pennies in each.
3. Silvia found a box of thirty-six jacks. When she played jacks she counted out 6 groups of jacks with the same number of jacks in each group.
4. Paul lives 8 miles from his cousin's house and 14 miles from his niece's house. Paul, his cousin, and his niece all live along the same highway in southern Illinois.
5. Four boys on Tom's baseball team each hit 6 home runs while 5 boys on Sal's team each hit 6 home runs.
6. There are 270 children at Uni Elementary School. There are 9 classrooms at Uni. Each of the classrooms holds the same number of children.

Consider using as a drill sheet page 40 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:

Add and Subtract on the NUMBER LADDER



A 17 + 20 = 37

B 17 + 22 = ____

C 17 + 24 = ____

D 17 + ___ = 51

E 27 + 26 = ____

F ___ + 15 = 45

G 15 + ___ = 51

H 35 + 32 = ____

I 41 − 20 = ____

J 47 − 21 = ____

K 55 − ___ = 29

L ___ − 42 = 23

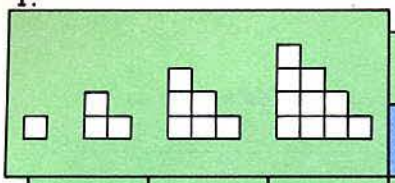
M 79 − 37 = ____

N 71 − 36 = ____

Here is another suggestion for a drill sheet:

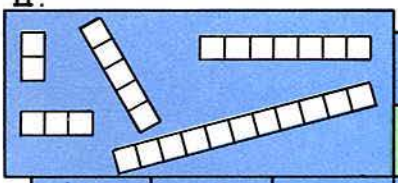
1. 326 + 541	**2.** 526 + 270	**3.** 132 + 457
4. 302 + 720	**5.** 639 + 50	**6.** 273 + 516
7. 700 + 600	**8.** 531 + 546	**9.** 643 + 624
10. 643 + 753	**11.** 539 + 820	**12.** 705 + 872
13. 43 15 + 51	**14.** 61 25 + 32	**15.** 40 34 + 2
16. 72 13 + 41	**17.** 38 25 + 14	**18.** 83 52 + 41

I.



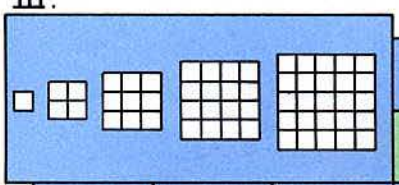
			+15	−6
			35	14
+28	−10	+36	+45	−15
48	10	56	65	5

II.



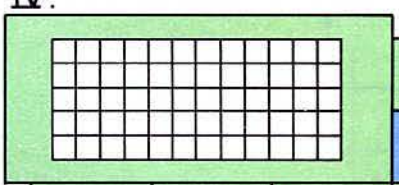
			+13	−11
			41	17
+17	−7	+19	−13	+23
45	21	47	15	51

III.



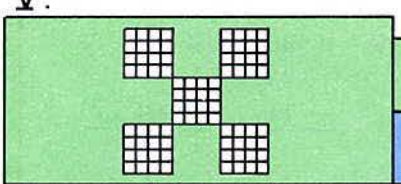
			+36	−25
			91	30
−16	+49	+64	−49	−9
39	104	119	6	46

IV.



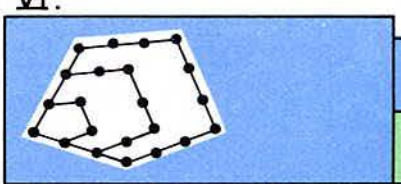
			+18	+48
			78	108
−18	+36	−12	+72	+84
42	96	48	132	144

V.



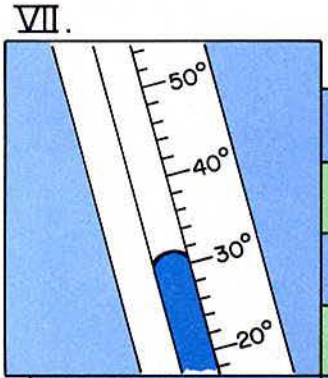
			−24	+32
			56	112
+96	−48	+120	−32	+160
176	32	200	48	240

VI.



			+13	+29
			35	51
−12	+48	−17	+70	+95
10	70	5	92	117

VII.



8° above freezing	
40°	
20° below freezing	
12°	
32° below freezing	40° above freezing
0°	72°
67° above freezing	180° above freezing
99°	212°

VIII.

Time shown	
9:30 AM	
40 minutes later	
10:10 AM	
3 hrs., 40 min. later	1 hr., 40 min. earlier
1:10 PM	7:50 AM
2h.,25m. later	9h.,45m. later
11:55 AM	7:15 PM

42 A variety of situations which provide practice in addition and subtraction.

Tasks: *Pupils determine the number represented by each picture and determine the missing numbers and operations.*

Purpose: To further perfect computational skills in a semi-concrete setting.

Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Thermometer; clockface.

Materials for Pupils: None.

The Lesson:

This page is a prototype of many pages used in LEVEL C. Each may be treated as a test. There are over 50 responses on a typical page. You will find it an easy page to imitate when you need meaningful independent work for pupils.

A special merit of the design of this page is that it involves thinking about a picture in a number of ways, which cultivates mental arithmetic. Also, it presents opportunities to search for relationships among operations. For example, in exercise IV, suppose that a pupil has written a 78 under the + 18 and then encounters the 108 with a blank space above it. He might say to himself that he needs to add 30 to 78 to get 108; so, he ought to add 30 more than 18 to the number of tiny squares. Thus, he writes a + 48 in the blank.

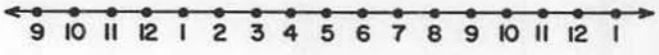
Measuring Instruments

Ask pupils to collect instruments used in measuring, or pictures of measuring devices. These may include: thermometer, light meter, scales (weight), sundial, ruler, micrometer, yardstick (in inches), measuring cup, meterstick (in centimeters), tape measure, egg timer, clock. Pupils should observe that the scale on a measuring instrument is very much like a number line.

The scale on a clockface is a kind of number line but jumps on this scale do not always illustrate ordinary addition and subtraction. If it is 10 o'clock now, what time was it 3 hours ago? What time will it be 3 hours from now? Put three headings on the chalkboard like those below. Record the time in any column. Have pupils complete the row. Let pupils use a clockface to help them in their computations.

3 hours ago	now	3 hours from now
7	10	1
8	11	2
9	12	3
10	1	4
11	2	5
12	3	6

Ask your most able pupils if they could draw a kind of number line which would be useful in answering questions about "jumps" in time. What we have in mind is a scale such as the following:



With this scale it is easy to see that if it is 11 o'clock now, it will be 3 o'clock 4 hours from now and that it was 10 o'clock 13 hours ago.

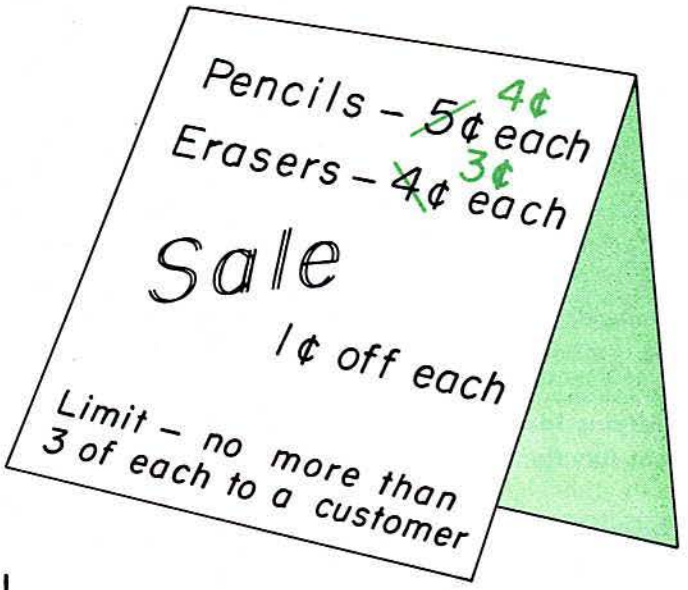
Drill

Here are some stories for headline practice:

1. A trucker had to bring 38 loads of topsoil with his truck. The first day he brought 19 loads of topsoil.
2. Rachelle said 34 of the 69 children in her class are girls.
3. Abigail picked up 5 jacks in each of 3 games.

Headline for the Day: Tell a story to fit this fact team: $200 + \square = 212$, $\square + 200 = 212$, $212 - \square = 200$, $212 - 200 = \square$

Mary's mother gave her a dime to buy a pencil and an eraser. They usually cost 5¢ and 4¢. When she got to the store, she found that there was a sale going on. Here is the sale sign on the right.



a. How much would Mary now spend for a pencil and an eraser? 7¢

b. If she bought something else, could she spend all of her dime? YES. (2 ERASERS AND 1 PENCIL.)

c. Bill came in the store and asked for 5 pencils. What did the clerk say? YOU CAN ONLY BUY 3.

d. Alice gave the clerk a dime and a nickel. She received 1¢ change. What did she buy? 2 PENCILS AND 2 ERASERS.

e. What is the amount of the smallest purchase in this sale of pencils and erasers? 0¢

f. What is the amount of the largest sale? 21¢

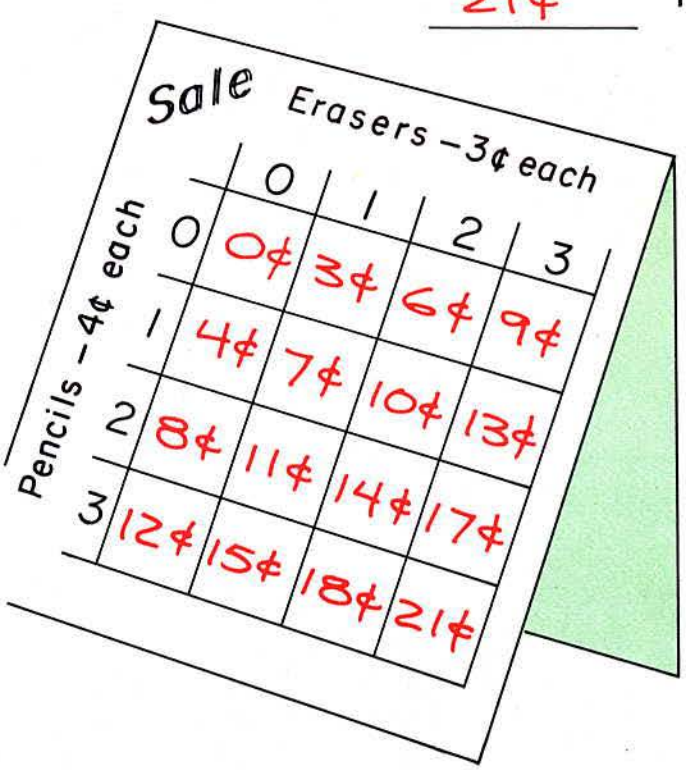
g. Frances had a dime and two nickels. She spent as much as she could. How much did she spend? 18¢

h. What did she buy? 3 PENCILS AND 2 ERASERS

i. Jimmy spent 13¢. What did he buy? 3 ERASERS AND 1 PENCIL

j. Could anyone spend exactly 19¢? No.

k. What purchase amounts to 17¢? 3 ERASERS AND 2 PENCILS



On the back of the "Sale" sign, the clerk had worked out a "time saver" chart.

Complete the chart showing all possible purchases using the rule of the sale, as set out in the sign above.

Can you know exactly what was purchased if you know the amount of the purchase? YES

What kind of chart would the clerk need before the sale — when the prices were 5¢ and 4¢? SINCE THERE WOULD BE NO LIMIT, HE COULDN'T MAKE A CHART TO SHOW ALL POSSIBLE PURCHASES.

43 Application problems dealing with purchases, making change, and the oonstruction of time-saver charts for possible purchases.

Tasks: *Pupils follow the conditions of the sale to determine the appropriate responses to questions* a-k. *For the bottom of the page, pupils construct the "time saver" chart required.*

Purpose: To explore an application requiring the use of a variety of computational skills.

Unifying Ideas: Addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupil page 43 grew out of a "story problem" found in a not-too-ancient textbook: *Mary bought a pencil for 4¢ and an eraser for 3¢. How much did she spend?* Certainly not much of a challenge here. If a pupil doesn't know that $3 + 4 = 7$ then don't ask him how much Mary spent. If he knows that $3 + 4 = 7$ then don't ask him such a trivial question . . . he would rightfully be bored with it all. If we must ask questions like this, we should ask them in a way that will deepen his understanding of how mathematical sentences are used to express conditions in everyday life. If we wish pupils to answer the question: How much is 3¢ and 4¢ altogether? and similar questions, we can at least present them in a challenging way. For instance, consider the following question: Under the conditions given in the sign (on pupil page 43), if you spent a certain amount of money would it be possible to get more than one combination of pencils and erasers?

Now, about every third-grader could work out one combination after another and would probably reach the conclusion that the answer is No. An adult might not take quite as long.

Another disarmingly simple question would be: Under the conditions given in the sign, how many different combinations would be possible in pur-

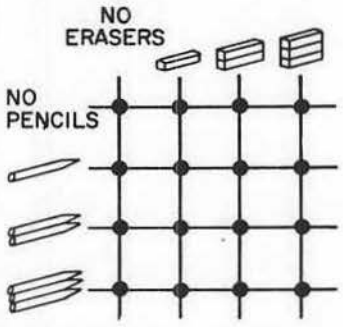
more

Headline for the Day: Suppose that, instead of the sale (on page 43), the price of pencils was increased from 5¢ to 6¢ and the price of erasers was increased from 4¢ to 5¢ but the limit of no more than 3 of each to a customer remained the same. Make a time-saver chart for the clerk.

chasing pencils or erasers?

Again, pupils might begin by listing the possibilities at random—searching for new combinations, rejecting some trials as repetitions. Eventually they would reach the correct conclusion that there are 16 possible combinations if 0-erasers-and-0-pencils was included as a possibility. This conclusion would almost certainly come from a group of 3 or 4 children working together.

The same conclusion could be reached if pupils used a sketch something like this:



Each marked intersection suggests a purchase—none violating the condition of no more than 3 of each to a customer. There are 4 × 4, or 16, combinations possible. On pupil page 43, the time-saver chart in the lower left-hand corner follows this same plan. In place of an intersection, there is a box in which the total price of that particular purchase is noted. Once that chart is completed, the answer to the first question above is completely obvious: No price is repeated in the chart; every possible purchase has a different price.

Example *i* on pupil page 43 states: Jimmie spent 13¢. What did he buy? To answer this, pupils would need only to glance at the time-saver chart. Likewise, in considering example *j*—Could anyone spend 19¢?—the answer is: Of course not, because 19¢ is not written in the time-saver chart. In fact, as soon as the time-saver chart is completed, the 11 questions can be answered by just glancing at the chart. However, before pupils complete the chart, have them answer the questions.

Many simple problem situations can be created to provide an activity like the one on pupil page 43. Note the importance of the sales limit in making each possible amount correspond with a unique purchase. Without the given limitation, 12¢ would purchase 0-pencils-and-4-erasers or 3-pencils-and-0-erasers. Let pupils make a time-saver chart for the old prices but remove the limitation on sales. Then ask if you can know exactly what was purchased if you know the amount of the purchase. Try 20¢, 24¢, 25¢, 29¢, etc.

Drill

Here are some stories for headline practice:

1. The Cardinals had a 30-man football squad. In one of their home games only 14 players got into the game.
2. Cynthia is reading a book with 375 pages. She has read 235 pages.
3. Yancy said there are 36 girls and 18 boys in his class.

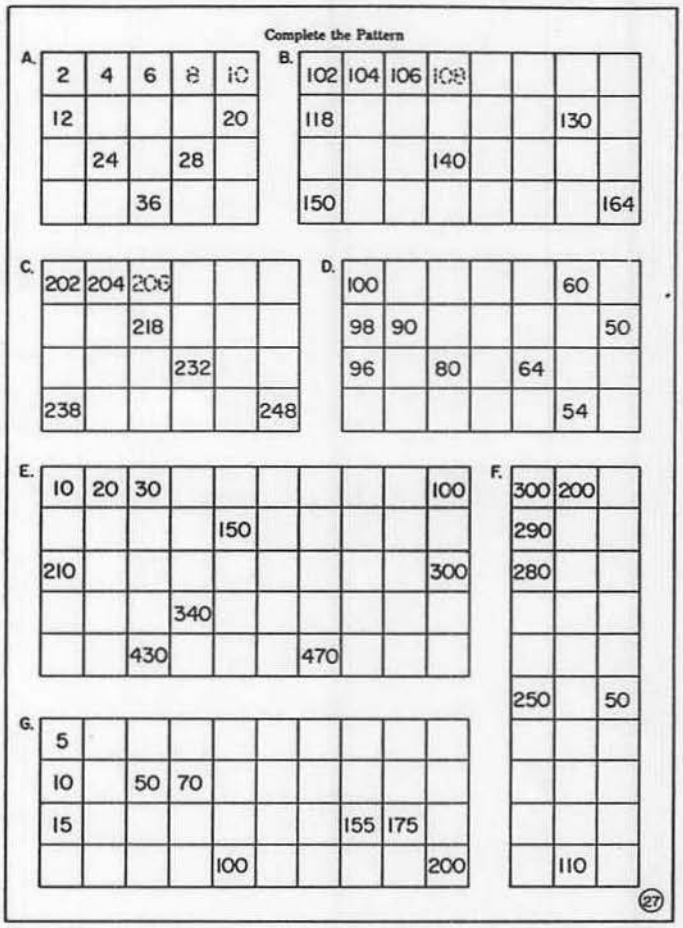
Here are suggestions for two drill sheets:

	I	II	III	IV	V	VI
	6	50	56	3	40	43
A	+ 4	+ 20	+ 24	+ 7	+ 40	+ 47
	30	1	31	50	8	58
B	+ 10	+ 9	+ 19	+ 50	+ 2	+ 52
	40	5	45	8	30	38
C	+ 60	+ 5	+ 65	+ 7	+ 70	+ 77
	7	30	37	50	3	53
D	+ 9	+ 40	+ 49	+ 60	+ 7	+ 67
	3	6	5	8	3	9
E	+ 8	+ 7	+ 9	+ 4	+ 6	+ 9
	20	50	70	30	50	40
F	+ 30	+ 50	+ 40	+ 70	+ 70	+ 80
	23	57	75	74	73	49
G	+ 38	+ 56	+ 49	+ 38	+ 56	+ 89
	35	42	17	58	46	46
H	+ 35	+ 38	+ 92	+ 58	+ 54	+ 64
I	59, , ,	47,	43, ,	35, , ,	,	

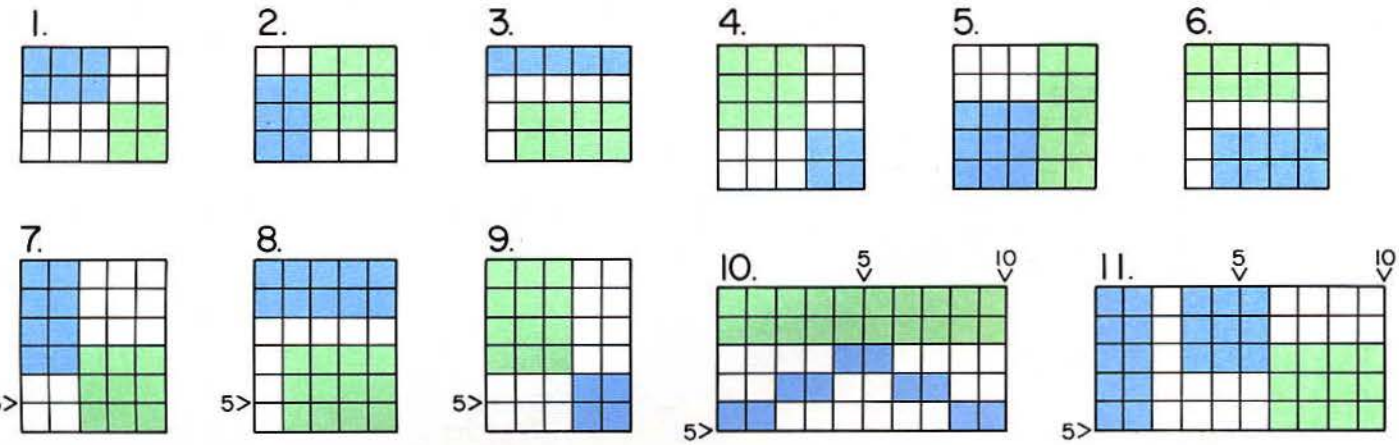
1.	$70 - 30 = \square$ $75 - 30 = \triangle$ $75 - 31 = \bigcirc$	**2.**	$60 - 10 = \square$ $68 - 10 = \triangle$ $68 - 12 = \bigcirc$
3.	$110 - 70 = \square$ $113 - 70 = \triangle$ $113 - 71 = \bigcirc$	**4.**	$80 - 60 = \square$ $88 - 60 = \triangle$ $88 - 63 = \bigcirc$
5.	$90 - 10 = \square$ $96 - 10 = \triangle$ $96 - 16 = \bigcirc$	**6.**	$150 - 80 = \square$ $159 - 80 = \triangle$ $159 - 84 = \bigcirc$

7.	67 − 21	**8.**	58 − 23	**9.**	77 − 43
10.	76 − 54	**11.**	87 − 63	**12.**	547 − 235
13.	137 − 34	**14.**	365 − 123	**15.**	734 − 534

Consider using as a drill sheet page 27 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

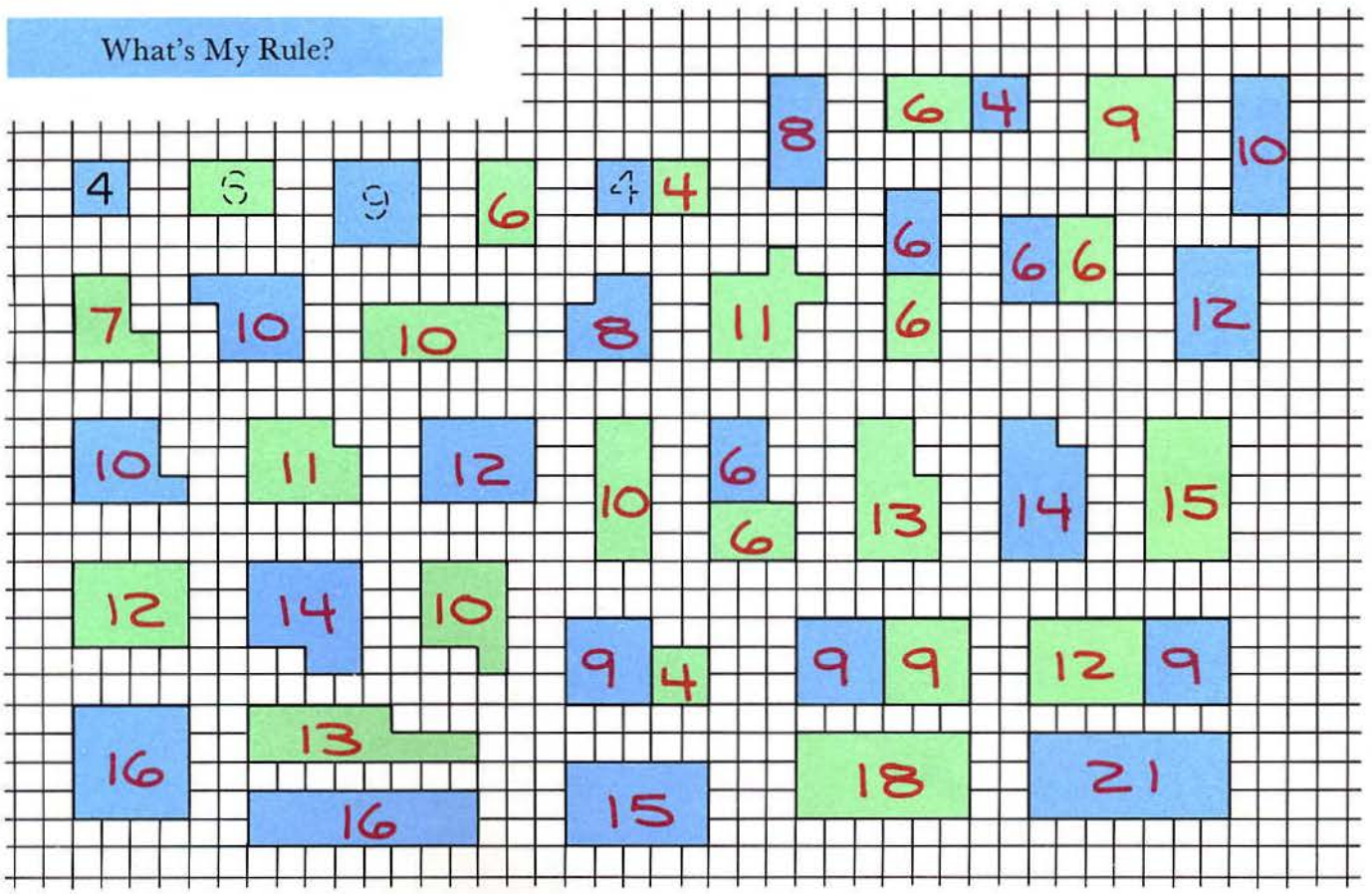


How many tiles? . . . How many blue, how many green, how many white?



	1	2	3	4	5	6	7	8	9	10	11
Number blue	6	6	5	4	9	8	8	10	4	10	19
Number green	4	9	8	9	10	8	9	12	12	20	12
All	20	20	20	25	25	25	30	30	30	50	50
Blue *or* green	10	15	13	13	19	16	17	22	16	30	31
Number white	10	5	7	12	6	9	13	8	14	20	19

What's My Rule?



44 Counting and tabulating problems which encourage pupils to use counts for rectangular arrays; addition and subtraction - What's My Rule?

Tasks: *For each rectangular array of tiles, pupils count or use shortcuts to counting to record in the table the numbers of blue tiles, green tiles, all tiles, blue or green tiles, and white tiles. For the game at the bottom of the page, pupils must determine the number of tiles hidden behind each tinted region.*

Purpose: To encourage the use of shortcuts in counting and to perfect computational skills.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

It is very important that pupils determine counts in these exercises by spotting familiar rectangular arrays. Develop speed by doing similar exercises on a graph board. Also, ask pupils to draw diagrams on graph paper for given numbers.

Notice that in the *What's My Rule?* game at the bottom of the page the exercises are clustered to encourage the use of one result in finding another. The sequence 10, 6 + 6, 13, 14, 15 is an example of such a clustering.

Open Number Sentences

Here is nice enrichment work. Pupils can complete the tables and then graph each number sentence.

A. $\square + \bigcirc = 4$

$\square$	1	0					
$\bigcirc$	3						

B. $\triangle - \hexagon = 3$

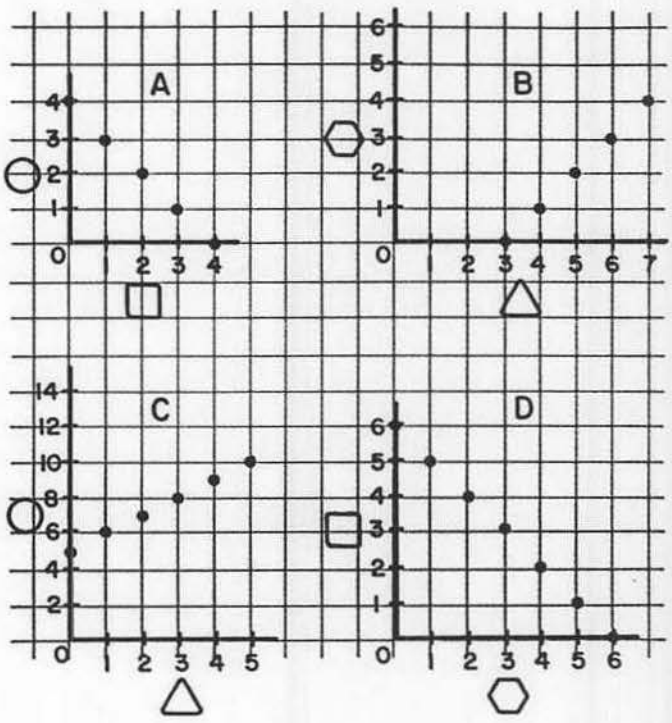
$\triangle$	7						
$\hexagon$	4						

C. $\triangle + 5 = \bigcirc$

$\triangle$	3						
$\bigcirc$	8						

D. $6 - \hexagon = \square$

$\hexagon$	2						
$\square$	4						



(See "Enrichment Activity 3" in Notes on Pupil Page 5.)

Drill

Here are some stories for headline practice:

1. Zachariah had 116 marbles in a box. He put them in some sacks. Each sack holds 5 marbles.

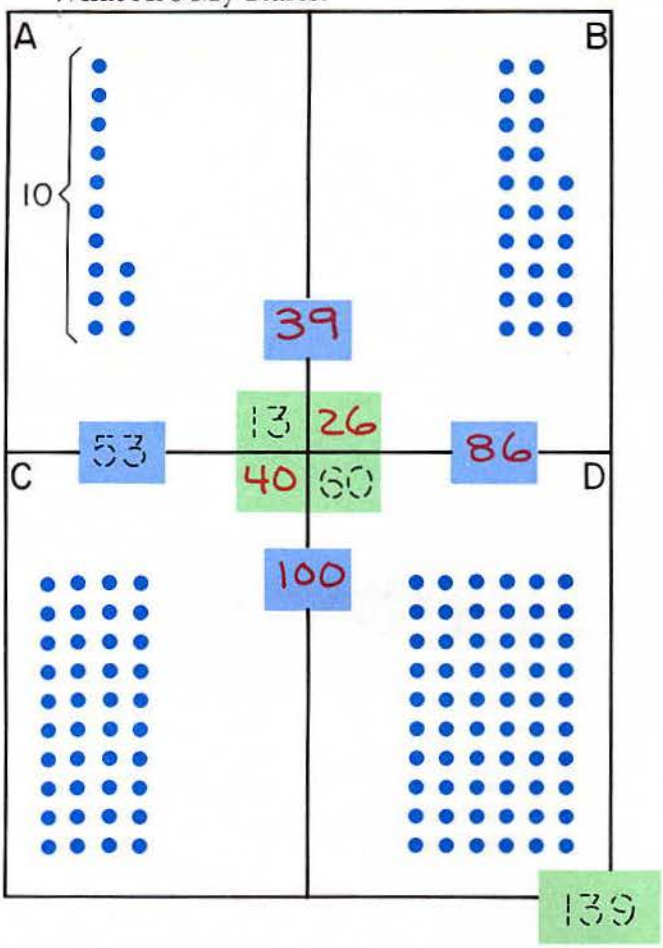
$$(\square \times 5) + \triangle = 116$$

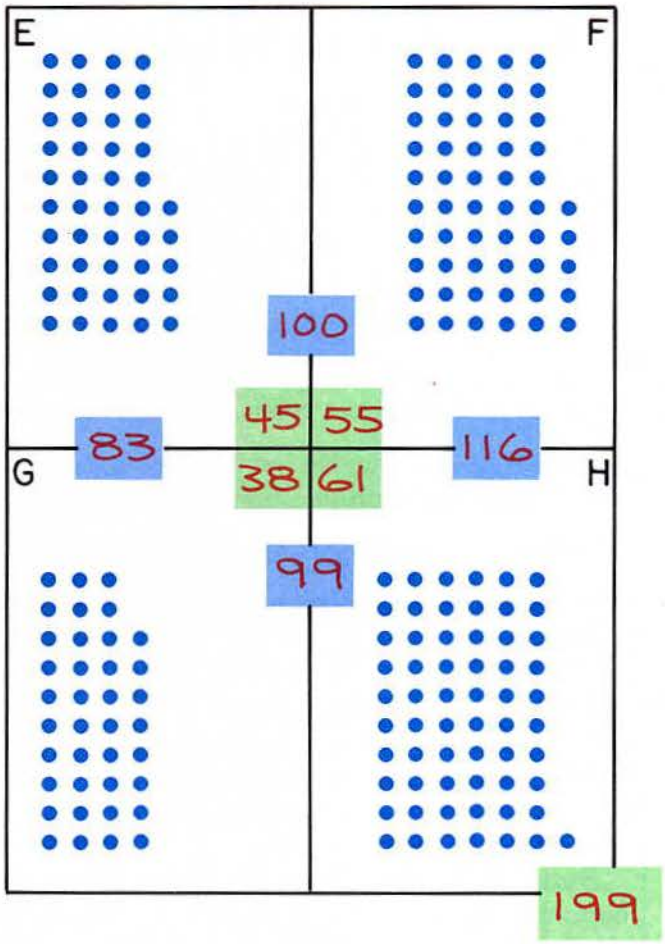
2. Our school has sold 337 tickets. We must sell 676 altogether.
3. Tallulah is 18 years old. She has a sister named Maxine. The sum of their ages is 32.
4. Wanda and Gazelle were counting pickets along the picket fences on their block. Wanda counted 497 on her side of the street. Gazelle counted 353 on her side of the street.

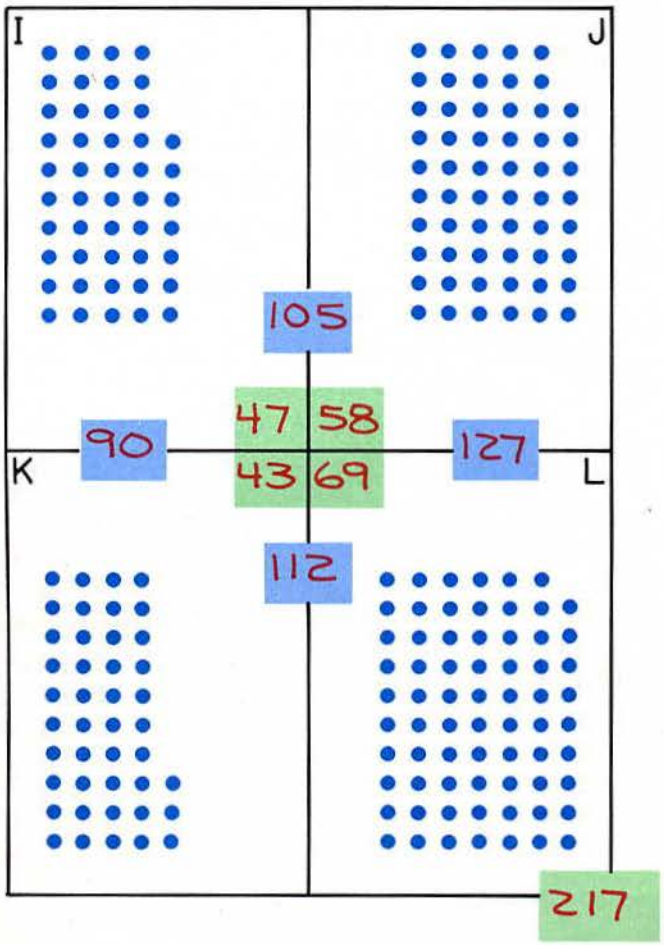
Headline for the Day: Extend the table on page 44 to include two more rows.

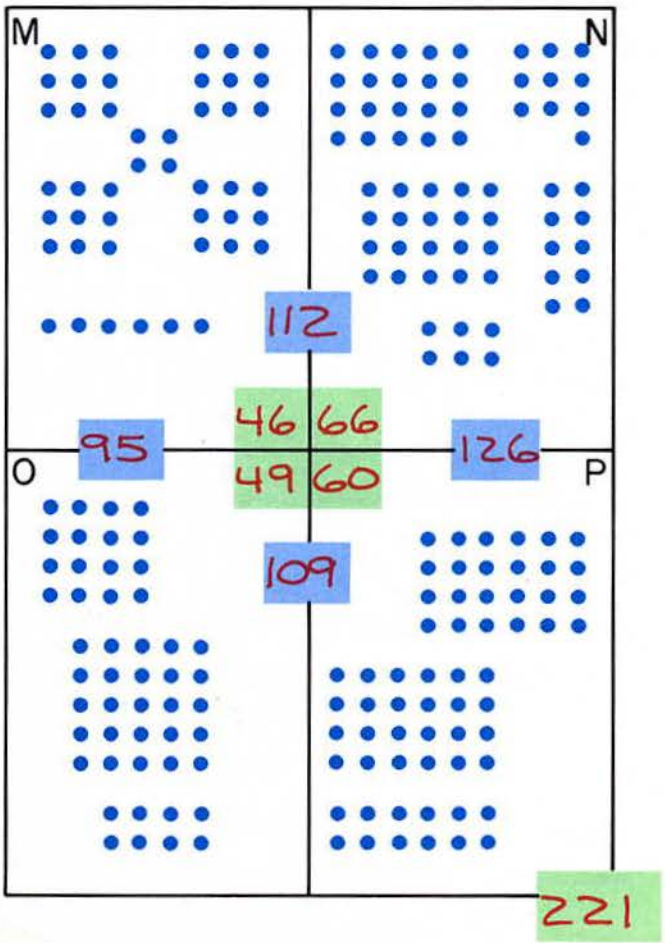
Blue or white										
Green or white										

Complete this new table.


A
B
10
39
13
26
53
86
C
40
60
D
100
139


E
F
100
45
55
83
116
G
38
61
H
99
199


I
J
105
47
58
90
127
K
43
69
L
112
217


M
N
112
46
66
95
126
O
49
60
P
109
221

45 Counting problems which enoourage pupils to use decimal groupings to determine total oounts; addition.

Tasks: *Pupils record in the green box on each card the number of dots pictured on that card and they record in the green box at the lower right of each group of four cards the total number of dots pictured on the cards in the group. In the blue boxes pupils record the sum of the numbers of dots pictured on each pair of adjacent cards in each group of four cards.*

Purpose: To encourage the use of shortcuts to counting, in particular that of counting by 10's, in order to carry out the tasks described above.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Perhaps the only guidance pupils need on this page is to be told to look for shortcuts to counting.

Developing Independence

Whenever you feel that pupils are ready to attempt a page without any preparatory explanation, try it. You can always retreat and insert whatever class discussion and work you feel that pupils need.

One very effectivo method is to partition the class into small work groups. Let each group talk among themselves about the page. Urge them to think: What work do you think needs to be done on this page? Talk about it together. If you reach an agreement, go ahead. I am ready to help if you are unable to get started on your own. Help each other as much as you can.

This kind of introduction to a new page provides a note of challenge, and suggests the idea of a puzzle. These are two of the essential ingredients of creative mathematics.

In mathematics, as in other subject matter areas, there has been a long tradition which condones spoon-feeding. Traditionally, teachers are asked to point out clearly what is to be done next each step of the way. The result is that pupils become dependent on directions, instructions, and orders of procedure. This textbook provides many opportunities to encourage pupils to think. Have them consider a problem that may not be clear to them at first. Where does one begin? Is there a part of the problem that looks familiar? If so, can this be a clue to the less familiar areas? What's going on?

Almost all "secrets" within the pages of this textbook can be uncovered by pupils working in small groups and eager to prove their ability. Often, a pupil working alone gets lost on a side road which does not help him toward the goal. This is seldom the case with small groups.

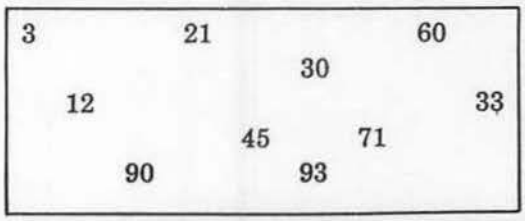
It is essential, of course, that pupils stand a good chance of succeeding. For this reason, be quite sure that your class is prepared for any pages you may ask them to try on their own. Within a few minutes you can easily tell whether success is in the wind. You can then offer what help you think will be most beneficial.

Drill

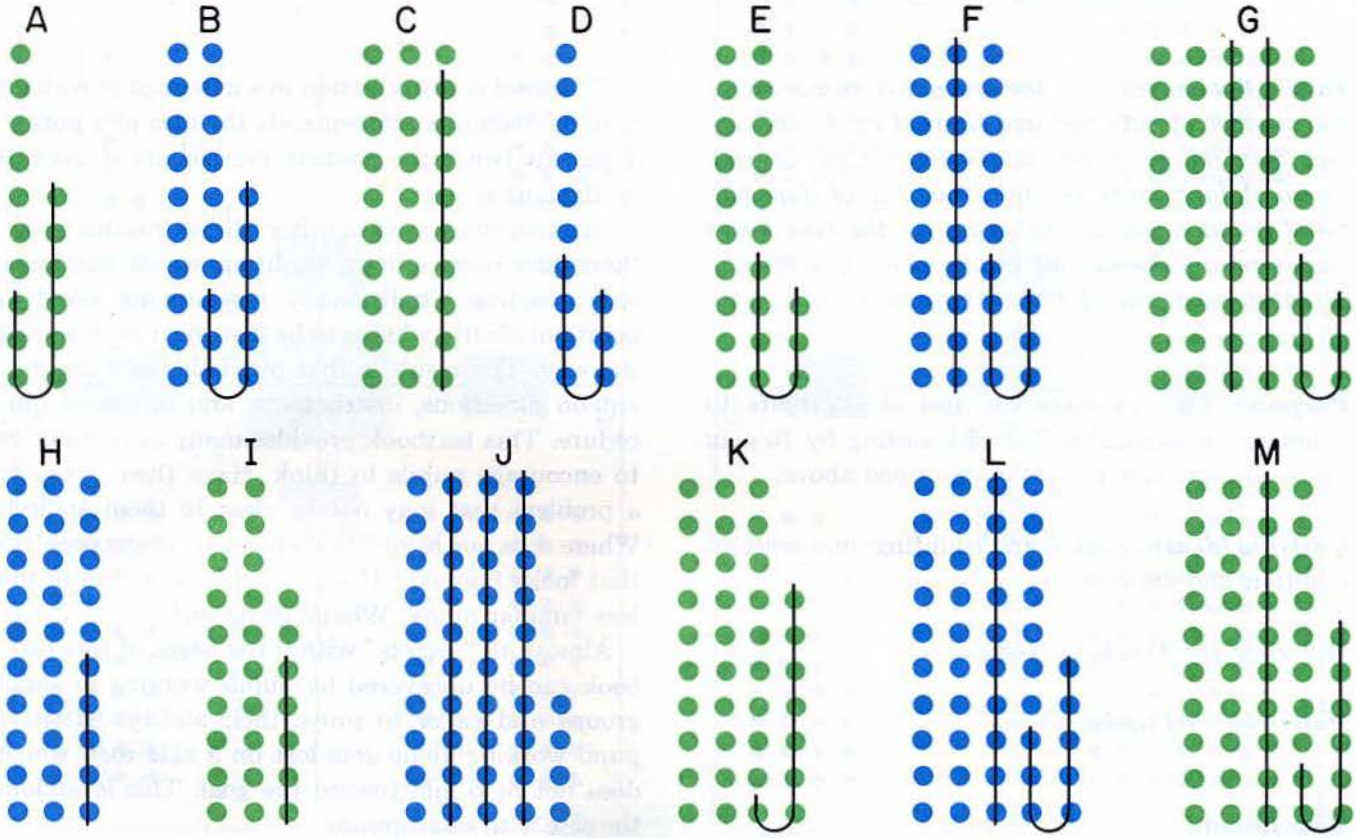
Here are some stories for headline practice:

1. Elsie's brother has a newspaper route. One week he collected 492 cents. Each paper costs 6 cents.
2. Zazu needs 79¢. She has 27¢.
3. Three girls and 4 boys went fishing. They each caught 3 fish.

Headline for the Day: Circle the number which does not belong with the others.



How many dots? How many crossed out? How many not crossed out?



	A	B	C	D	E	F	G	H	I	J	K	L	M
All dots	16	26	30	13	23	33	53	30	27	44	37	45	46
Crossed out	9	9	9	7	7	17	27	5	5	30	8	18	18
Not crossed out	7	17	21	6	16	16	26	25	22	14	29	27	28

Chain Reactions

a → 10 | −6 | 4 | −3 | 1

b → 28 | −8 | 20 | −10 | 10

c ↓	d ↓	e ↓	f ↓	g ↓	h ↓	i ↓
17	17	25	25	32	32	32
−6	−2	−6	−8	−3	−10	−7
11	15	19	17	29	22	25
−3	−7	−10	−8	−14	−7	−10
8	8	9	9	15	15	15

46-47 Counting and tabulating problems which encourage pupils to use decimal groupings to determine total counts; subtraction - chain subtractions.

Tasks: *In each of the exercises A-M pupils count the total number of dots in each representation, the number of dots crossed out, and the number of dots not crossed out. They record these data in the table. For the lower part of the page pupils record the missing entries in each of the chain reaction patterns* a-i.

Purpose: To perfect computational skills while simultaneously exploring procedures for developing a subtraction algorithm.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Ditto masters.

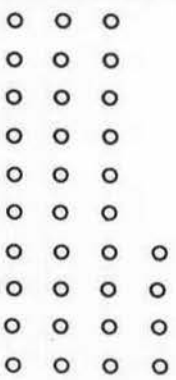
Materials for Pupils: Ditto sheets.

The Lesson:

Subtraction is represented on this page and the next by "cross out" problems. Pupils should be helped to discover that shortcuts are possible – the dot columns are arranged in groups of ten.

The chain reaction problems at the bottom might be illustrated for pupils by using the cross out technique. Put examples like the following on the board:

34	-10		-4	



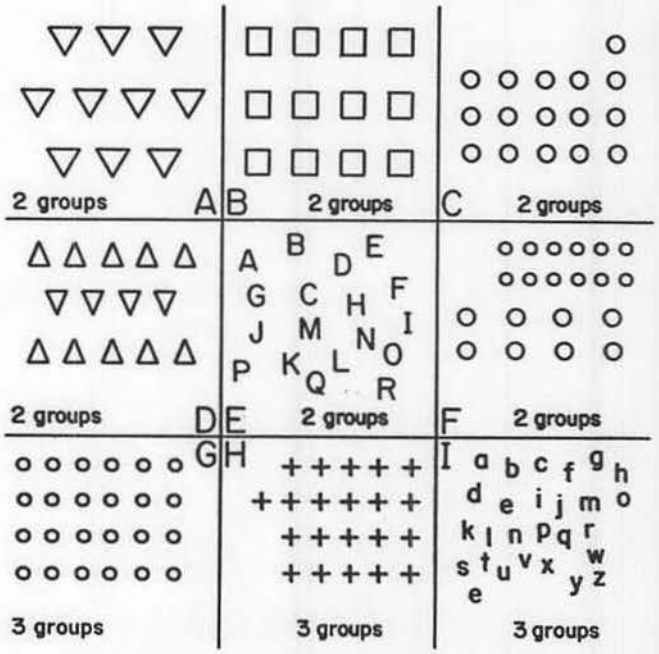
As each step is taken to complete the chain reaction problem, a pupil should cross out an appropriate group of dots.

Note that exercises *c* and *d* show two ways of subtracting 9 from 17; exercises *e* and *f* show two ways of subtracting 16 from 25; exercises *g*, *h*, and *i* show three ways of subtracting 17 from 32.

Possible Supplementary Activities

Here are some suggestions for Ditto sheets.

I. Draw lines to show groups with the same number of objects in each group.



	A	B	C	D	E	F	G	H	I
Number of groups	2	2							
Number in each group									
Total number of objects									

Headline for the Day: Fill in the blanks to continue the pattern.

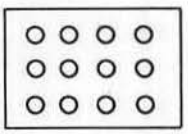
44, , , , 32, 29, , , , 17, 14, , , , 2

II. Complete the table.

Puzzles 6¢ each

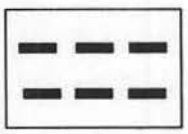
Number of Puzzles	1								
Total Cost	6¢								

III. Write the sentences suggested by the pictures.



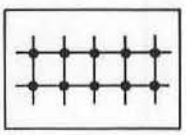
3 x 4 = 12 12 ÷ 4 = 3

x = ÷ =



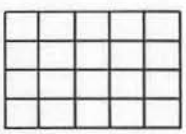
x = ÷ =

x = ÷ =



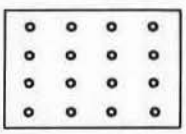
= x = ÷

= x = ÷



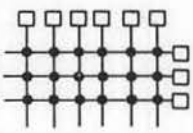
= x ÷ =

x = = ÷



x = ÷ =

÷ = x =



÷ = = ÷

= x x =

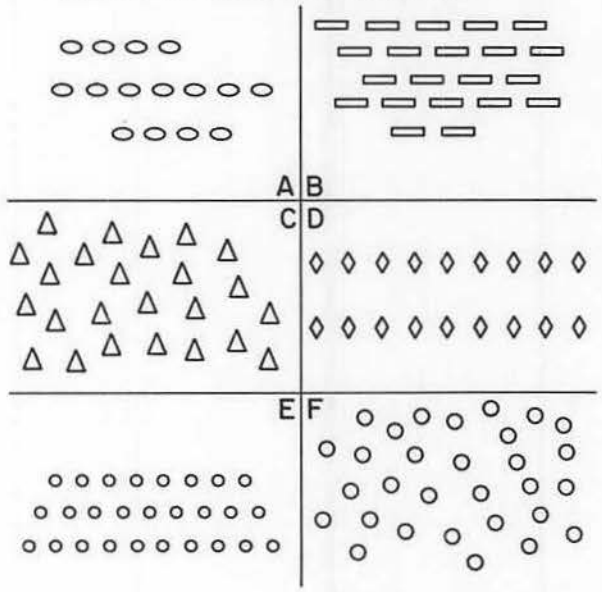
IV. Complete these tables.

+	1	7	9
9			
7			
1			

X	3	4	5
3			
4			
5			

6	4	2	+
			9
			6
			3

V. Draw lines to show 3 groups with the same number of objects in each group.



Draw or imagine a picture.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
No. of groups	3						3	4	6	2				5	
No. in each group							11				7	6	5	6	3
Total number								12	24	18	28	36	25		60

VI. Find the missing addends.

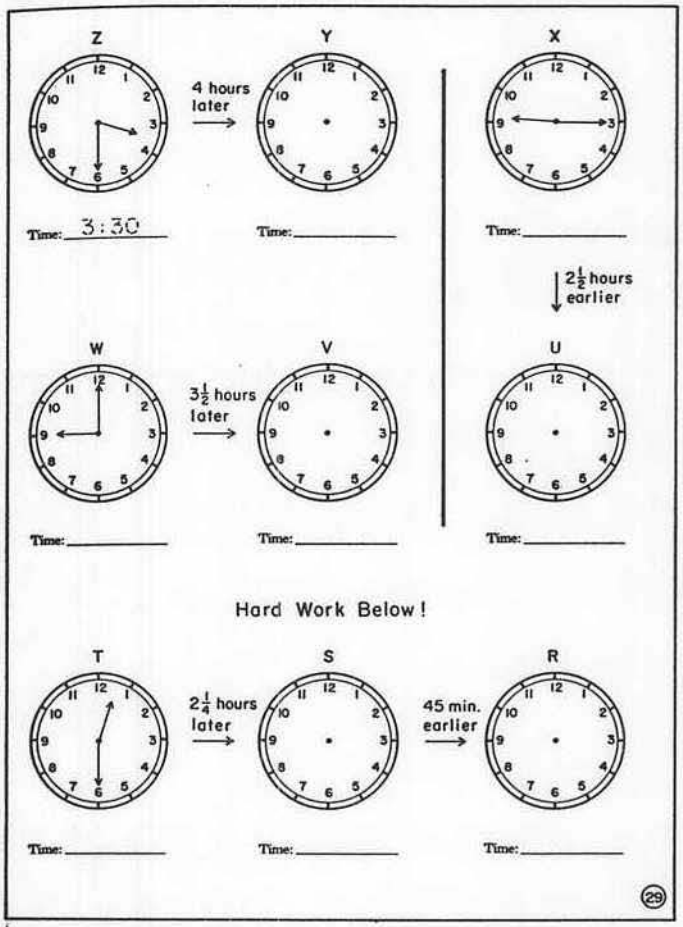
	Addends		Sum
a	3		6
b		5	7
c	4		7
d	3		3
e		2	4
f	3		4
g	2		5
h		4	4
i	4		8
j		2	6

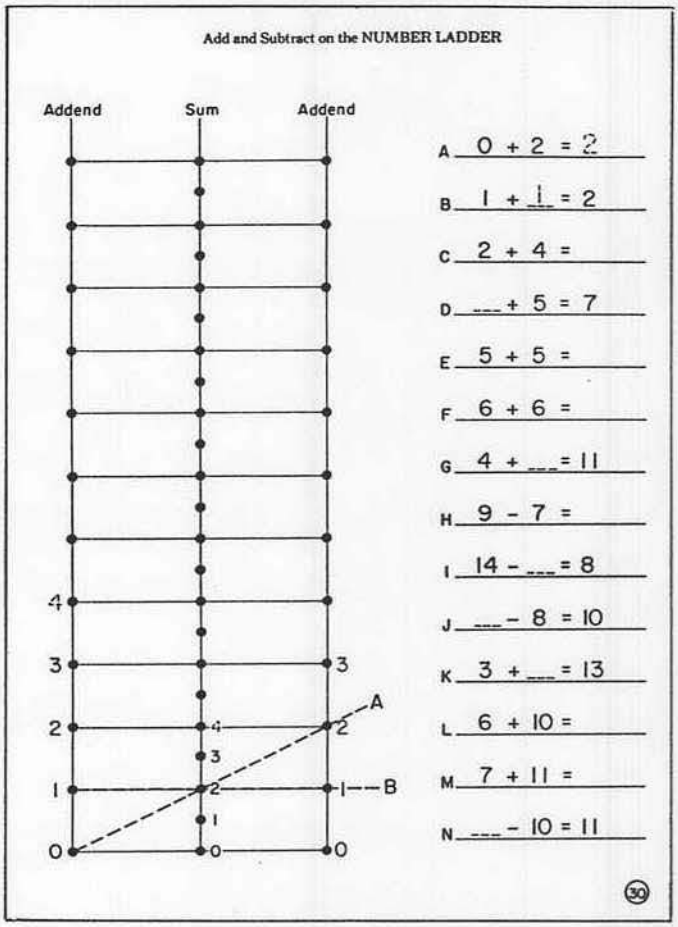
Find the missing factors.

	Factors		Product
k	3		9
l		5	10
m	4		12
n	3		0
o		2	4
p	3		3
q	2		6
r		4	0
s	4		16
t		2	8

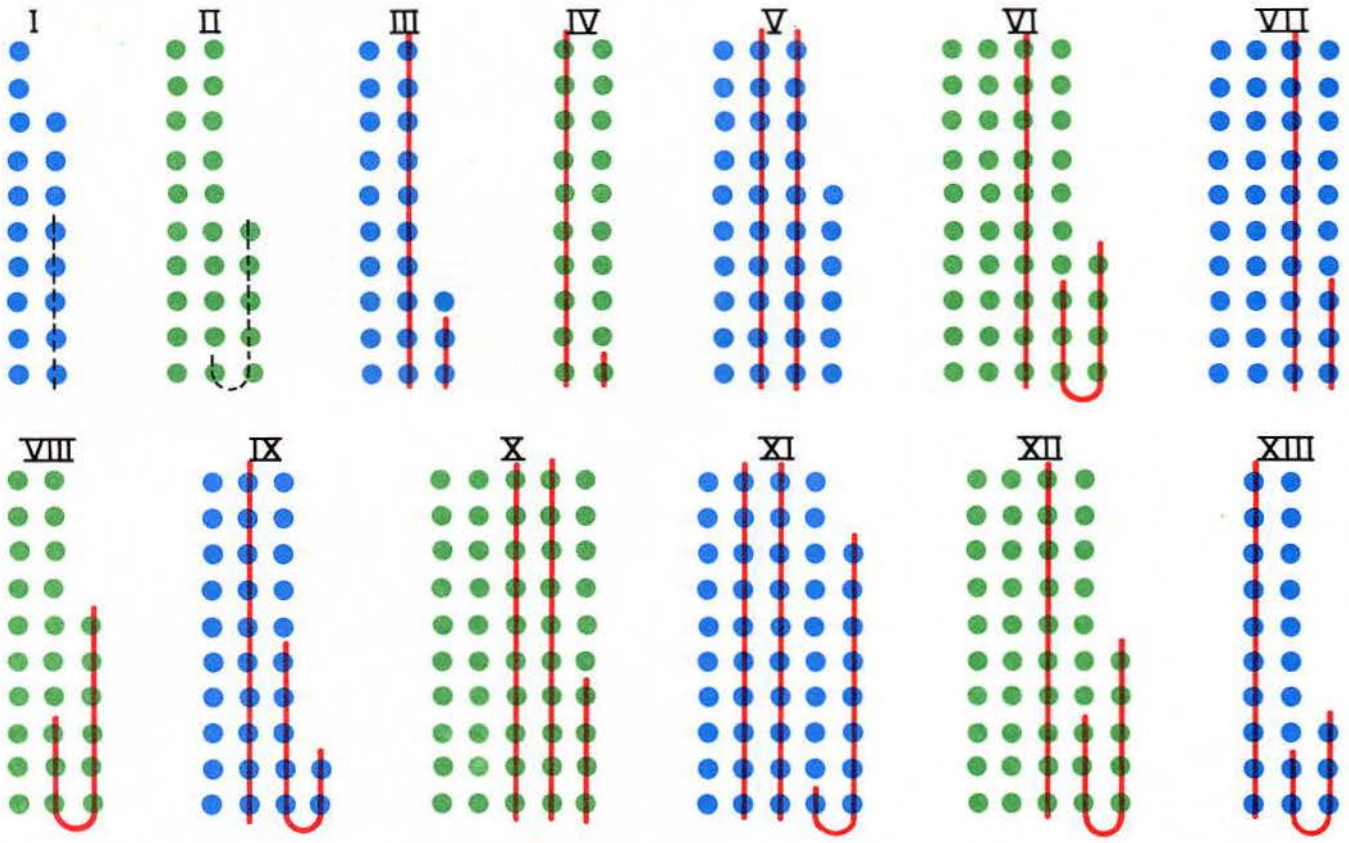
Drill

Consider using as drill sheets pages 29 and 30 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown on the next page.


Z
Y
X
4 hours later
Time: 3:30
Time:
Time:
2½ hours earlier
W
V
U
3½ hours later
Time:
Time:
Time:
Hard Work Below!
T
S
R
2¼ hours later
45 min. earlier
Time:
Time:
Time:
29


Add and Subtract on the NUMBER LADDER
Addend
Sum
Addend
A
B
A 0 + 2 = 2
B 1 + 1 = 2
C 2 + 4 =
D ___ + 5 = 7
E 5 + 5 =
F 6 + 6 =
G 4 + ___ = 11
H 9 - 7 =
I 14 - ___ = 8
J ___ - 8 = 10
K 3 + ___ = 13
L 6 + 10 =
M 7 + 11 =
N ___ - 10 = 11
30

You cross out the dots. Notice what the record tells you.



	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
All dots	18	25	23	20	36	44	40	26	32	50	48	45	23
Crossed out	5	6	12	11	20	17	13	9	17	24	29	18	15
Not crossed out	13	19	11	9	16	27	27	17	15	26	19	27	8

Chain Reactions

A

23	-2	21	-10	11

B

20	-1	19	-10	9

C

44
-7
37
-10
27

D

40
-3
37
-10
27

E

32
-7
25
-10
15

F

50
-4
46
-20
26

G

48
-9
39
-20
19

H

45
-8
37
-10
27

I

23
-5
18
-10
8

Notes on Pupil Page 47

Tasks: *In each of exercises I-XIII pupils count (or use shortcuts to counting) the total number of dots in each representation, the number of dots crossed out, and the number of dots not crossed out. They record these data in the table. For the lower part of the page pupils record the missing data in each of the chain reaction patterns A-I.*

Purpose: To continue to perfect computational skills while exploring procedures for developing a subtraction algorithm.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

There are many ways to cross out the dots in each example. Some facilitate counting. Discuss with pupils possible approaches in similar examples on the board. A good way to begin work on this page is to count and record the number of dots in the top row (All dots) for each of the 13 pictures.

The nine chain reaction problems at the bottom of the page correspond to nine of the cross out problems on the page. Have pupils find these pairs of corresponding examples. (A with III, B with IV, C with VI, D with VII, E with IX, F with X, G with XI, H with XII, I with XIII.)

Possible Supplementary Activities

I. Complete the tables.

5			+
	13	11	10
	11		
11			

+		5	10
			20
		10	
	14		

		9	+
		18	
	14	16	
10			5

×	0	2	4
0			
2			
4			

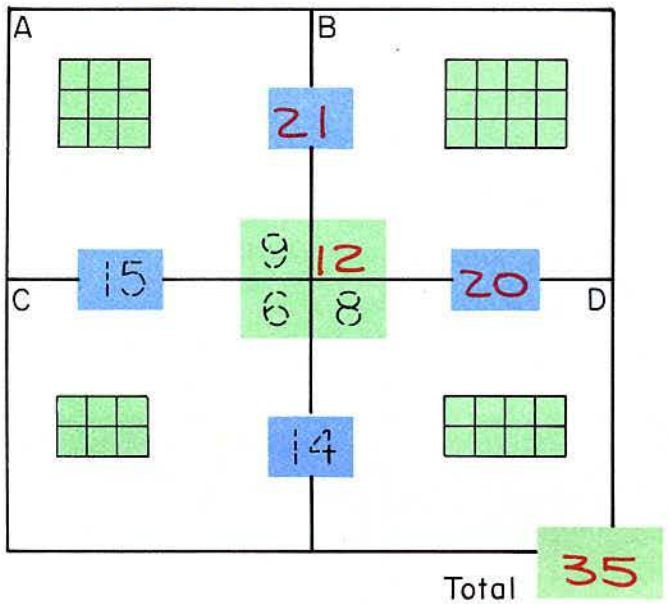
×	1	3	5
1			
3			
5			

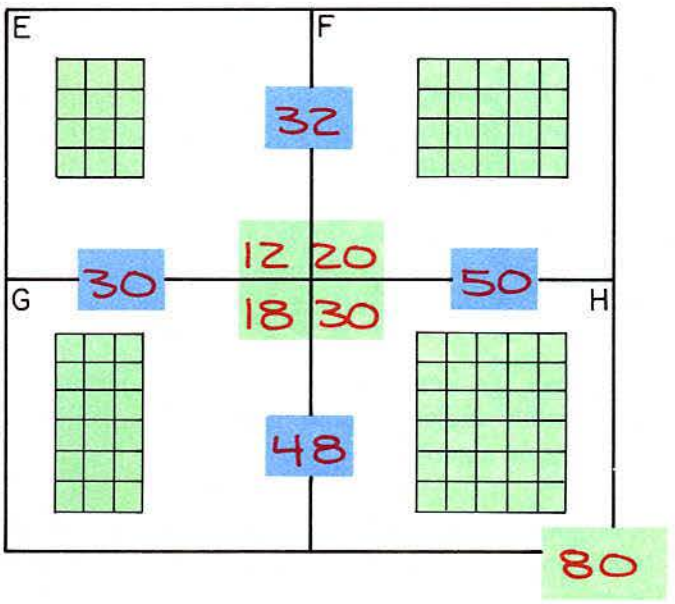
×	6	7	8
0			
2			
4			

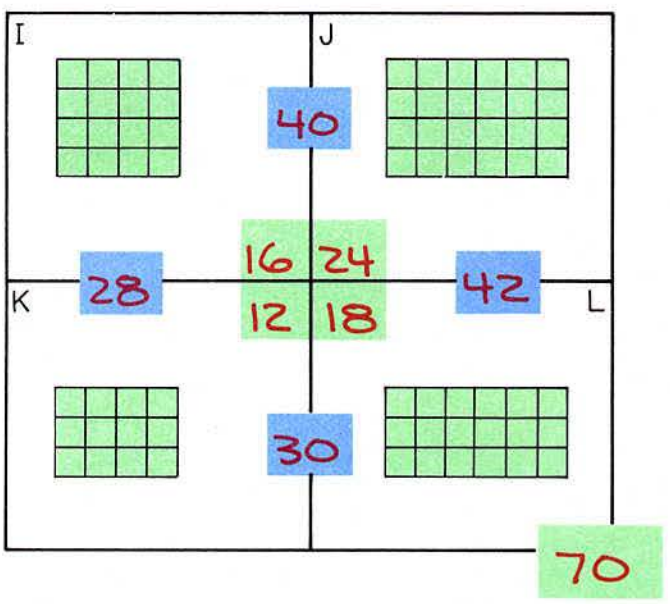
II. Find the missing numbers.

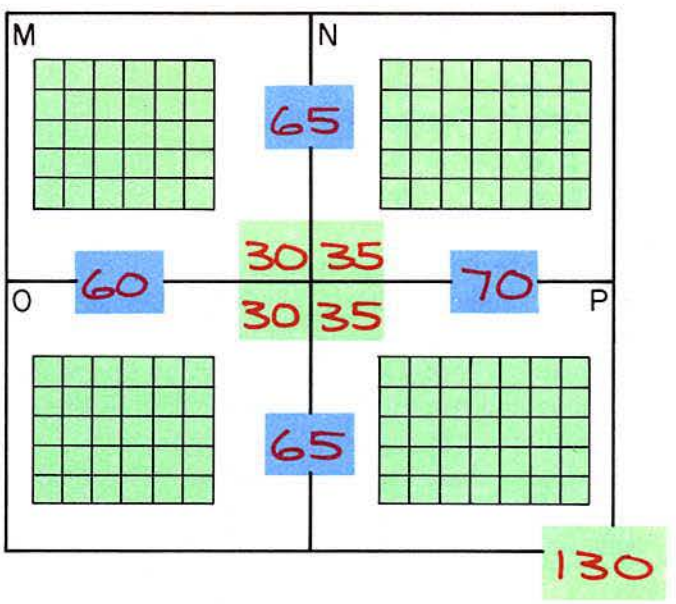
$3 \times 5 = \_\_$	$21 \div 3 = \_\_$	$2 \times 10 = \_\_$
$4 \times \_\_ = 16$	$18 \div \_\_ = 2$	$1 \times \_\_ = 10$
$\_\_ \times 3 = 18$	$\_\_ \div 5 = 5$	$\_\_ \times 10 = 0$
$5 \times \_\_ = 20$	$24 \div \_\_ = 8$	$7 \div 1 = \_\_$
$6 \times 4 = \_\_$	$12 \div 2 = \_\_$	$8 \div \_\_ = 4$
$8 = \_\_ \div 2$	$\_\_ = 5 \times 7$	$8 \div \_\_ = 8$
$30 = 5 \times \_\_$	$\_\_ \times 8 = 0$	

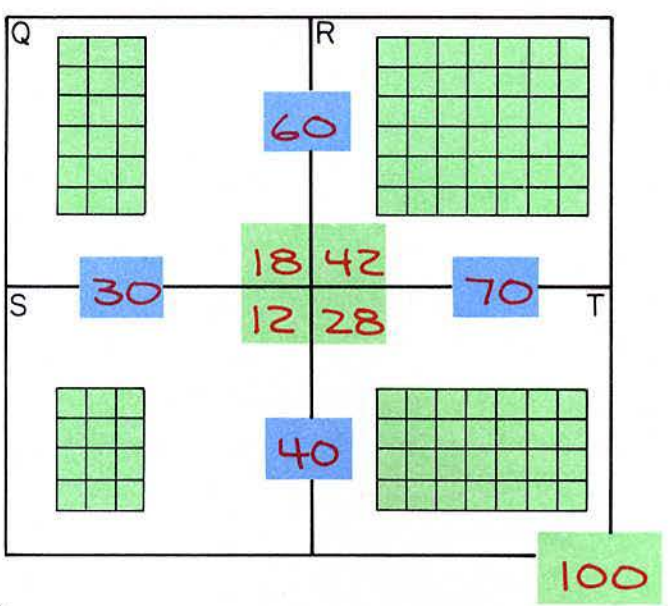
Headline for the Day: Nicole bought a present for her brother Pierre. She gave the clerk $3.00. The cost of the present was $2.23. The clerk gave her the correct change using the fewest coins possible. What was the change Nicole received?

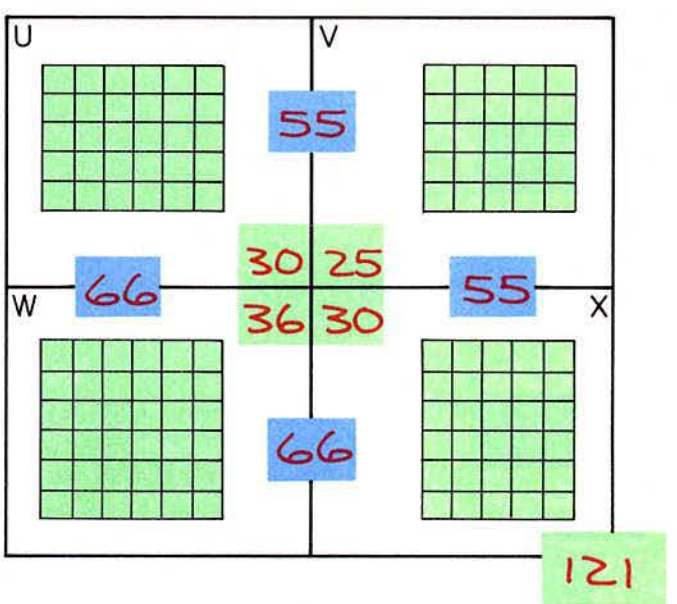

A
B
21
9
12
15
C
6
8
20
D
14
Total
35


E
F
32
12
20
30
G
18
30
50
H
48
80


I
J
40
16
24
28
K
12
18
42
L
30
70


M
N
65
30
35
60
O
30
35
70
P
65
130


Q
R
60
18
42
30
S
12
28
70
T
40
100


U
V
55
30
25
66
W
36
30
55
X
66
121

48-49 Counting problems which compel pupils to rely on shortcuts (i.e. multiplication) to determine total counts of squares on adjacent cards.

Tasks: *Pupils count, or use shortcuts to counting, to record in the green box on each card the number of little squares pictured on that card. They record in the green box at the lower right of each group of four cards the total number of squares pictured on the cards in the group. In the blue boxes pupils record the sum of the numbers of little squares on each pair of adjacent cards in each group of four cards.*

Purpose: To encourage the use of shortcuts to counting in order to carry out the tasks described above.

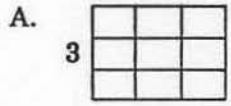
Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board; Ditto masters.

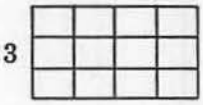
Materials for Pupils: Graph paper; Ditto sheets.

The Lesson:

As a first step in the study of this page, have pupils record the multiplication facts suggested by the array in each labeled box. This can be done on the page, directly below each array. Have them also record the number of columns and the number of rows of each array. Thus:

A. 

$3 \times 3 = 9$

B. 

$3 \times 4 = 12$

Note that the four arrays, in any of the six large boxes, could be put together to make a larger rectangular array. (For example, the arrays in A, B, C, and D give you a 5-by-7 array.) Doing so gives

more

Headline for the Day: Odette and Hume bought a wristwatch for Horace. They gave the clerk 9 dollars. The price of the watch was $8.37. The clerk, using the fewest coins possible, gave them the correct change. Describe the change they received.

---

you a fast way of determining the total for the four arrays.

Here are further activities which relate to and extend the ideas on this page.

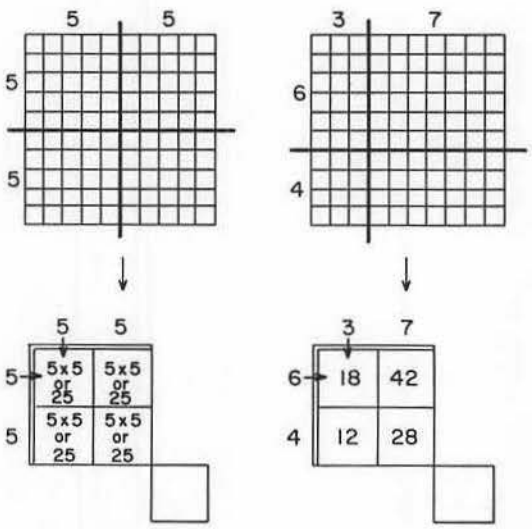
Using 10 as a Factor

See if pupils can discover the pattern of multiplication when one factor is ten.

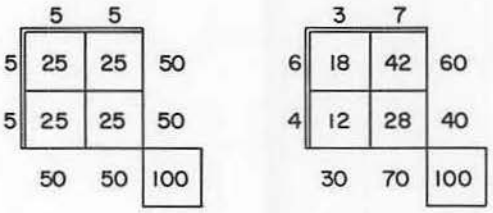
$$1 \times 10 = 10$$
$$2 \times 10 = 20$$
$$3 \times 10 = 30$$
$$4 \times 10 = 40$$
$$\vdots$$
$$10 \times 10 = 100$$
$$11 \times 10 = 110$$
$$16 \times 10 =$$
$$85 \times 10 =$$
$$\times 10 = 680$$

Cross Number Puzzle Arrays

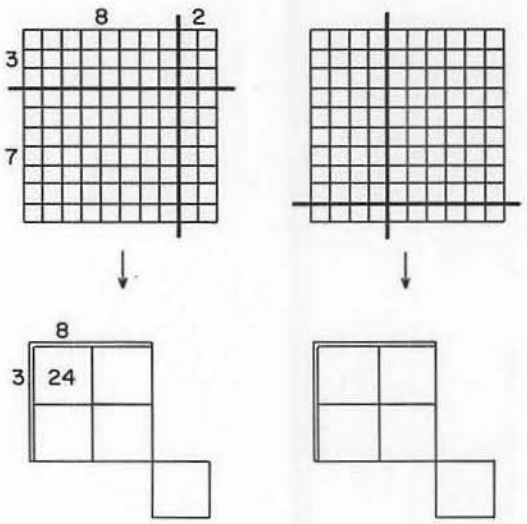
Divide a 10 × 10 array into four arrays by using a horizontal line and a vertical line. Record the numbers of columns and rows in each array. Then record the numbers of little squares in the arrays in *Cross Number Puzzle* form:



Now we can complete these as *Cross Number Puzzles* in the usual way, as shown at the top of the next column.



Make up examples which pupils can complete on their own:



Looking for Sums of Ten

Present on a Ditto sheet such problems as:

$4 + 3 + 6 + 7 = \boxed{20}$

$7 + 2 + 3 = \square$ $\qquad 8 + 6 + 2 = \square$

$2 + 5 + 3 + 8 + 7 + 5 + 4 = \square$

$7 + 9 + 13 + 1 + 10 = \square$

$4 + 7 + 16 + \square = 30$

$9 + 9 + 9 + 11 + 11 + 11 = \square$

Drill

Here are some stories for headline practice:

1. Manny's mother bought groceries which cost $7.03. She gave the clerk a 10-dollar bill.
2. Biff had 53 pop-bottle tops. He gave Jeremiah 17 of them.
3. Dawn, Joel, and Frieda went to the bookstore. They each bought 2 pens and 3 pencils.

Consider using as drill sheets pages 36 and 37 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown on the following page.

Find the Numbers in the Shaded Blocks. Write down as few as you can in the white blocks.

A.

1	2	
4		
	11	
22		

B.

10	20	
40		
	110	
220		

C.

2	8						44		
4			22						

D.

1									
2									
								35	

E.

1	2								
	12								
			34						

F.

12		6							
				30					40
						94			

Find the Numbers in the Shaded Blocks. Write down as few as you can in the white blocks.

A.

1	3	5		
11				19
31				

B.

137				153			
138			150		158		
139							

C.

43			31		
42					
40	36				

D.

143			131			
141						
140						

E.

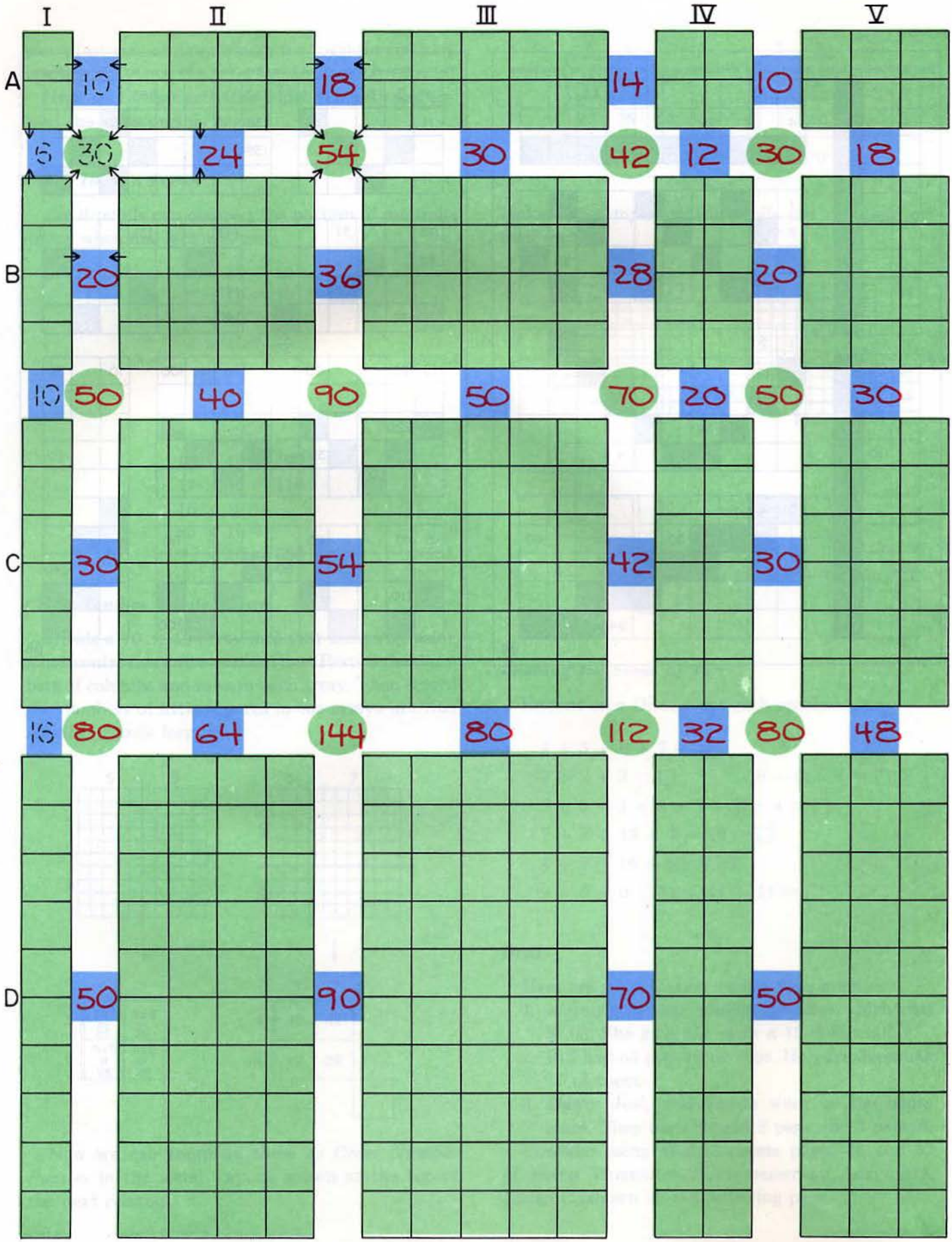
10	20			50					100
	220								
				350					

F.

10		30
		120
		240

G.

5	10			25					50
					80				
	110								
									200


I
II
III
IV
V
A
B
C
D
10
18
14
10
5
30
24
54
30
42
12
30
18
20
36
28
20
10
50
40
90
50
70
20
50
30
30
54
42
30
16
80
64
144
80
112
32
80
48
50
90
70
50

Notes on Pupil Page 49

Tasks: *Pupils count, or use shortcuts to counting, and record in each blue box the total number of small green squares in the two rectangles joined by the blue box. Pupils record in each green circular region the total number of small green squares in the surrounding four rectangles.*

Purpose: To encourage the use of shortcuts to counting (i.e., multiplication facts) to find the required sums.

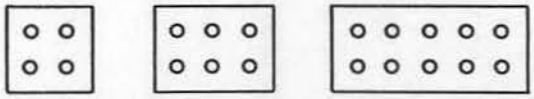
Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board; Ditto masters.

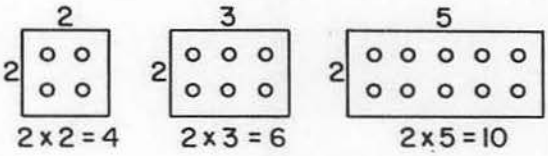
Materials for Pupils: Graph paper; Ditto sheets.

The Lesson:

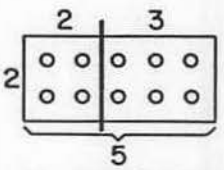
Before working on this page, it would be well to explore at the chalkboard the dot arrays shown at the top of the following column.



Have pupils label these arrays and note the relation between the arrays and the mathematical sentences.



Do pupils notice that the two smaller arrays make up the larger one? Here is a way to see the relationship:



$(2 \times 2) + (2 \times 3) = 2 \times 5 = 2 \times (2+3)$

Now begin with a large array of dots on the chalkboard and ask questions such as on the next page.

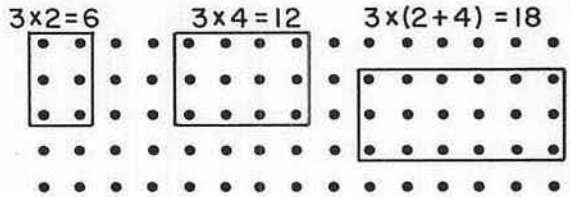
more

Headline for the Day: Houston spent 9¢ more than Ninette. Make a table to record what you can say about this situation.

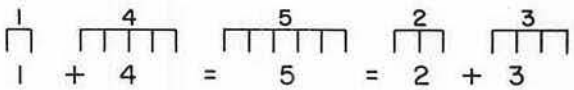
Can someone draw a rectangle around a group of dots to suggest 3×2?

Can someone draw a rectangle around a group of dots to suggest 3×4?

Can someone draw a rectangle around a group of dots to suggest $3 \times (2 + 4)$?

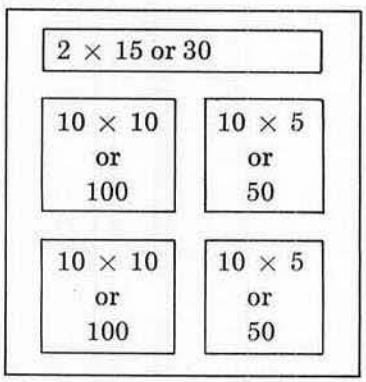


This entire pupil page was designed to illuminate this relationship. Note this in the pattern of vertical columns:



Note the pattern of horizontal rows.

An interesting problem which anticipates an algorithm for long multiplication is posed in the question: How many small green squares are there on the page? If small groups are asked to explore and report on a strategy, some may come up with this grouping:



100
100
50
50
30
330

Note that when the sum of the numbers of small green squares in (A, II), (A, III), (B, II), and (B, III) has been determined (54), you automatically have the sum of the numbers of small green squares in (C, II) and (C, III). Etc. Pupil page 49 abounds in such patterns. Consider investigating some of the groupings suggested by the other small group reports. Exploring these alternatives should make a final algorithm more meaningful.

Here are additional supplementary activities which support and extend the ideas explored on this page.

Activity I

o. A A A A B B B B B B C C C C
p. A A A A B B B B B B C C C C
q. A A A A B B B B B B C C C C
r. A A A A B B B B B B C C C C
s. A A A A B B B B B B C C C C
t. A A A A B B B B B B C C C C
u. D D D E E E E E E E C C C C
v. D D D E E E E E E E C C C C
w. D D D E E E E E E E F F F F
x. D D D E E E E E E E F F F F
y. D D D E E E E E E E F F F F
z. D D D E E E E E E E F F F F

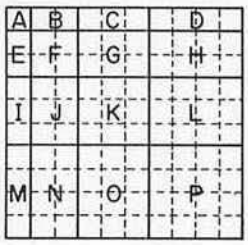
How Many?

Letter	Rows	Total letters
B	p, q, s, t	24
C	o, q, s, t, p	
E	x, y, w, q, r	
A	q, r, s, t, u, v	
D	t, u, v, w, x, y, z	
F	u, v, w, x, y	
B	q, r, s, t, u	
E	u, p, z, r, x	

Activity II

$$N(F) + N(G) = 10$$

Read: The number of little squares in F plus the number in G is 10.



1) $N(A) + N(C) =$ ___
2) $N(E) + N(F) =$ ___
3) $N(M) - N(K) =$ ___
4) $N(S) - N(F) =$ ___
5) $N(G) + N(H) =$ ___
6) $N(R) + N(H) =$ ___
7) $N(M) - N(H) =$ ___
8) $N(L) - N(B) = N(\ )$
9) $N(H) + N(\ ) = N(F)$
10) $N(\ ) - N(G) = N(C)$
11) $N(M) + N(T) =$ ___

Further examples for the front runners:

12) $N(\text{L}) + N(\ ) = N(\text{P}) + 3$
13) $N(\text{U}) - N(\text{T}) = N(\ ) + 0$
14) $N(\text{C}) - N(\text{B}) = N(\text{G}) - \underline{\quad}$
15) $N(\text{B}) + N(\text{C}) = N(\ ) - N(\text{A})$
16) $N(\text{N}) - N(\text{S}) = N(\text{F}) - N(\ )$
17) $N(\ ) - N(\ ) = N(\text{T}) + N(\text{P})$

Drill

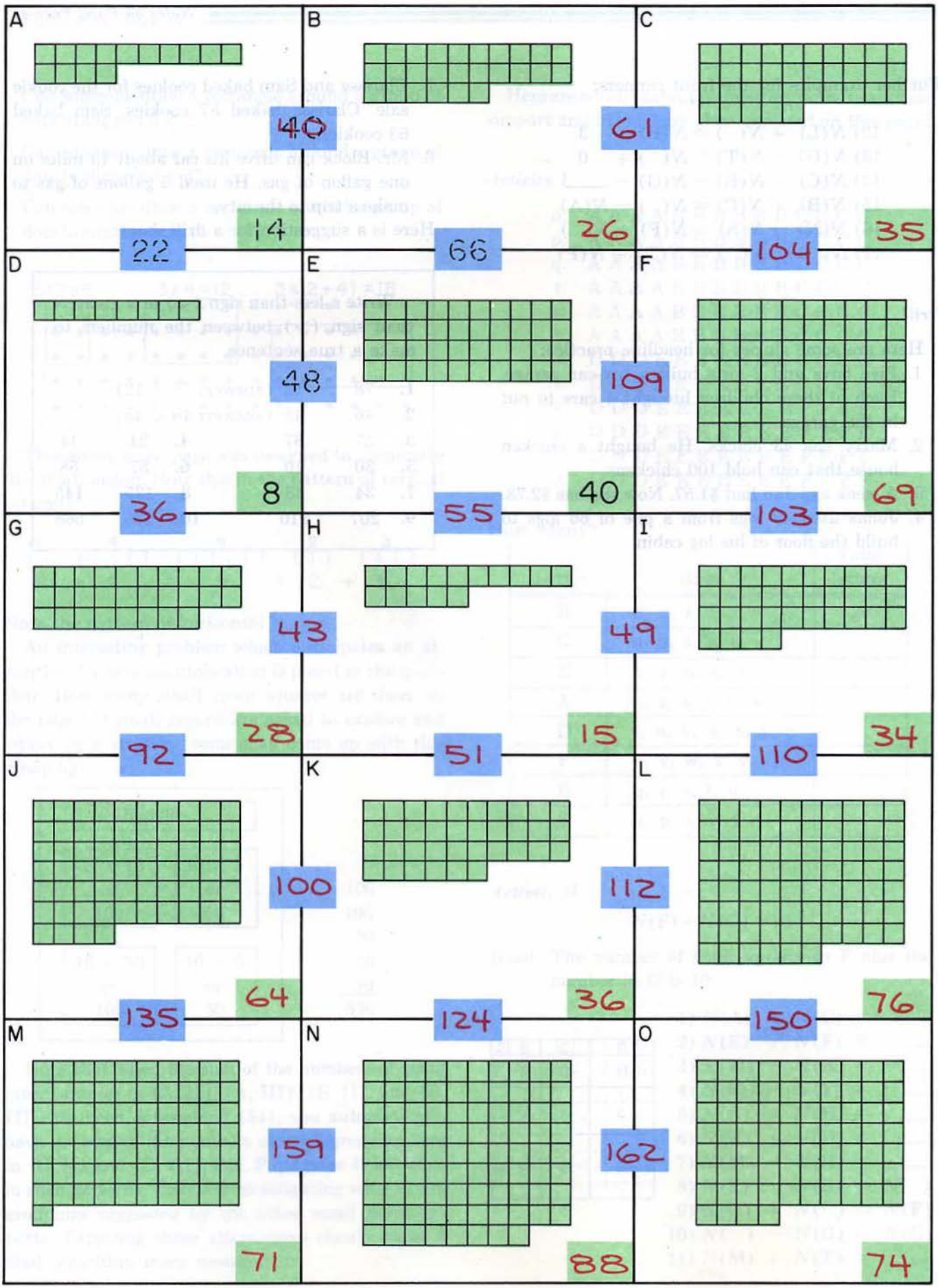
Here are some stories for headline practice:

1. Five boys and 2 girls built a toy-car garage. Each of these children brought 4 cars to put in the garage.
2. Marty has 43 chicks. He bought a chicken house that can hold 100 chickens.
3. A week ago Jan had $1.57. Now she has $2.73.
4. Jonas used 49 logs from a pile of 66 logs to build the floor of his log cabin.
5. Clarissa and Sam baked cookies for the cookie sale. Clarissa baked 87 cookies. Sam baked 63 cookies.
6. Mr. Block can drive his car about 13 miles on one gallon of gas. He used 5 gallons of gas to make a trip to the city.

Here is a suggestion for a drill sheet:

Write a less-than sign ($<$) or a greater-than sign ($>$) between the numbers to make a true sentence.

1.	78	12	(answer: $78 > 12$)			
2.	46	48	(answer: $46 < 48$)			
3.	57	67		**4.**	24	14
5.	30	10		**6.**	57	58
7.	34	43		**8.**	135	145
9.	207	210		**10.**	576	566


A
40
22
14
B
61
66
26
C
104
35
D
48
36
8
E
109
55
40
F
103
69
G
43
92
28
H
49
51
15
I
110
34
J
100
135
64
K
112
124
36
L
150
76
M
159
71
N
162
88
O
74

50 Counting problems which compel pupils to use decimal groupings to determine numbers of squares on cards and totals of numbers of squares on adjacent cards.

Tasks: *Pupils use shortcuts to counting to record in the green boxes the number of small green squares. They record in each blue box the total number of small green squares in the two blocks joined by the blue box.*

Purpose: To encourage the use of shortcuts to counting (i.e., multiplication facts) to carry out the required additions.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

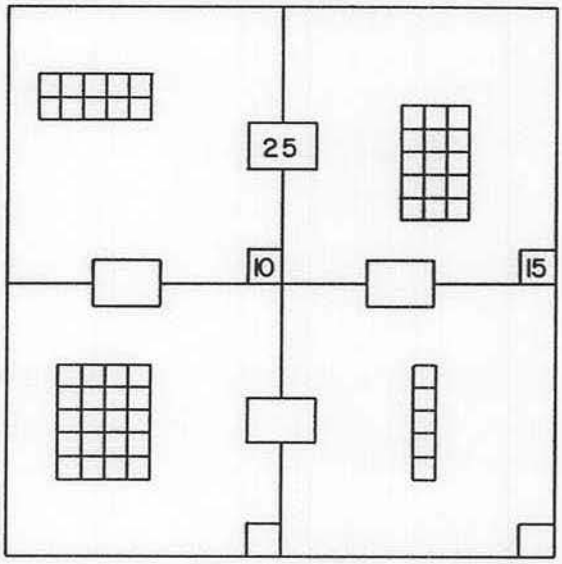
Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

This is the third in a series of pages designed to help pupils develop increased insight into shortcuts to counting leading to competence in handling algorithms for addition and multiplication. The main emphasis in this work is placed upon patterns of ten in arranging things to count.

Much practice in counting and adding is provided by the technique we use on this page for keeping records. While such a technique is not unfamiliar to pupils in this program, it would be well to review the procedure through several examples on the chalkboard. Have pupils consider examples like the following and, with the clues given, complete the record of "how many."



more

Headline for the Day: Hayes spent 9¢ less than Juno. Make a table to record what you can say about this situation.

As pupils work on the page, or beforehand, discuss possible strategies for counting rapidly. Any of the following activities afford further useful experience related to the major objectives of this page.

Graph Paper Rectangles

Have pupils draw rectangles consisting of 50, 80, 100, 170, 250, etc., little squares.

Sums of Connected Blocks

Have pupils make a record in the usual algorithm form of the addends and sums of the counts in adjacent blocks on pupil page 50 as indicated below:

A+B	14	B+E	26	D+G	8	
	+26		+40		+40	
	40		66		48	etc.

Practice with Cross Number Puzzles

→	10	20		20	80		50	20	
→	30	40		20	30		10	70	
	90	90		60	10		90	30	
	60	10		50	70		40	30	
	60	50		70	40		40	10	
	80	70		50	50		20	20	
	70	80		80	60		80	90	
	60	40		30	40		10	90	
	20	70		30	40		50	50	
	10	20		40	50		10	60	
	70	30		90	70		60	30	
	30	70		10	90		40	60	

Look for 10's

$3 + (5 + 7) = \triangle$

$\bigcirc = 5 + (9 + 5)$

$4 + 6 + \bigcirc = 13$	$6 + \square + 3 = 13$
$8 + 2 + 5 = \square$	$2 + \triangle + 9 = 19$
$\triangle + 5 + 8 = 18$	$7 + 3 + 9 = \bigcirc$
$\hexagon + 9 + 4 = 14$	$3 + \hexagon + 2 = 12$
$5 + 5 + \bigcirc = 12$	$9 + 1 + 7 = \square$
$1 + 3 + 7 = \square$	$5 + 2 + \triangle = 12$

Drill

Here is a suggestion for a developmental drill sheet which deals with an algorithm for column subtraction:

	I	II	III	IV	V	VI
A	30+7 − 3 30 + 4	37 − 3	20+17 − 8	37 − 8	57 − 3	57 − 8
B	40+5 − 2	45 − 2	30+15 − 7	45 − 7	65 − 4	65 − 7
C	60+9 − 6	69 − 6	39 − 7	89 − 2	59 − 9	99 − 4
D	30+6 − 4	36 − 4	36 −14	20+16 − 7	36 − 7	36 −27
E	80+1 − 1	81 − 1	81 −21	70+11 − 3	70+11 −(30+ 3)	71 −33
F	30+14 −(10+ 6)	44 −16	70+11 − 2	81 − 2	70+14 −(10+ 6)	84 −16
G	50+7 − 4	57 − 4	40+17 − 8	57 − 8	30 − 5	30 −15
H	90+8 − 8	98 − 8	70+13 −(20+ 8)	83 −28	92 − 5	92 −45

I

Rule: Subtract 5

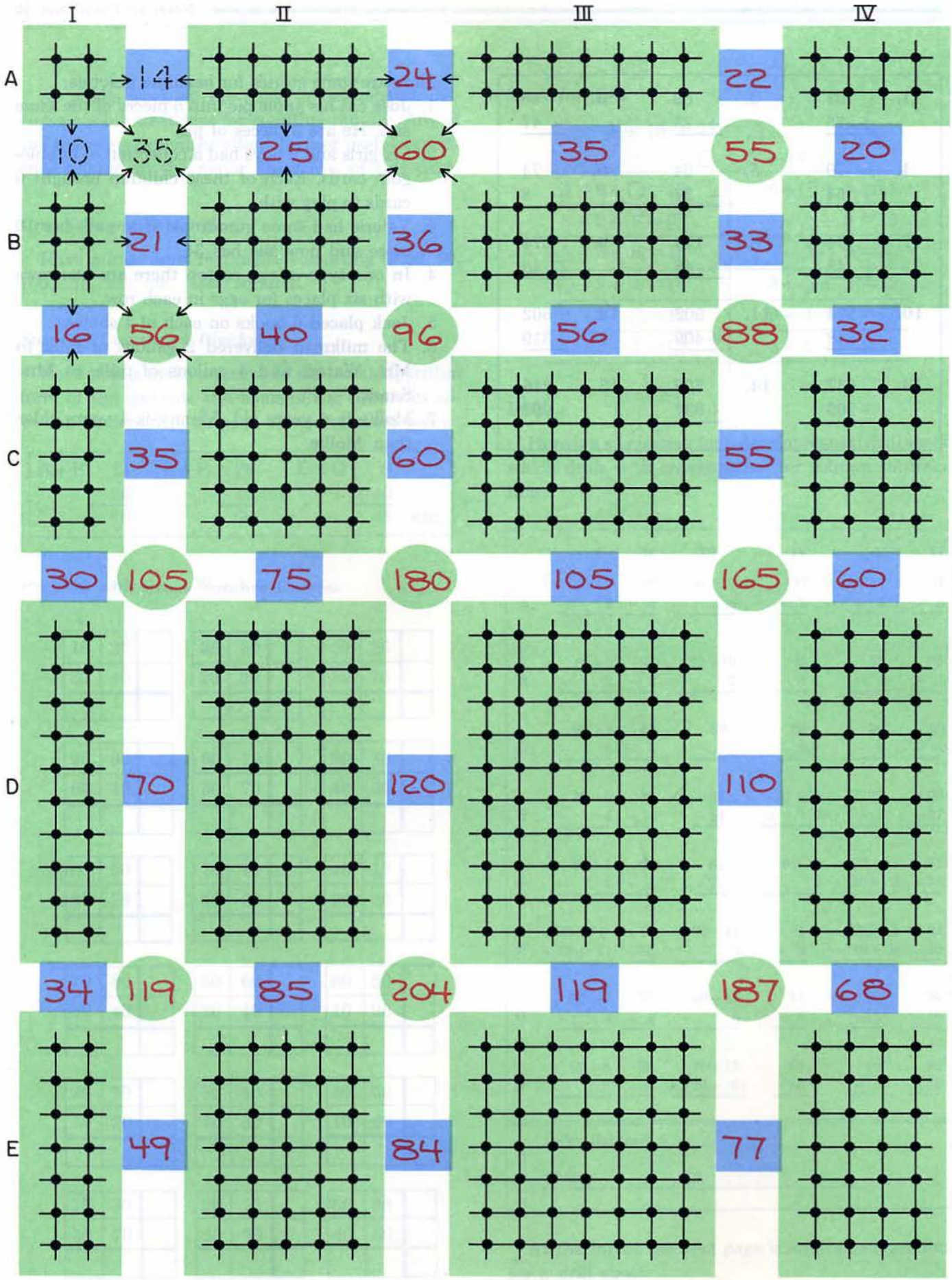
87, 82, , , , 62, , , , , ,

At the top of the next page is another suggestion for a drill sheet:

1.	67 + 75	**2.**	89 + 23	**3.**	68 + 47
4.	79 + 54	**5.**	64 + 27	**6.**	74 + 8
7.	74 + 48	**8.**	174 + 48	**9.**	574 + 48
10.	574 +648	**11.**	902 +409	**12.**	602 +319
13.	637 +205	**14.**	729 +837	**15.**	216 +997

Here are some stories for headline practice:

1. Jose cut his apple pie into 5 pieces of the same size. He ate 2 pieces of pie.
2. Six girls and 2 boys had a collection of bubble-gum cards. Each of these children brought 6 cards to play with.
3. Valerie had some gumdrops. Kim gave her 16 more and now she has 34.
4. In one type of egg carton there are two rows with six places for eggs in each row.
5. Jack placed 6 books on each of 4 shelves.
6. The milkman delivered 2 gallons of milk to Mrs. Waters and 4 gallons of milk to Mrs. Sands.
7. Mollie is 6 years old. Manny is 4 years older than Mollie.


I
II
III
IV
A
B
C
D
E
14
24
22
10
35
25
60
35
55
20
21
36
33
16
56
40
96
56
88
32
35
60
55
30
105
75
180
105
165
60
70
120
110
34
119
85
204
119
187
68
49
84
77

51 Counting problems which compel pupils to rely on shortcuts (i.e. multiplication) to determine total counts of squares on adjacent cards.

Tasks: *Pupils count or use shortcuts to counting to record in each blue box the total number of crossing points in the two rectangular regions joined by that box. They record in each green circular region the total number of crossing points in the four surrounding rectangular regions.*

Purpose: To encourage the use of shortcuts to counting (i.e., multiplication facts) to carry out the required additions.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets (see below).

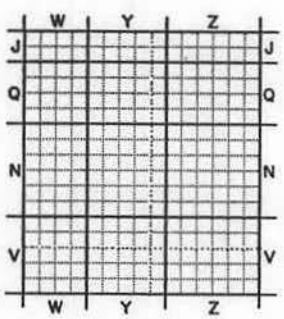
The Lesson:

This pupil page is quite similar to pupil page 50. Further activities related to the work on this page include those suggested in Notes on Pupil Page 50 plus the following.

Activity I

$$N(Y, J)=10$$

Read: The number of tiny squares in the cell where column Y and row J meet is 10.

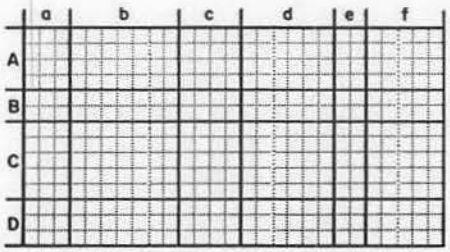


1) $N(Z, J) = \_\_$
2) $N(W, N) = \_\_$
3) $N(Z, V) = \_\_$
4) $N(W, Q) = \_\_$
5) $N(Y, V) = \_\_$
6) $N(Z, N) = \_\_$
7) $N(W, J) = \_\_$
8) $N(Y, Q) = \_\_$
9) $N(Y, N) = \_\_$
10) $N(Z, Q) = \_\_$
11) $N(W, V) = \_\_$

Further examples for the front runners:

12) $N(Z, Q) - 20 = 14 - \_\_$
13) $N(W, J) + 25 = 40 - \_\_$
14) $N(W, Q) + N(Y, V) = N(\ ) + 5$
15) $N(W, V) + N(Y, Q) = N(Y, N) + \_\_$

Activity II



Picture Suggests

1. Aa → 4 × = __
2. Bc → × = __
3. __ → 3 × = __
4. __ → 5 × = __
5. Ad → __ = × __
6. __ → 6 = 12 ÷ __
7. __ → 10 = 5 × __
8. __ → 18 ÷ 3 = __
9. Aa, Ab → 4 × 10 = (4 × 3) + (4 ×) = + = __
10. Bc, Bd → × = (×) + (×) = + = __
11. Dd, De → × = (×) + (×) = + = __

Drill

Here are two stories for headline practice:

1. Three girls and 3 boys collected grasshoppers. They each caught 4 grasshoppers.
2. There were 42 trees on one block. On another block there were 66.

Headline for the Day: Fill in the blanks to continue the pattern.

121, , , 64, 49, , , 16, 9, , 1

Mark the correct answer. Mark the furthest from the correct answer. Difference between the answers you marked.

$$\begin{array}{r} 15 \\ +\ 5 \\ \hline \end{array}$$

(1) 16 (2) 20 (3) 10 (4) 25

1 ○ 2 ● 3 ⊗ 4 ○ 10

1 2

$$\begin{array}{r} 34 \\ +11 \\ \hline \end{array}$$

(1) 35
(2) 44
(3) 45
(4) 41

1 ⊗ 2 ○ 3 ● 4 ○ 10

$28 + 37 =$

(1) 55 (3) 57
(2) 58 (4) 65

1 ⊗ 2 ○ 3 ○ 4 ● 10

3

$$\begin{array}{r} 47 \\ -38 \\ \hline \end{array}$$

(1) 9
(2) 19
(3) 85
(4) 75

1 ● 2 ○ 3 ⊗ 4 ○ 76

4 5

$200 + 375 =$

(1) 175 (3) 500
(2) 575 (4) 695

1 ⊗ 2 ● 3 ○ 4 ○ 400

$$\begin{array}{r} 736 \\ +264 \\ \hline \end{array}$$

(1) 990
(2) 900
(3) 1000
(4) 532

1 ○ 2 ○ 3 ● 4 ⊗ 468

6

$193 - 90 =$

(1) 103 (3) 183
(2) 283 (4) 100

1 ● 2 ⊗ 3 ○ 4 ○ 180

TOY SALE

Models 14¢ each
Puzzles 18¢ ea.
Games 27¢ ea.

7 8

How much did Tony pay for a model and a puzzle?

(1) 22¢ (3) 45¢
(2) 32¢ (4) 41¢

1 ○ 2 ● 3 ⊗ 4 ○ 13¢

Shirley bought a puzzle and a game and gave the clerk $1.00. How much change should she get?

(1) 45¢ (3) 55¢
(2) $1.45 (4) $1.55

1 ○ 2 ○ 3 ● 4 ⊗ $1.00

9

Paul bought a model, a puzzle, and a game. He gave the clerk a dollar. How much change should he get?

(1) 51¢ (3) 49¢
(2) 59¢ (4) 41¢

1 ○ 2 ⊗ 3 ○ 4 ● 18¢

10 11

Paula bought a puzzle and a game. Betty bought a model and a puzzle. How much more did Paula spend than Betty?

(1) 13¢ (3) 45¢
(2) 32¢ (4) 77¢

1 ● 2 ○ 3 ○ 4 ⊗ 64¢

Lester bought two puzzles and two models. He gave the clerk a dollar. How much change should he get?

(1) 64¢ (3) 40¢
(2) 54¢ (4) 36¢

1 ⊗ 2 ○ 3 ○ 4 ● 28¢

12

Mona bought two games, two puzzles, and three models. How much did she spend?

(1) $1.32 (3) $1.18
(2) $1.22 (4) 98¢

1 ● 2 ○ 3 ○ 4 ⊗ 34¢

52 Test page - multiple-choice items - addition, subtraction, and applications.

Tasks: *Pupils use the sample problem solved in the blue region at the top of the page as a guide to selecting the correct answer and the furthest from the correct answer. They record the correct answer by shading the appropriate circle; they record the number that is furthest from the correct answer by placing a cross in the appropriate circle. They record in the blank provided the difference between the two selections they make in each problem.*

Purpose: To test achievement and diagnose strengths and weaknesses in applying computational skills and in knowledge of arithmetic concepts dealt with up to this point in the course.

Unifying Ideas: Addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Ditto master.

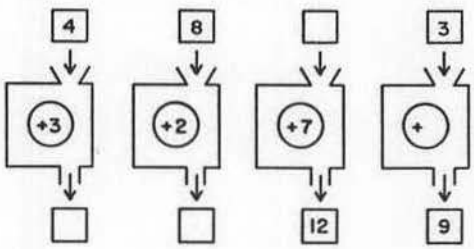
Materials for Pupils: Ditto sheets as described below.

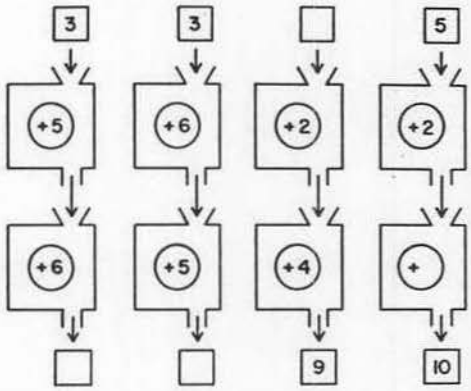
The Lesson:

We suggest that you count 3 points for each of the 12 items on this test.

Operation Machines

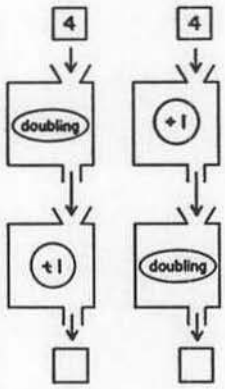
An interesting way to provide drill in addition and subtraction is to use machine notation. Here are some sample exercises which may be put on the chalkboard or on Ditto sheets:





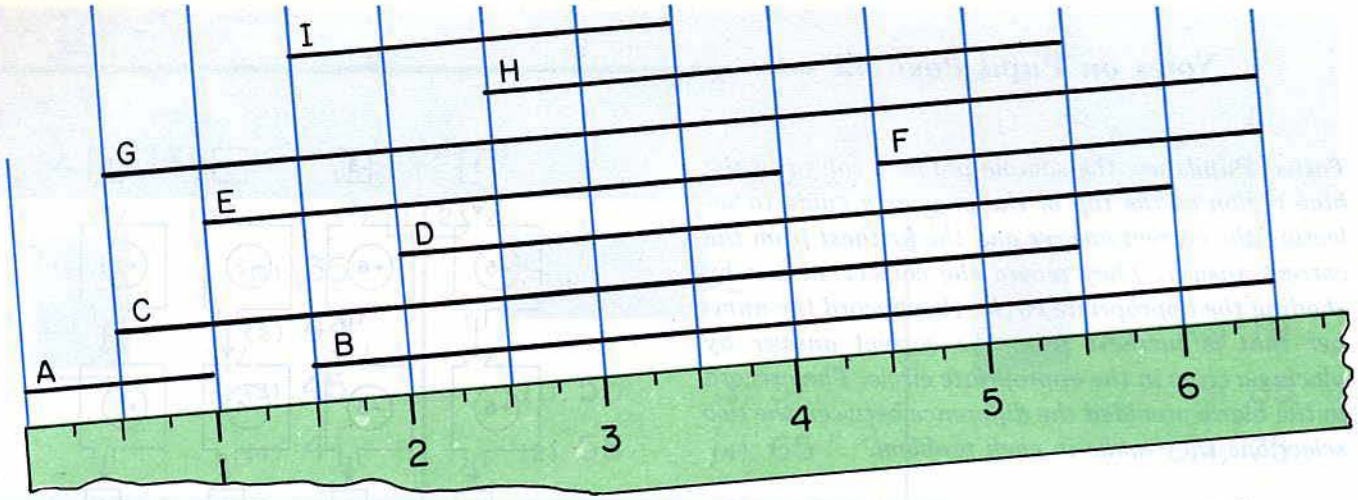
Mr. Marvin Karlin reports that a primary schoolboy in Jefferson County, Colorado, became so excited by the idea of machines that he built one. He came to school one morning with a large cardboard box having a slot cut in the top and a slot at the bottom. He climbed in the box, having first equipped himself with a pad, a pencil, and a flashlight. If he produced a wrong answer, the questioner was allowed to kick the machine. You can imagine the rest of the story.

Machine notation can be used to ask deep questions without depending upon language. For example: Do the answers differ if we double first and add 1 instead of adding 1 and then doubling?



(For an excellent discussion of machines see Marvin Karlin's article "Machines" in the May 1965 issue of *The Arithmetic Teacher*, pp. 327-334.)

Headline for the Day: Tell stories to fit the mathematical sentences 1-6 on page 52.



Lengths:

A: 1 inch C: 5 in. E: 3 in. G: 6 in. H: 3 in.

B: 5 inches D: 4 in. F: 2 in. I: 2 in.

SUMS of lengths:

A and E: 4 in. A, I, and F: 5 in.

G and C: 11 in. B, D, and G: 15 in.

B and H: 8 in. C, H, and A: 9 in.

D and F: 6 in. D, H, and B: 12 in.

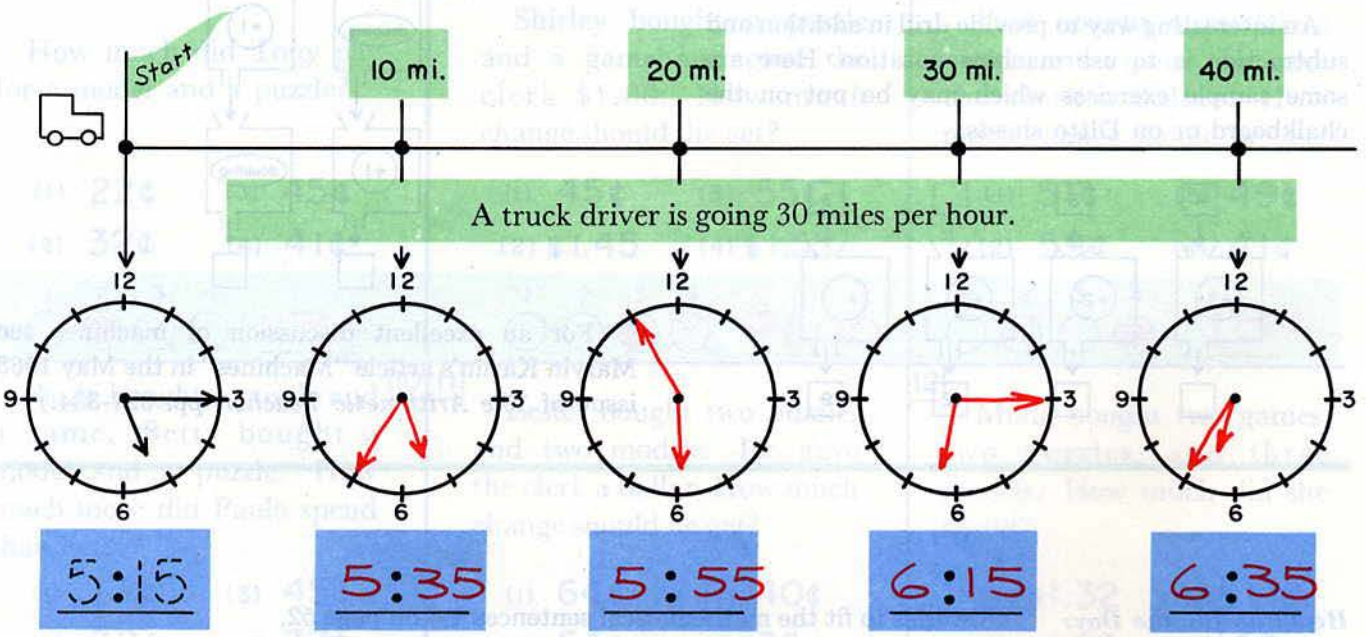
DIFFERENCES of lengths:

E and F: 1 in. H and I: 1 in.

G and D: 2 in. D and A: 3 in.

B and A: 4 in. G and F: 4 in.

F and I: 0 in. D and I: 2 in.



He would travel:

60 miles in 2 hrs.; 45 miles in 1½ hrs.; 150 miles in 5 hrs.

53 Finding lengths and sums and differences of lengths - time-rate-distance problems.

Tasks: *Pupils use the green ruler and the blue grid lines to determine the measure in inches of each of the line segments shown. They record these data in the blanks beneath the ruler. Pupils then compute and record the required sums and differences in the blanks provided. For the bottom of the page pupils draw the hands of the clock and record the time thus shown at each mileage point indicated. They record in the blanks at the bottom the distance traveled in the specified time intervals.*

Purpose: To explore several applications while simultaneously perfecting computational skills.

Unifying Ideas: Addition and subtraction; multiplication and division; geometry; measurement.

Materials for Teacher: Demonstration clock; foot-ruler; yardstick; meterstick; Ditto masters.

Materials for Pupils: Ditto sheets (see below).

The Lesson:

If preparation is necessary for this page, introduce some experiences with ruler measurement such as the following.

Measuring the lengths of various straight line segments on the board or on Ditto sheets.

Drawing line segments of specified lengths.

Drawing segments as long as two smaller segments put together.

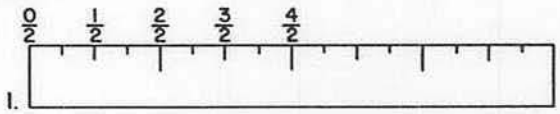
Drawing segments so many inches longer or shorter than given segments.

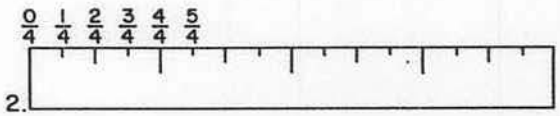
Comparing the lengths of segments and indicating the differences.

Measuring lengths without starting at the 0 on the ruler.

Discussing the half and quarter marks.

Upon completion of the page have pupils consider labeling some of the points on a ruler in various ways. Draw partially completed rulers on a Ditto master:





more

Headline for the Day: Jessica told her father she would be ready to go to the party in 10 minutes. What can you say about this situation? Make a table.

1 in. $= \frac{2}{2}$ (read: 2 half) inches

$= \frac{4}{4}$ (read: 4 quarter) inches

2 in. $= \frac{4}{2}$ (read: 4 half) inches

$= \frac{8}{4}$ (read: 8 quarter) inches

3 in. $= \frac{6}{2}$ inches $= \frac{12}{4}$ inches

Now try such problems as these:

What length is $\frac{2}{2}$ inches + $\frac{4}{2}$ inches?

(Answer: $\frac{6}{2}$ inches or 3 inches.)

What length is $\frac{3}{2}$ inches + $\frac{3}{2}$ inches?

What length is $\frac{3}{4}$ inches + $\frac{5}{4}$ inches?

Introduce the time and rate situations by having pupils consider such problems as:

A man travels steadily at 40 miles an hour.

(a) How far does he go in 2 hours? ½ hour? 15 minutes? Etc.

(b) If he begins at 1 P.M., how far does he go by 3 P.M.? 1:30 P.M.? 1:15 P.M.?

Have pupils demonstrate such time and distance situations on a clock. Then do problems similar to (b), but change the starting time to 2:30 or 5:15.

Enrichment

Consider the following problem.

Dad drove 50 miles per hour all the way to Grandmother's, except during the half hour we stopped for lunch. We arrived 2½ hours after we left home. How far is it to Grandmother's?

Begin a discussion about the trip to Grandmother's: How far did they drive before lunch? How far after lunch? How long a time did they drive before lunch? How long after lunch? Do you have enough information to answer these four questions?

Put the four questions on the chalkboard. Then say: If I answer two of the questions for you, can you answer the rest? (Yes) If I agree to give you the answer to any one of the questions and then ask you to answer the rest, which question would you wish me to answer? (Any will do because each provides enough information so that the others can be answered.)

Let's suppose that all the information in the printed story is correct, except the speed at which Dad drives. Now, can you find the actual distance to Grandmother's? (No) We don't know the speed; therefore let's see what the results would be for different speeds:

Speed in m.p.h.	Time traveled	Distance to Grandmother's
50 m.p.h.	2 hrs.	100 miles
40 m.p.h.	2 hrs.	80 miles
45 m.p.h.	2 hrs.	90 miles
46 m.p.h.	2 hrs.	92 miles

Now, let's suppose that the speed given in the story is accurate, but the time was in error. How far to Grandmother's? We don't know. If we knew the correct time, we would know the distance.

Speed in m.p.h.	Time traveled	Distance to Grandmother's
50 m.p.h.	3 hrs.	150
50 m.p.h.	2½ hrs.	
50 m.p.h.	2¼ hrs.	
50 m.p.h.	2 hrs. 30 mins.	

These are, of course, *If . . . then* ___ situations. In each of the charts above, we vary one of the conditions at a time and study its effect on the other conditions. In the first example, as the speed decreases, provided the time remains the same, the distance also decreases. In the second example, as the time increases, provided the speed remains the same, the distance also increases.

Finally, let's suppose that none of the information in the problem is reliable, and that, as a matter of fact, it is 120 miles to Grandmother's.

Speed in m.p.h.	Time traveled	Distance to Grandmother's
40	3 hrs.	120 miles
		120 miles
		120 miles
		120 miles
		120 miles

How can we fill in the chart? We have to say something like this: If the speed were 40 m.p.h. then it would take 3 hours. Etc.

Almost any story problem can be studied in the same way. How is the situation altered if one of the aspects is varied? Exploring situations in this way helps pupils to understand the relationship between arithmetic and the real world of events.

Drill

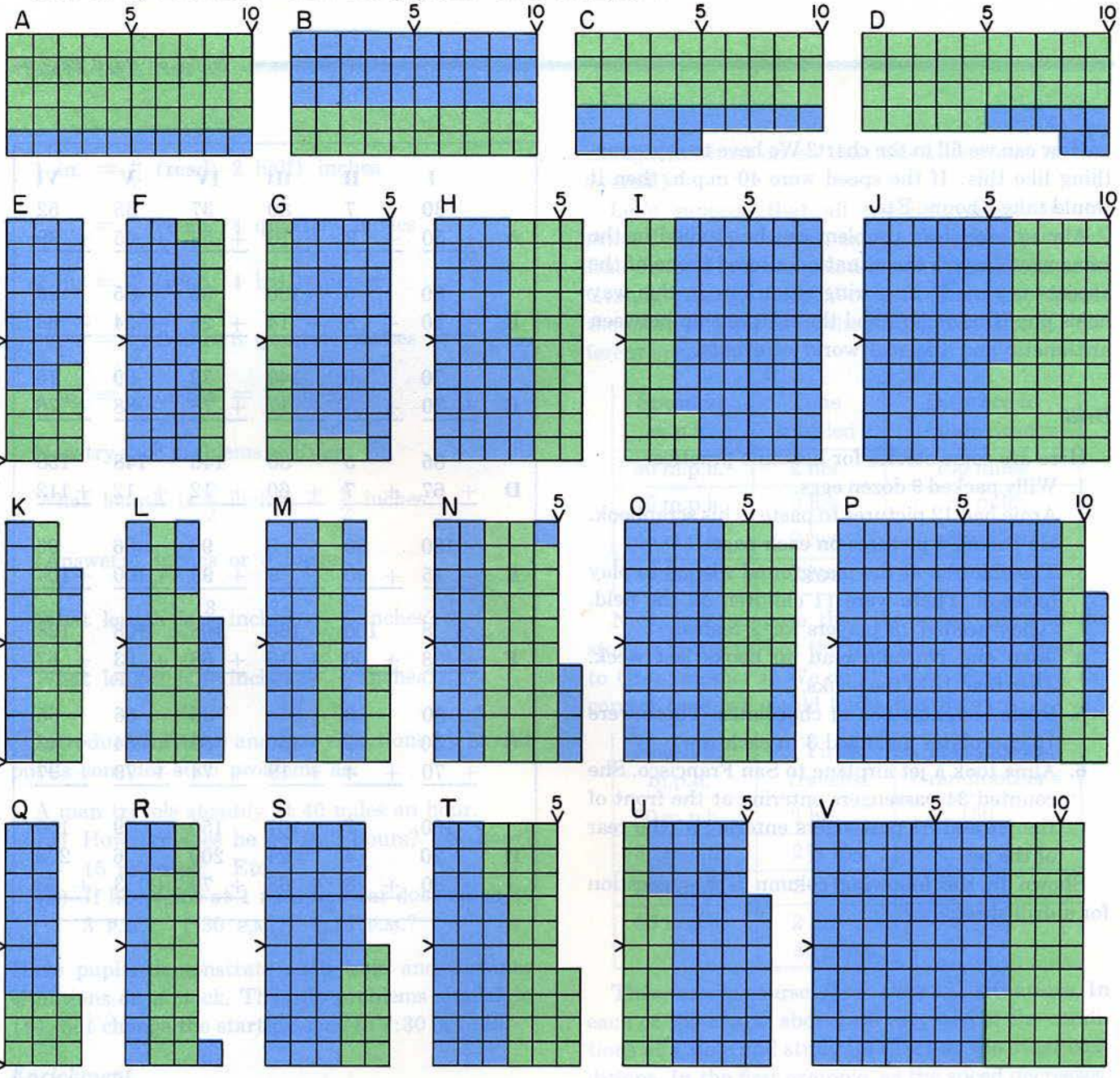
Here are some stories for headline practice:

1. Willy packed 9 dozen eggs.
2. Arnie had 12 pictures to paste in his scrapbook. He pasted 4 pictures on each page.
3. The children at the playground wished to play baseball. There were 11 children on the field. They needed 18 players for 2 teams.
4. Joan and Norman read 40 books last week. Joan read 18 of the books.
5. Rosie bought a box of chocolates. There were 15 chocolates arranged 3 in each row.
6. Alma took a jet airplane to San Francisco. She counted 34 passengers entering at the front of the jet and 81 passengers entering at the rear of the jet.

Shown in the following column is a suggestion for a drill sheet.

	I	II	III	IV	V	VI
	30	7	80	37	35	52
A	+ 50	+ 9	+ 16	+ 59	+ 55	+ 39
	80	6	100	86	85	29
B	+ 20	+ 8	+ 14	+ 28	+ 24	+ 83
	70	9	140	79	69	76
C	+ 70	+ 8	+ 17	+ 78	+ 88	+ 76
	85	5	80	140	148	158
D	+ 67	+ 7	+ 60	+ 12	+ 12	+112
	180	90	6	96	96	96
E	+ 15	+ 90	+ 9	+ 99	+100	+104
	8	100	160	108	108	128
F	+ 8	+ 60	+ 16	+ 68	+ 63	+ 67
	30	6		36	56	65
G	50	9	150	59	34	43
	+ 70	+ 4	+ 19	+ 74	+ 79	+ 97
	100	7	107	150	9	159
H	20	4	24	200	6	206
	+ 80	+ 8	+ 88	+ 70	+ 3	+ 73

How many tiles in all? How many green? How many blue?



	A	B	C	D	E	F	G	H	I	J	K
Total	50	50	45	42	30	40	50	60	80	100	28
Green	40	20	30	35	10	11	13	25	22	44	16
Blue	10	30	15	7	20	29	37	35	58	56	12

	L	M	N	O	P	Q	R	S	T	U	V
Total	36	44	54	69	107	23	31	45	54	57	112
Green	17	14	29	25	12	9	18	9	27	17	35
Blue	19	30	25	44	95	14	13	36	27	40	77

54 Counting and tabulation problems which encourage pupils to use decimal groupings to determine tile counts.

Tasks: *Pupils use shortcuts to counting and record, in the appropriate cells in the table at the bottom of the page, the total number of tiles, the number of green tiles, and the number of blue tiles in each of the pictures.*

Purpose: To encourage the use of shortcuts to counting in order to perfect computational skills.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

Instruct pupils to obtain all the totals first and record them in the table at the bottom of the page. Then have them obtain the counts of the green tiles all the way through and record those. Finally, they may obtain the counts of the blue tiles in any way they please. Hopefully, they will use subtraction to find the counts of the blue tiles.

Try the following enrichment activities to extend the idea of decimal partitioning with which we are concerned on pupil page 54.

Partitioning Drill

$86 + 6 = 80 + \square = 90 + \triangle = \bigcirc$
$59 + 2 = 50 + \square = 60 + \triangle = \bigcirc$
$35 + 7 = 30 + \square = 40 + \triangle = \bigcirc$
$24 + 9 = 20 + \square = 30 + \triangle = \bigcirc$
$77 + 6 = \square + 13 = \triangle + 3 = \bigcirc$
$84 + 8 = \square + 12 = \triangle + 2 = \bigcirc$
$79 + 9 = \square + 18 = \triangle + \bigtriangledown = \hexagon$

Building a Table

Suppose that I travel 10 miles every 20 minutes. Set up a table to show how far I will travel in 20 minutes, 40 minutes, 60 minutes, etc. How can I use the table to tell me how far I would go in 30 minutes? 50 minutes?

Drill

Here is a suggestion for a drill sheet on an algorithm for column subtraction:

1.	53 − 17	2.	44 − 25	3.	63 − 44
4.	63 − 18	5.	91 − 33	6.	76 − 12
7.	52 − 29	8.	85 − 67	9.	46 − 28
10.	78 − 73	11.	37 − 17	12.	37 − 19
13.	52 − 18	14.	67 − 18	15.	90 − 24

Here are some stories for headline practice:

1. At a party Virginia gave 2 apples to each guest. She gave 16 apples to her guests.
2. Pepe had to walk 9 blocks to school. Louise had to walk 13 blocks to school.
3. Wanda paid eighty-seven cents for three pints of ice cream.

Headline for the Day: Suppose that each of columns A-V in the table on page 54 is a headline for a story. Tell a story to fit each headline.

A

FRONT	BACK	
35	30	5
14	10	4
	21	49

B

FRONT	BACK	
47	40	7
10	10	0
	37	57

C

FRONT	BACK	
80	80	0
60	60	0
	20	140

D

FRONT	BACK	
34	20	14
17	10	7
	17	51

E

50	40	10
36	30	6
	14	86

F

81	80	1
31	30	1
	50	112

G

45	40	5
11	10	1
	34	56

H

30	20	10
13	10	3
	17	43

I

46	30	16
38	30	8
	8	84

J

76	70	6
51	50	1
	25	127

K

82	70	12
29	20	9
	53	111

L

40	30	10
14	10	4
	26	54

M

92	90	2
92	90	2
	0	184

N

35	20	15
18	10	8
	17	53

O

21	10	11
12	10	2
	9	33

55 WAMR cards featuring decimal partitioning to determine sums and differences and missing addends.

Tasks: *Pupils discover the rules for WAMR Set I and apply them in B-O.*

Purpose: To compare algorithms for column addition and column subtraction.

Unifying Ideas: Numeration; addition and subtraction.

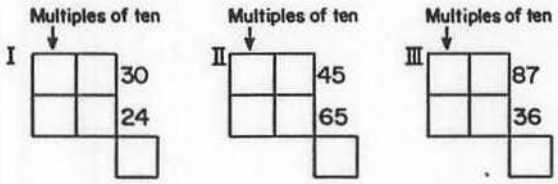
Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

On the white side of each card in WAMR Set I we partition each of the two given numbers (green side) so that we can perform column subtraction. The difference is written in the lower left of the six white cells; the sum is written in the lower right.

Cross Number Puzzles played according to a familiar partitioning rule is a good way to prepare pupils for this page.



Since the left-hand column specifies only that the cells contain multiples of 10, there are several ways of completing the *Cross Number Puzzles*. In example I, above, some of the possibilities are:

20	10	30
20	4	24
40	14	54

30	0	30
10	14	24
40	14	54

30	0	30
20	4	24
50	4	54

It might be interesting for a pupil to explore all the possibilities in any example and to prove that he has them all.

If the right-hand column of basic cells is restricted to numbers from 0 through 9, there is only one possible partitioning—the one most typically used in the common algorithm for column addition.

Regrouping or Repartitioning

The usual way of partitioning for algorithms for addition and subtraction must not obscure the fact that numbers may be partitioned in many ways. Frequently other ways of partitioning are more efficient and, therefore, are better shortcuts than the usual algorithms. Exercises like the following provide much practice in addition and subtraction and afford experiences designed to alert pupils to the freedom they have.

	a.	b.	c.	d.	e.
15	10 + 5	8 + 7	5 + 10	+ 6	- 5
+18	10 +	12 + 6	5 + 13	+	20 - 2
----	= 20 + 13	= +	= +	= + 13	= 40 - 7

	f.	g.	h.	i.	j.
34	9 +	+ 4	+	31 +	- 6
+ 9	4 +	+	3 + 6	+	10 -
----	= + 30	= 30 +	= + 20	= + 3	= 50 - 7

	k.	l.	m.	n.	o.
26	+ 1	+ 6	20 + 6	- 4	- 9
+26	+ 1	+ 4	+	30 -	- 9
----	= + 2	= +	= + 12	= - 8	= - 18

			p.	q.		
37	20 +	+ 7		14	7 +	4 +
+ 19	16 +	10 +		+ 28	14 +	+ 20
----	= +	= +		----	= +	= +

			r.	s.		
37	+ 30	15 +		----	+ 4	-
+	+ 10	10 +		+ 29	25 +	30 - 1
50	= +	= 25 +		58	= 50 + 8	= 60 - 2

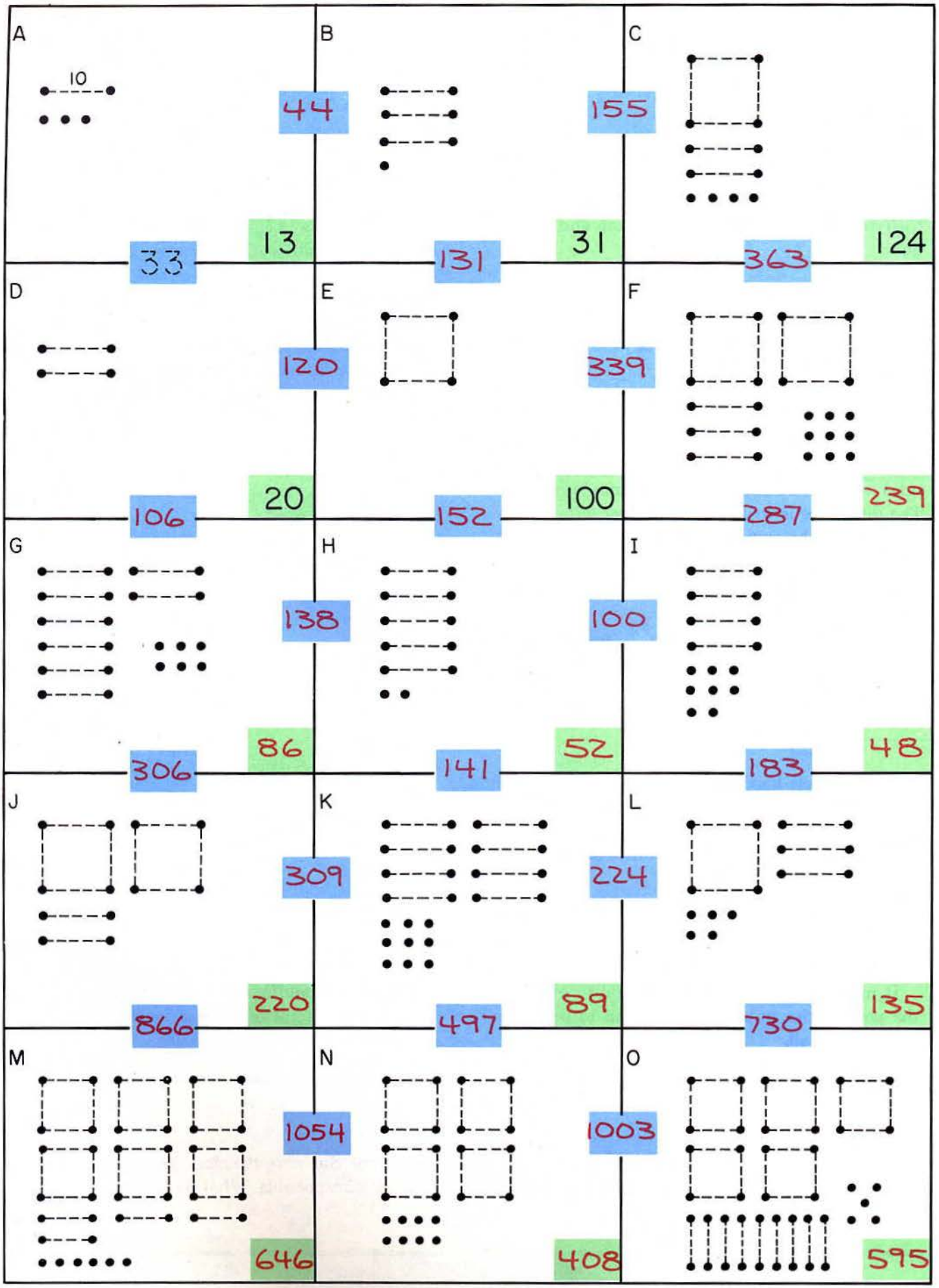
Drill

Here is a headline story:

1. Laura bought one stereo record for $3.49 and another record for $2.35.

Headline for the Day: Joyce paid $1.41 to buy a gift for George. She gave the clerk $2.00. The clerk gave her the correct change using the fewest coins possible. What was the change he gave to Joyce?

What Are My Rules? . . . Try to figure out the shorthand!



56 Translating shorthand notation to decimal notation and finding sums of numbers denoted on adjacent cards.

Tasks: *Pupils use decimal notation to record in each of the green boxes the number represented by the shorthand symbols in the white region containing that box. In each of the blue boxes pupils record the sum of the two numbers represented in the two white regions joined by that box.*

Purpose: To continue to explore numeration systems while simultaneously perfecting computational skills.

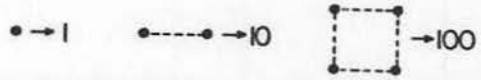
Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Ditto masters.

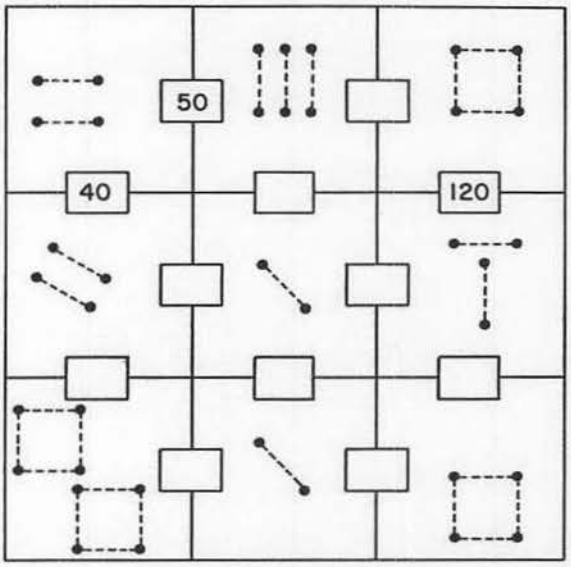
Materials for Pupils: Ditto sheets.

The Lesson:

In LEVEL B pupils were introduced to the shorthand notation for units, tens, and hundreds used on pupil page 56, as shown in the next column.



If needed, the following on a Ditto sheet or on the chalkboard will introduce pupil page 56 in a simple fashion:



more

Headline for the Day: In each row of exercises on page 56 there are 3 green boxes. In each row find the sum of the numbers in these boxes and record your results in the shorthand shown in the exercises.

You may wish to follow up pupil page 56 with oral drill:

30, ... plus 6 is ___.
30, ... plus 20, ... plus 6 is ___.
30, ... plus 26 is ___.
50, ... plus 30 is ___.
50, ... plus 38 is ___.
etc.

Try some rapid chalkboard drill:

30 + 16 = ___	26 + 37 = ___
20 + 16 = ___	36 + 37 = ___
40 + 16 = ___	46 + 37 = ___
60 + 16 = ___	46 + 47 = ___
70 + 16 = ___	56 + 37 = ___
80 + 16 = ___	56 + 47 = ___

Fewest Questions

An interesting game which has proved to be very exciting for youngsters is *Fewest Questions.*

Draw a number line chunk on the board listing the numbers from 1 through 32. Someone selects one of these 32 numbers and then the job is to guess the selected number by asking questions which must be answered either with a Yes or a No. Suppose that a pupil picks the number 9 and informs the class of his choice without letting you in on the secret. You ask:

1. Is your number greater than 16? (The pupil says No; and you erase the 17-32 section of the chunk.)
2. Is your number greater than 8? (Erase the 1-8 section when the pupil says Yes.)
3. Is your number greater than 12? (Erase the 13-16 section.)
4. Is your number greater than 10? (Erase the 11-12 section.)
5. Is your number 10? (The pupil says No; so you say that it must be 9.)

This questioning strategy will guarantee singling out the pupil's choice with no more than five questions. How many questions will guarantee a correct guess for a choice made from the numbers 1 through 16? 1 through 64? 1 through 8?

See "The Game of Fewest Questions" on pages 30 and 31 of the booklet *Games and Enrichment Activities.*

Practice with Cross Number Puzzles

400	60	8	
100	20	3	

500	40	5	
200	30	2	

600	70	9	
500	10	5	

700	70	7	
500	20	3	

800	50	7	
700	50	6	

900	80	3	
400	70	3	

300	90	9	
200	90	9	

600	60	6	
300	60	6	

			432
200	20	1	

			457
			324

Drill

Here are some stories for headline practice:

1. Otto saved 25¢. He wishes to buy a toy tea set for Melissa. The set costs 59¢.
2. Warren fell asleep at 4 o'clock. He awoke 7 hours later.
3. Andrea's new baby sister weighs only 8 pounds. Andrea weighs 37 pounds.
4. Betty's sister is 16 years old. Betty is 6 years older than her sister.
5. Sixty-three children were in the school orchestra. There were 80 children in the school who played musical instruments.

A suggestion for a developmental drill sheet on column subtraction of 3-digit numbers is shown on the following page.

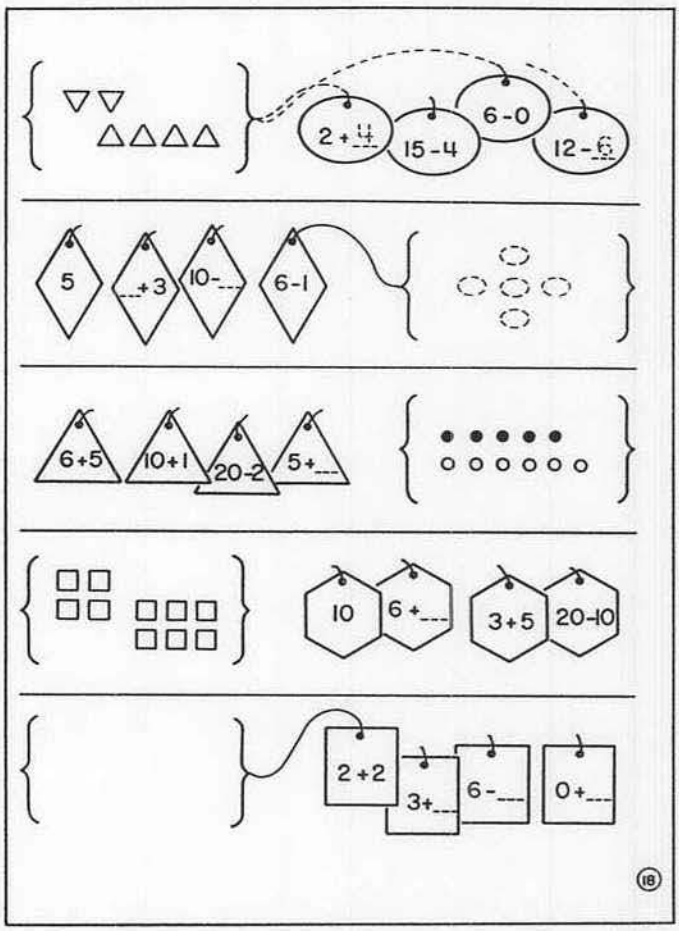
	I	II	III	IV	V
A	500 + 70 + 8 −(200 + 30 + 6) 300 + 40 + 2	578 − 236 342	400 + 30 + 4 −(100 + 20 + 1)	434 − 121	567 − 253
B	400 + 20 + 13 −(100 + 20 + 9)	433 − 129	800 + 60 + 11 −(200 + 10 + 4)	871 − 214	534 − 108
C	700 + 40 + 10 −(400 + 0 + 6)	750 − 406	300 + 0 + 13 −(100 + 0 + 4)	313 − 104	725 − 319
D	800 + 70 + 4 −(100 + 30 + 3)	874 − 133	600 + 40 + 10 −(500 + 10 + 7)	650 − 517	950 − 115
E	200 + 30 + 11 −(10 + 8)	241 − 18	500 + 20 + 13 −(100 + 20 + 8)	533 − 128	425 − 310
F	300 + 50 + 10 −(50 + 3)	360 − 53	200 + 100 + 6 −(200 + 90 + 6)	306 − 296	700 − 25
G	100 + 100 + 15 −(100 + 90 + 8)	215 − 198	700 + 50 + 11 −(200 + 10 + 9)	761 − 219	403 − 353
H	900 + 20 + 14 −(600 + 20 + 7)	934 − 627	500 + 100 + 4 −(200 + 70 + 1)	604 − 271	300 − 181

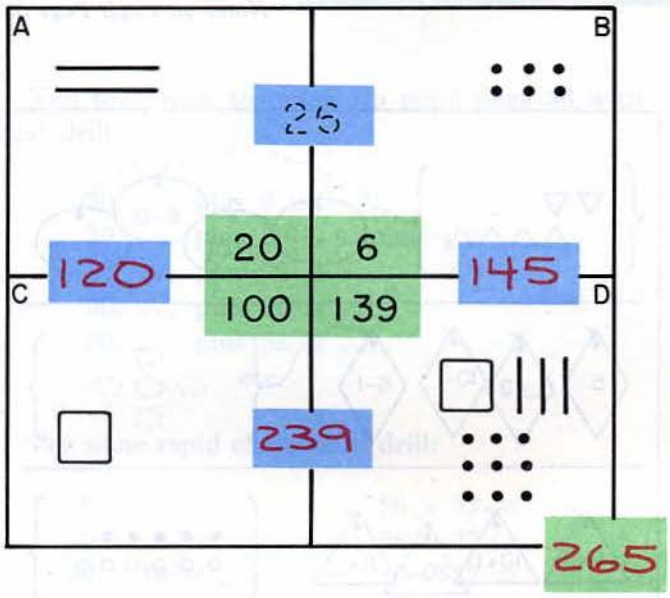
Rule: Subtract 4

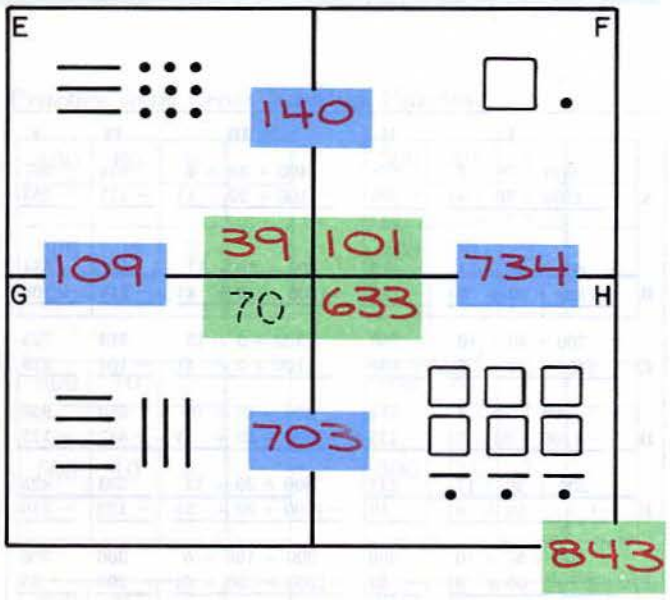
I

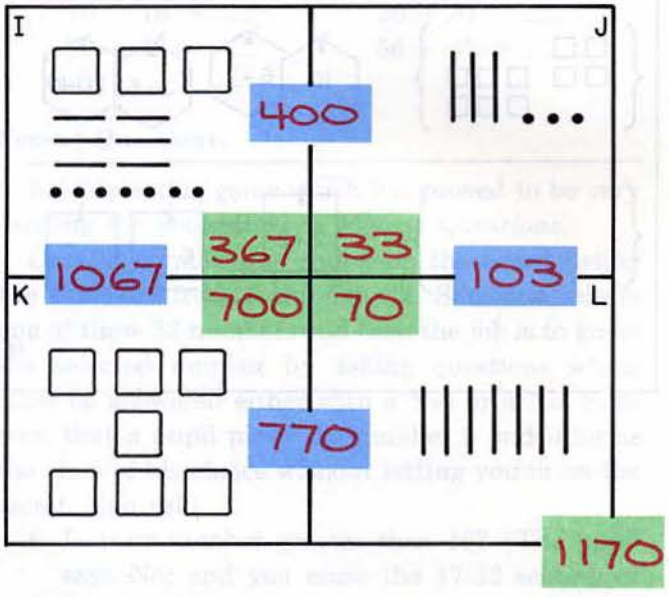
87, 83, , , , , 63, , , , ,

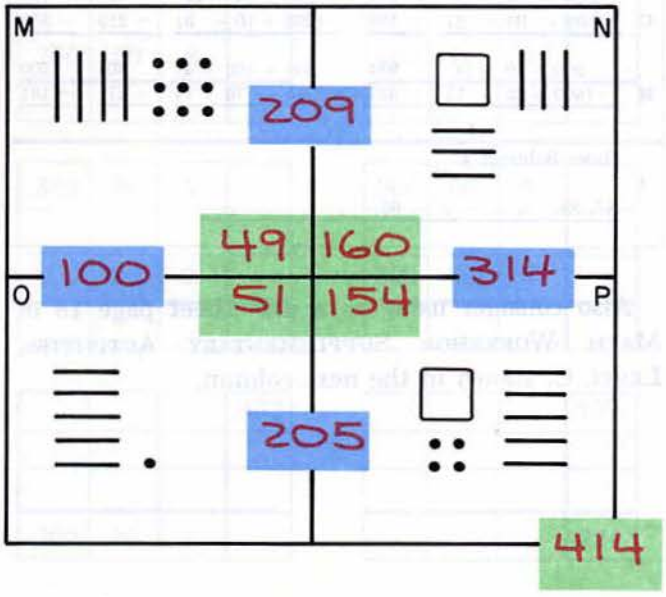
Also consider using as a drill sheet page 18 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown in the next column.

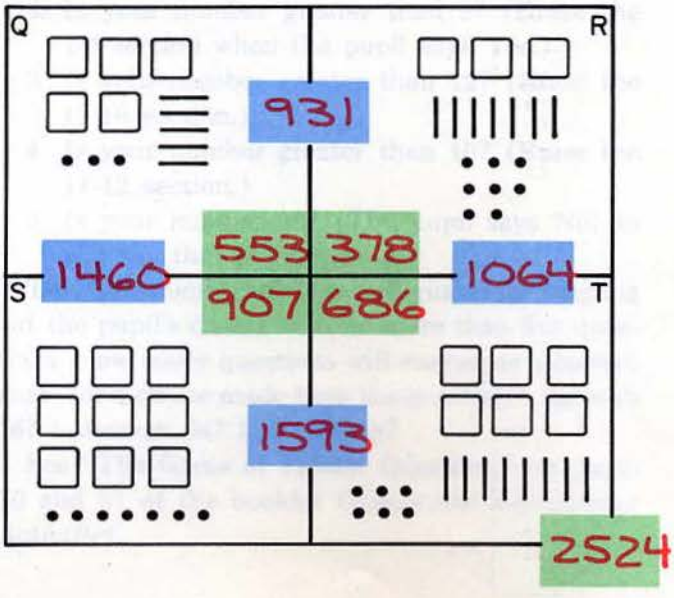


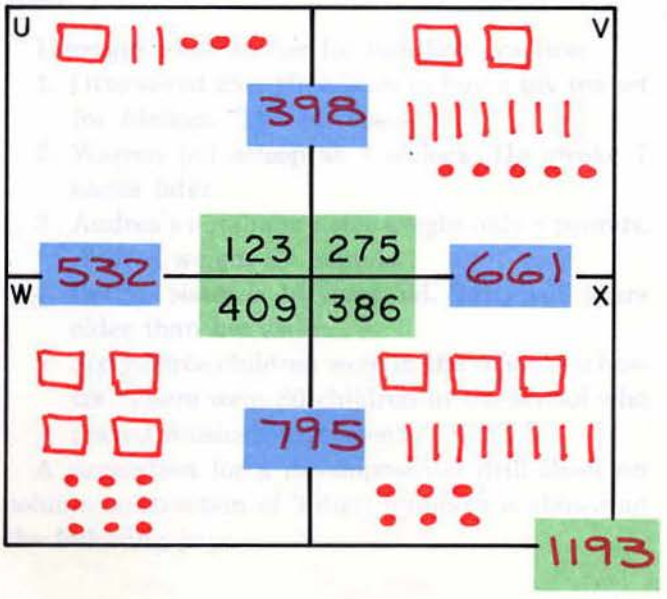

A
B
26
20
6
120
145
C
D
100
139
239
265


E
F
140
39
101
109
734
G
H
70
633
703
843


I
J
400
367
33
1067
103
K
L
700
70
770
1170


M
N
209
49
160
100
314
O
P
51
154
205
414


Q
R
931
553
378
1460
1064
S
T
907
686
1593
2524


U
V
398
123
275
532
661
W
X
409
386
795
1193

57 Translating shorthand to decimal notation and conversely, and finding sums of numbers denoted on adjacent cards.

Tasks: *Pupils use decimal notation to record in the green center boxes the numbers represented by the shorthand symbols in the white regions. In the green box at the lower right corner of each group of four white regions pupils record the sum of the numbers pictured in those regions. In each of the blue boxes pupils record the sum of the two numbers pictured in the two white regions joined by that box.*

Purpose: To continue to explore numeration systems while perfecting computational skills.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

Note the condensed shorthand used on this page. Compare it with that used on pupil page 56.

Here are further activities which support and extend the ideas developed on this page.

Make the Sentences True

○ + 0 = 0 + 100	30 + △ = 60 + 30
100 + 50 = △ + 100	⬡ + 50 = 50 + 60
90 + 60 = 60 + ⬡	30 + 50 = 50 + ○
80 + ○ = 40 + 80	20 + 10 = □ + 20
40 + 70 = □ + 40	130 + □ = □ + 130

Partitioning and Combining

54 + 5 = 50 + □ = △
25 + 3 = 20 + □ = △
72 + 6 = 70 + □ = △
56 + 3 = 50 + □ = △
63 + 4 = 60 + □ = △
15 + 2 = □ + 7 = △
28 + 1 = □ + 9 = △
46 + 3 = □ + 9 = △
35 + 24 = □ + 9 = △
86 + 12 = □ + 8 = △
16 + 31 = □ + 7 = △
24 + 14 = □ + 8 = △
32 + 47 = 70 + □ = △
58 + 21 = 70 + □ = △
43 + 34 = 70 + □ = △
65 + 21 = □ + ○ = △

Drill

Here is a suggestion for a drill sheet on addition:

	I	II	III	IV	V	VI
A.	3	9	6	7	4	9
	7	1	4	5	5	5
	2	5	2	5	2	1
	8	7	8	3	8	1
	6	3	3	8	1	5
	+ 4	+ 5	+ 3	+ 4	+ 9	+ 9
B	12	25	17	19	28	72
	8	14	15	15	1	3
	1	5	3	15	29	2
	9	6	2	15	17	3
	17	3	12	16	14	9
	+ 3	+ 3	+ 8	+14	+ 6	+ 9
C	17	54	18	84	65	21
	27	5	23	24	48	22
	33	51	32	5	5	23
	3	9	13	5	34	24
	16	1	23	56	1	25
	+16	+ 1	+33	+ 6	+75	+26
D	52	81	38	45	57	49
	54	62	38	42	57	59
	56	43	34	49	56	67
	+58	+24	+33	+49	+59	+72
E	43	89	54	63	28	97
	49	2	54	63	28	97
	+47	+ 9	+54	+63	+28	+97

Here are two stories for headline practice:

1. Arlene had 57 pictures for her new album. She put 6 pictures on each page.
2. Paulette baked 3 dozen cookies. She needed 5 dozen for the class party.

Headline for the Day: Make a headline with the numbers in the five green boxes for white cards A, B, C, D on page 57. Tell a story for the headline.

What Are My Rules?

A chain reaction — to the right.

a.

10	+5	15	-2	13

b.

13	+4	17	+3	20

c.

17	-7	10	-4	6	+10	16

d.

11	-4	7	+3	10

e.

9	+2	11	+4	15

f.

3	+10	13	+5	18	+2	20	-5	15	+10	25

A chain reaction — down.

g.

18
+ 4
22
+10
32

h.

30
+20
50
-10
40

i.

24
- 7
17
- 9
8

A chain reaction — down to the right.

k.

20				
	-3			
		17		
			+5	
				22

l.

15				
	-6			
		9		

j.

25				
	-5			
		20		

A chain reaction — 3 ways!

m.

20	- 4	16	+ 3	19
- 8	-14	- 10	+13	+10
12	-6	6	+23	29

n.

15	+5	20	+ 8	28
- 5	+10	+ 5	+15	+ 7
10	+ 15	25	+10	35

o.

30	- 10	20	-10	10
- 5	-15	- 5	-15	- 5
25	- 10	15	-10	5
-4	-14	-4	- 14	-4
21	-10	11	-10	1

p.

12	+10	22	+ 9	31
+9	+ 18	+8	+19	+10
21	+ 9	30	+11	41
-7	+ 2	-7	+ 0	-11
14	+ 9	23	+7	30

58 Addition and subtraction Chain Reactions to provide computational practice.

Tasks: *Pupils use the given directions and the arrows to determine and record the missing numbers and operations. For the bottom of the page pupils use all three rules (as indicated by the arrows) that were used in the individual exercises in the top half of the page in order to complete the four charts* m-p.

Purpose: To continue perfecting computational skills via the vehicle of chain reaction games.

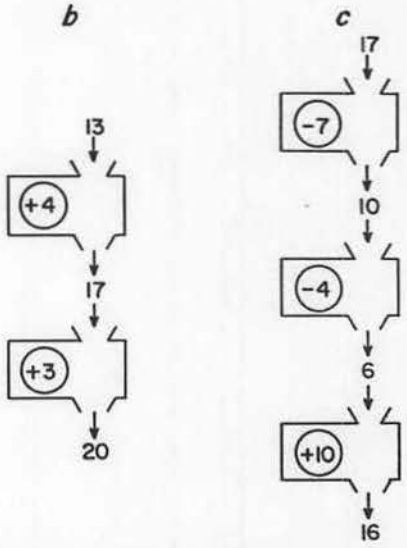
Unifying Ideas: Structure; addition and subtraction; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

A "chain reaction" is nothing more than a compact representation of a "hookup" of two or more addition or subtraction machines. For example, chain reactions *b* and *c* are equivalent to the following hookups:



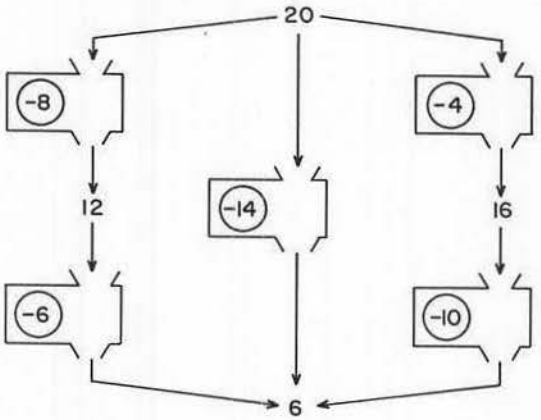
As far as initial input and final output are concerned, any such hookup can be replaced by a single machine or by another hookup. For example, the hookup for *b* can be replaced by a + 7 machine or

more

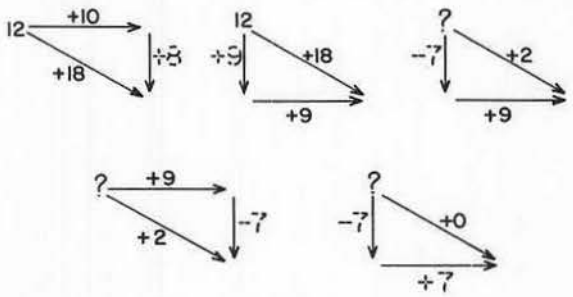
Headline for the Day: Tell a story to fit the fact team: $225 + 75 = \square$ $\square - 75 = 225$ $75 + 225 = \square$ $\square - 225 = 75$

by a hookup of a + 10 machine followed by a − 3 machine. In each case the initial input 13 is transformed into the final output 20.

The interchangeability of hookups is played upon in chain reactions *m-o*. For example, restricting our attention to the first three columns of *m*, we use them as a compact representation of the fact that the − 14 machine does the same job as the hookup of a − 8 machine followed by a − 6 machine and as the hookup of a − 4 machine followed by a − 10 machine:



If you represent an addition or subtraction operation by an arrow then a path of arrows from one number to another is equivalent to any other path from the first number to the second. For example, in chain reaction *p*, we can use this idea to find several of the missing operations (white boxes) without computing any of the numbers for the tinted boxes:



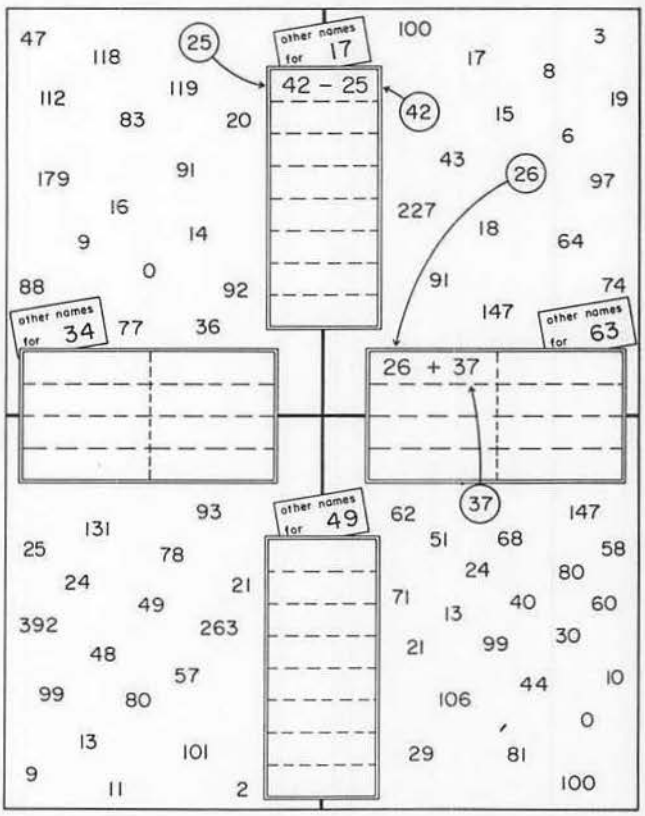
We can now record these five operations (ghosted printing) in the table for *p*:

12	+10		+9	
÷9	+18	÷3	A	B
	+9		C	41
−7	+2	−7	+0	D
	+9		÷7	

The fact that we are given the 41 in that position determines the problem completely. Even without that condition, there are still limitations on the operations A, B, C, and D. For example, C and D must be inverses of each other, for hooking up C and D must be equivalent to adding 0. Also, whatever operation C is, it must amount to adding 1 more than B because adding 9 amounts to adding 1 more than adding 8.

Numbers Have Many Names

Here is an independent exercise you can put on a Ditto sheet. It explores the idea that there are many different ways to obtain a number. The pupil is to select a pair of numbers, one from each of two neighboring groups, such that their sum or difference names the numbers designated:



Drill

Here are some stories for headline practice:

1. Kate's mother received 23 cents change from the grocery store clerk. She gave the clerk $3.50.
2. There were 37 children in the class. Seventeen of these children were girls.

3. Kevin's grandfather lived to be 94 years old. Kevin thought that in 84 years he would be that old.

Here is a suggestion for a drill sheet:

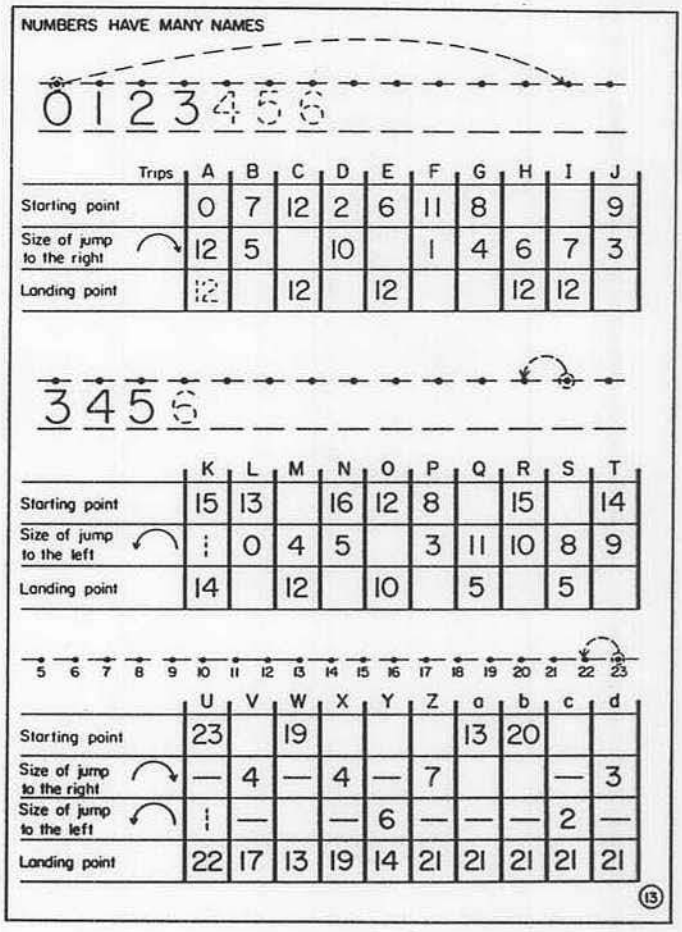
	I	II	III	IV	V
A	400+100+7 −(100+ 80+6) + +	507 −186	300+100+0 −(80+0)	400 − 80	200 − 40
B	700+110+3 −(200+ 90+1)	813 −291	400+120+5 −(300+ 60+4)	525 −364	254 − 63
C	800+140+13 −(100+ 60+ 7)	953 −167	500+ 60+10 −(200+ 10+ 7)	570 −217	482 −109
D	500+ 30+10 −(200+ 20+ 2)	540 −222	700+ 60+9 −(400+ 60+2)	769 −462	375 − 80
E	700+100+8 −(600+ 90+1)	808 −691	90+10 −(70+ 2)	100 − 72	300 −165
F	900+100+0 −(200+ 80+0)	1000 − 280	500+140+17 −(100+ 90+ 9)	557 −199	657 −200
G	7035 −3029	15,700 − 8,650	125,487 −115,309	125 − 65	217 − 19
H	500 − 38	318 −120	745 − 39	745 − 58	745 −259

Rule: Subtract 6

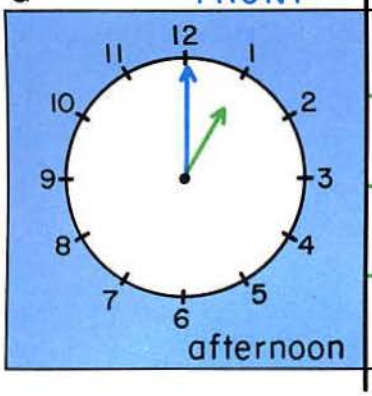
I

179,173, , , , , ,137, , , ,

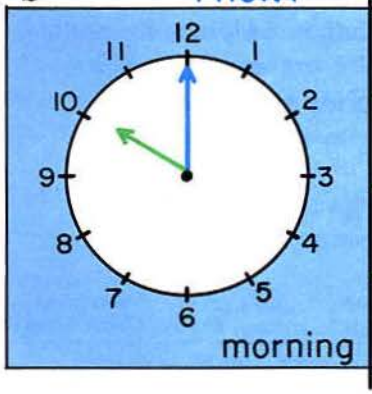
Also consider using as a drill sheet page 13 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:


NUMBERS HAVE MANY NAMES
0 1 2 3 4 5 6
Trips | A | B | C | D | E | F | G | H | I | J
Starting point | 0 | 7 | 12 | 2 | 6 | 11 | 8 | | | 9
Size of jump to the right | 12 | 5 | | 10 | | 1 | 4 | 6 | 7 | 3
Landing point | 12 | | 12 | | 12 | | | 12 | 12 |
3 4 5 6
| K | L | M | N | O | P | Q | R | S | T
Starting point | 15 | 13 | | 16 | 12 | 8 | | 15 | | 14
Size of jump to the left | ? | 0 | 4 | 5 | | 3 | 11 | 10 | 8 | 9
Landing point | 14 | | 12 | | 10 | | 5 | | 5 |
5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
| U | V | W | X | Y | Z | a | b | c | d
Starting point | 23 | | 19 | | | | 13 | 20 | | |
Size of jump to the right | — | 4 | — | 4 | — | 7 | | | — | 3
Size of jump to the left | ? | — | | — | 6 | — | — | — | 2 | —
Landing point | 22 | 17 | 13 | 19 | 14 | 21 | 21 | 21 | 21 | 21
13

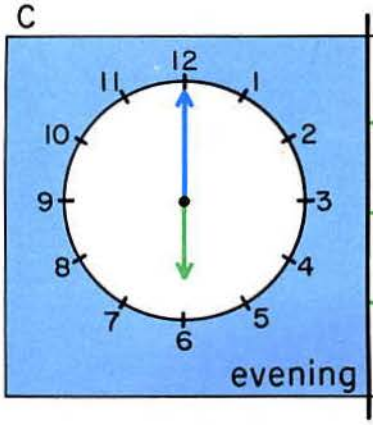
a

FRONT	BACK
 afternoon	I 1:00 PM II 2:00 PM III 12:30 PM IV 3:15 PM

b

FRONT	BACK
 morning	I 10:00 AM II 11:00 AM III 9:30AM IV 12:15 PM

c

FRONT	BACK
 evening	I 6:00 PM II 7:00 PM III 5:30 PM IV 8:15 PM

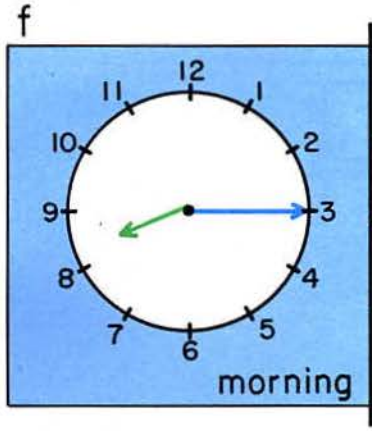
d

FRONT	BACK
afternoon	I 1:15 PM II 2:15 PM III 12:45 PM IV 3:30 PM

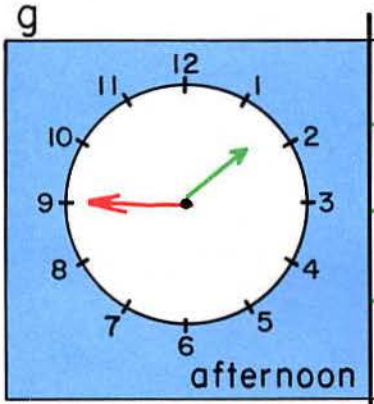
e

FRONT	BACK
afternoon	I 2:30 PM II 3:30 PM III 2:00 PM IV 4:45 PM

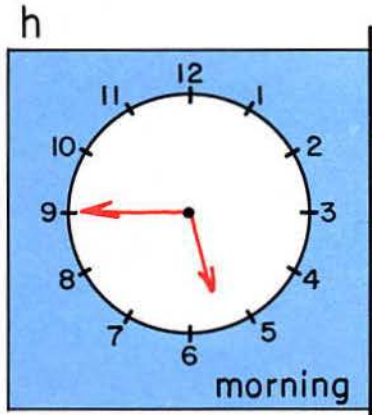
f

FRONT	BACK
 morning	I 8:15 AM II 9:15AM III 7:45AM IV 10:30AM

g

FRONT	BACK
 afternoon	I 1:45 PM II 2:45 PM III 1:15 PM IV 4:00 PM

h

FRONT	BACK
 morning	I 5:45AM II 6:45 AM III 5:15AM IV 8:00AM

59 WAMR cards of time problems dealing with the +1, -30, and +2 1/4 operations.

Tasks: *Pupils use the clues provided in* a *to complete* b-h.

Purpose: To review and extend concepts of time.

Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Demonstration clock.

Materials for Pupils: Desk-size clocks.

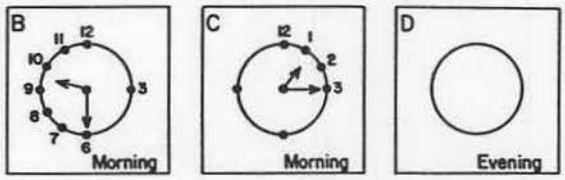
The Lesson:

If pupils are not ready to do this page on their own or in teams, you may wish to begin by doing some work at the chalkboard that parallels the work on pupil page 59.

	Rule I	Rule II	Rule III
A	3 P.M.	3:30 P.M.	6 P.M.
B			
C			
D		9:15 P.M.	

As pupils study this picture and chart, ask: How do we get the time under Rule I by studying the clock? (It's the indicated time.) Rule II?. (30 minutes past the indicated time.) Rule III? (3 hours past the indicated time.)

Now draw three new clockfaces and see who can fill in the times under each rule above.



Pupils should now be ready for pupil page 59 with few needing additional clues.

Drill

Here is a suggestion for a drill sheet:

$(30+6)+(20+9)=\square+(6+9)=\triangle+5$
$(20+3)+(40+9)=\square+(3+9)=\triangle+2$
$(30+5)+(20+8)=\square+(5+8)=\triangle+3$
$(40+3)+(30+8)=\square+(3+8)=\triangle+1$
$(60+4)+(20+8)=\square+(4+8)=\triangle+2$
$(30+5)+(40+7)=70+(5+\square)=\triangle+2$
$(20+8)+(60+7)=80+(\square+7)=90+\triangle$

Here are some stories for headline practice:

1. Mollie made 8 shoes for her dolls.
2. Janet went on a boat trip. She traveled 86 miles to an island and 87 miles back to the pier.
3. Three girls and 2 boys looked for insects. They each found 5 beetles and 2 ladybugs.

Headline for the Day: Tell a story to fit one of cards *a-h* on page 59.

3-Way Chain Reactions

a.

8	+7	15	−2	13
+5	+15	+8	+18	+20
13	+10	23	+10	33

b.

15	+10	25	+7	32
+5	+25	+15	+27	+20
20	+20	40	+12	52

c.

18	+9	27	+8	35
+5	+15	+6	+18	+10
23	+10	33	+12	45

d.

25	+2	27	+20	47
+10	+18	+16	+15	−5
35	+8	43	−1	42
+6	+16	+8	+19	+20
41	+10	51	+11	62

Helen bought 2 things from the school supplies listed. They were all different. What can you say?

School Supplies

Paper	8¢
Crayons	15¢
Paints	18¢
Notebook	25¢
Pencil Box	27¢

If she bought crayons and a notebook then she spent 40¢.

If she bought paper and crayons then she spent 23¢.

Use the chart below to explore the purchases of 2 different items Helen might have made.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
Paper — 8¢	1	1	1	1	0	0	0	0	0	0	
Crayons — 15¢	1	0	0	0	1	1	1	0	0	0	
Paints — 18¢	0	1	0	0	1	0	0	1	1	0	
Notebooks — 25¢	0	0	1	0	0	1	0	1	0	1	
Pencil Box — 27¢	0	0	0	1	0	0	1	0	1	1	
Total of purchases	23¢	26¢	33¢	35¢	33¢	40¢	42¢	43¢	45¢	52¢	¢

60 3-way Chain Reactions involving addition and subtraction - application problems dealing with tabulations of possible purchases.

Tasks: *Pupils use the clues provided by the arrows to complete each chain reaction table. For the lower part of the page pupils record in the table the numbers of various items which can be purchased and the cost of these purchases.*

Purpose: To explore an application via the vehicle of *What Can You Say?* To continue to perfect computational skills by using chain reaction games.

Unifying Ideas: Addition and subtraction; measurement; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Comments in Notes on Pupil Page 58 are pertinent to the activities at the top of pupil page 60.

Review our earlier discussion on *What Can You Say?* activities (see Notes on Pupil Page 19) before proceeding with this page.

One important thing for you to note and to help pupils note is that the cost of most of the combinations of two items is given in the chain reaction problems.

Consider some of the combinations with the class as mental arithmetic and then check to see if the corresponding arithmetic can be found in the chain reaction problems.

Now ask: What two-item combinations cost 26¢? 33¢? 45¢? 52¢?

As a final step discuss the method of recording possible purchases in the chart. (Note the way we have ordered the answers: 11000 > 10100 > 10010 > 10001 > 1100 > 1010 > 1001 > 110 > 101 > 11.)

We are interested in answers to the following questions in completing the table:

How many possible two-item combinations are there? (10)

Is any amount repeated? (Yes—33¢)

Drill

Here is a suggestion for a drill sheet:

$(80+4)-6=(70+\square)-6=70+\bigcirc$
$(90+3)-5=(80+\square)-5=80+\bigcirc$
$(50+7)-8=(40+\square)-8=40+\bigcirc$
$(20+9)-3=(10+\square)-3=10+\bigcirc$
$(20+9)-3=20+(\square-3)=20+\bigcirc$
$(20+2)-7=(10+\square)-7=10+\bigcirc$
$(50+6)-9=(40+\square)-9=40+\bigcirc$
$(70+3)-8=(\square+13)-8=\bigcirc+5$

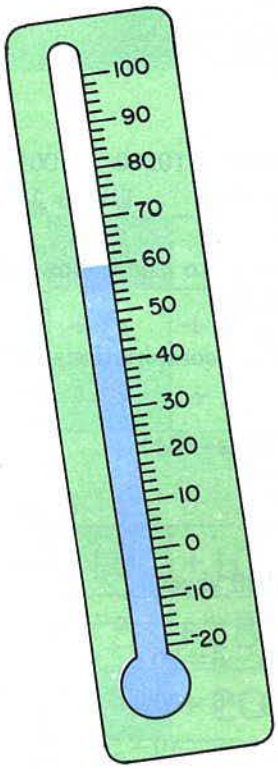
Here are some stories for headline practice:

1. Janice had 42 balloons for her birthday party. There were 8 children at the party. Each child got the same number of balloons when the party was over.

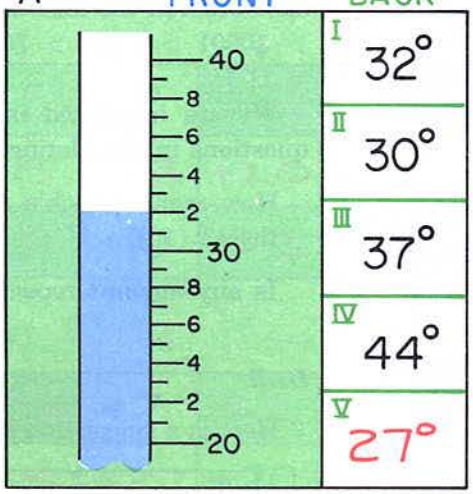
   $(8\times\square)+\bigcirc=42$ $\qquad$ $42=(8\times5)+2$

2. Jan's third-grade class owns 4 turtles. Jan fed each turtle 5 flies in the morning and 3 flies in the afternoon.
3. Polly gave her friend a dollar's worth of nickels.

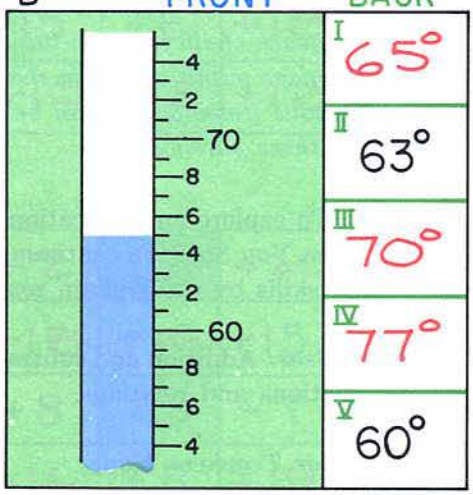
Headline for the Day: For the possible purchases listed in each of columns I-XI in the table on page 60, and using the fewest coins possible, tell the correct change Helen should receive if she gave the clerk one dollar.



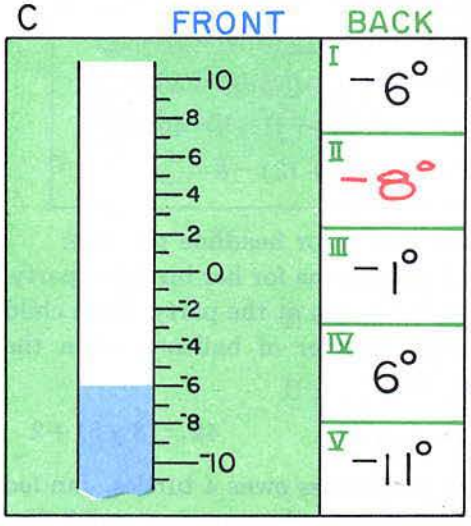
A



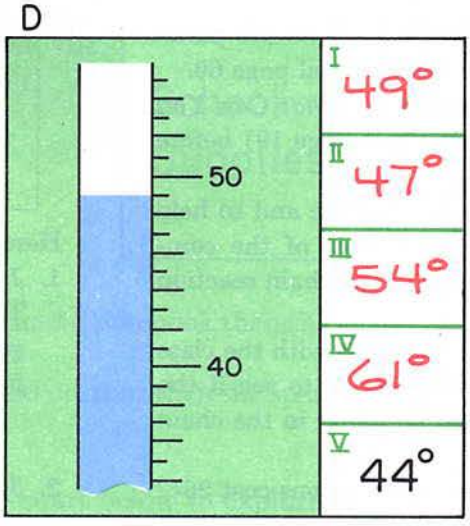
B



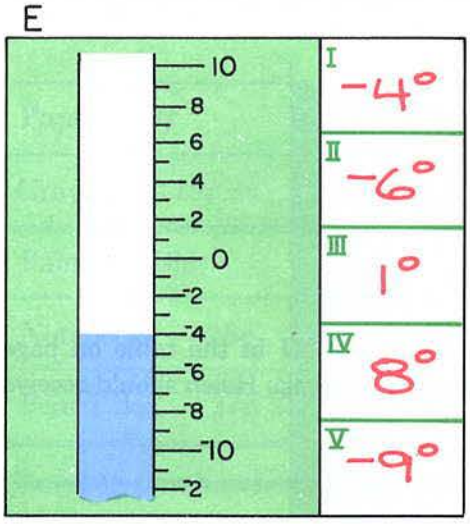
C



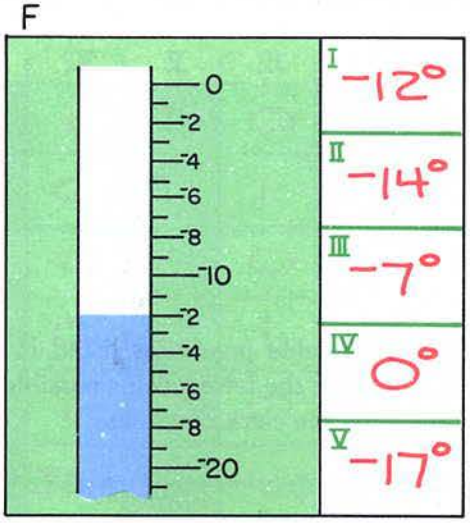
D



E



F



Extra –

Another game of What's My Rule?

5	10
7	14
10	20
15	30
12	24
0	0
25	50
50	100
m	2m

61 WAMR cards of temperature readings dealing with the -2, +5, +12, and -5 operations - What's My Rule?

Tasks: *Pupils use the clues provided to complete A-F. For the game at the lower right of the page, pupils use the clues provided by the first two rows of the table, or the pattern indicator in the last row, to complete the table.*

Purpose: To explore another application, temperature measurement.

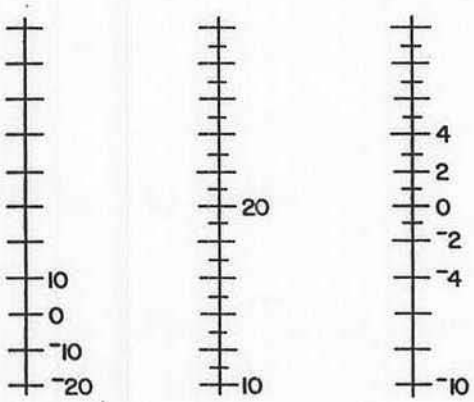
Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Demonstration-size thermometer with movable red tape to show different temperature readings; Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

Preparation for this page might include a discussion of the numbering technique found on a thermometer. It would be well to have pupils attempt to describe this. Then have pupils help you number some lines on the board or on prepared Ditto sheets.



A classroom-size thermometer made with a red tape can be most helpful in this work.

Indicate on a number line or on the classroom thermometer a given temperature. Have pupils record what happens when the temperature goes up 10°; up 15°; down 5°; down 12°. Keep a record:

Temperature	Rule I up 10°	Rule II up 15°	Rule III down 5°	Rule IV down 12°
0°	10°	15°	−5°	−12°
10°	20°	25°	5°	−2°
5°				
−10°				

With this practice, pupils should be ready for pupil page 61.

Drill

Here is some developmental drill work:

$$(80+4)-(20+6)=(70+\square)-(20+6)$$
$$=(70-20)+(\square-6)$$
$$=\triangle$$
$$(60+2)-(30+9)=(50+\square)-(30+9)$$
$$=(50-30)+(\square-9)$$
$$=\triangle$$
$$(70+2)-(40+8)=(60+\square)-(40+8)$$
$$=(60-40)+(\square-8)$$
$$=\triangle$$

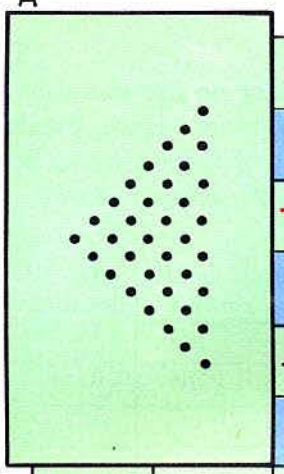
$34-16=\triangle$ $\quad$ $52-29=\triangle$ $\quad$ $82-58=\triangle$

Here are some stories for headline practice:

1. Jody's class had 5 fish bowls. Each fish bowl held 5 guppies and 2 angelfish.
2. Jason has 12 marbles. He grouped his marbles by 2's and said, "1 out of every 2 is red."
3. Mr. Jackson had 18 roses. He put 6 in each vase.

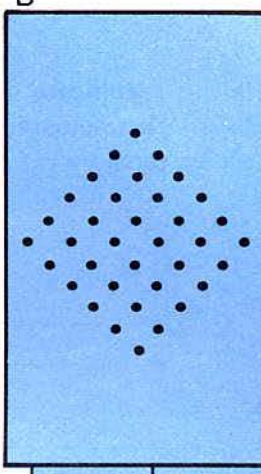
Headline for the Day: Tell a story to fit each of cards A-F on page 61.

A



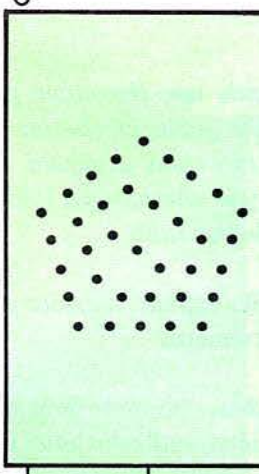
		+45
		81
		+78
		114
		+120
		156
+136	+171	+210
172	207	246

B



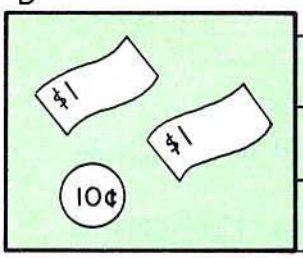
		+49
		85
		+100
		136
		+169
		205
+225	+324	+400
261	360	436

C



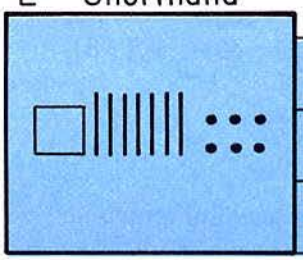
		+35
		70
		+51
		86
		+70
		105
+92	+117	+145
127	152	180

D



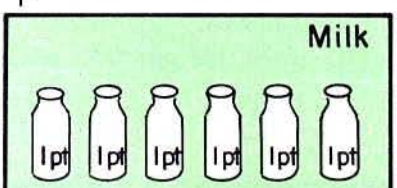
twice as much	$4.20	enough more to make $5.00	$2.90
half as much	$1.05	nine dimes more	$3.00
a half-dollar less	$1.60	nine dimes less	$1.20

E Shorthand



twice as many	352	enough more to make 250	74
half as many	88	seventy-six more	252
one hundred thirty-seven less	39	seventy-seven less	99

F



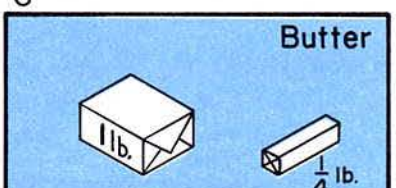
twice as much

1 gal., 2 qts.

half as much

1 qts., 1 pts.

G



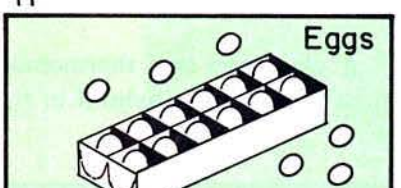
twice as much

2 lbs., 8 oz.

half as much

0 lbs., 10 oz.

H



twice as much

3 doz., 0 eggs

half as much

0 doz., 9 eggs

62 A variety of practice situations involving addition, subtraction, multiplication, division, and applications.

Tasks: *In each exercise pupils first determine the number or quantity represented by the picture and then carry out the required computational tasks to fill the blanks.*

Purpose: To explore a variety of situations, many of which are applications, while simultaneously perfecting computational skills.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: Pint, quart, and gallon containers; egg cartons; play money.

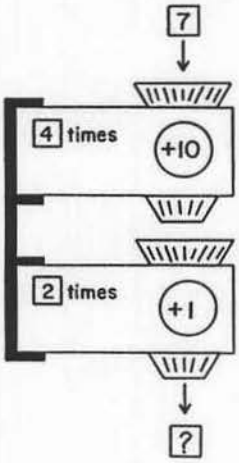
Materials for Pupils: Scratch paper.

The Lesson:

Pupils will get a real workout in computation in exercises A-E. They will probably need scratch paper to carry out their computations.

Adding Machines

An interesting class enrichment activity is to use a hooked-up pair of imaginary machines for adding 10 and adding 1. Billy sets the "times dial" on the machine for adding 10 and Susan sets it on the machine for adding 1. (The machines are now ready to add 42.) Fred picks the starting number. Charles describes the sequence of steps:



7, 17, 27, 37, 47, 48, 49

Amy writes the long number sentence:

$$7 + 10 + 10 + 10 + 10 + 1 + 1 = 49$$

Neal writes the short number sentence:

$$7 + 40 + 2 = 49$$

Michael writes the very short number sentence:

$$7 + 42 = 49$$

This activity can be varied by interchanging the positions of the two machines, by putting a 0 on a times dial, or by using machines for subtracting.

Some questions which arise during this activity are:

What times dial settings should you make if you wish to add 62? 26? 33?

Suppose that the times dials can't go higher than 9. How would you add 100 to a given number? (First set both dials at 9 and let the machines work on the given number. Then set the dial at 0 on the machine for adding 10, set the dial at 1 on the machine for adding 1, and let the machines work on the previous output.)

Drill

Here are some stories for headline practice:

1. Natalie invited 9 friends to her birthday party. She gave each friend 5 balloons to hang.
2. Yolanda had in her bank 2 quarters, 3 dimes, and a nickel.
3. Pedro has 12 marbles. He grouped his marbles by 4's and said, "2 out of every 4 are blue."

Headline for the Day: Fill in the blanks to continue the pattern.

55, 45, 36, 28, ___, ___, ___, ___, 3, 1

Alice had five silver coins, all alike, and each coin was less than a dollar. She spent some of this money. What can we say?

First, we know that she had either

5 nickels worth 25 ¢, or

5 dimes worth 50 ¢, or

5 quarters worth $1.25 or 125 ¢, or

5 half-dollars worth $2.50 or 250 ¢.

If Alice had 5 nickels	
and spent ¢	then she has _____ ¢ left.
20	5
4	21
10	15
6	19
15	10
9	16
12	13
18	7
7	18
0	25
21	4
24	1

If Alice had 5 dimes	
and spent ¢	then she has _____ ¢ left.
20	30
10	40
5	45
2	48
7	43
12	38
19	31
9	41
50	0
41	9
18	32
35	15

If Alice had 5 quarters	
and spent ¢	then she has _____ ¢ left.
20	105
100	25
50	75
15	110
115	10
17	108
30	95
35	90
40	85
53	72
60	65
98	27

If Alice had 5 half-dollars	
and spent ¢	then she has _____ ¢ left.
20	230
200	50
100	150
50	200
75	175
80	170
55	195
150	100
51	199
242	8
149	101
125	125

What Are My Rules?

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	
17	30	22	20	34	70	27	36	15	24	0	x+y
10	25	13	14	26	50	20	31	8	12	0	x
7	5	9	6	8	20	7	5	7	12	0	y
3	20	4	8	18	30	13	26	1	0	0	x−y

63 Applications dealing with purchases and change making - What Are My Rules?

Tasks: *Pupils record at the top of the page the values of the various possibilities for coins suggested in the story. Pupils then complete a table for each of the four possibilities. At the bottom of the page pupils complete the table by using clues provided by the first three columns or by using the pattern indicators in the extreme right-hand column.*

Purpose: To explore an application via the vehicle of *What Can You Say?*

Unifying Ideas: Structure; addition and subtraction; functions and relations; measurement.

Materials for Teacher: See Notes on Pupil Page 15.

Materials for Pupils: Graph paper.

The Lesson:

Prepare pupils for this page with a simpler *What Can You Say?* game: If I have ten pennies in my two pockets, what can you say? Perhaps someone will start by saying that if you had all the pennies in the left pocket then you would have no pennies in your right pocket. Begin keeping a record:

Left	Right	
10	0	$10 + 0 = 10$
5	5	$5 + 5 = 10$
1	9	$1 + 9 = 10$

Of course there will be 11 possible arrangements.

Suppose that I made a purchase in a store and gave the clerk a dollar. The clerk gave me change in silver. What can you say? Begin keeping a record to show the possible purchases.

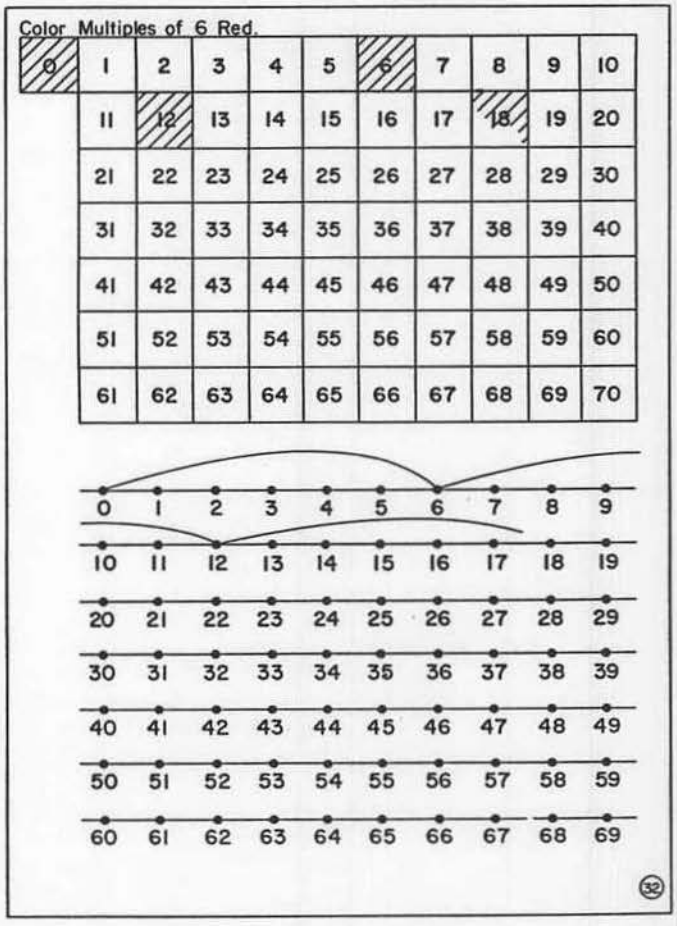
Change

Nickels	Dimes	Quarters	Half-dollars	Amount of change	Cost of purchase
1	1	1	1	90¢	10¢
0	0	3	0	75¢	25¢
etc.					

As we described in Notes on Pupil Page 15, these exercises lend themselves readily to graphing.

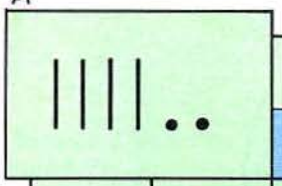
Drill

Consider using as a drill sheet page 32 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



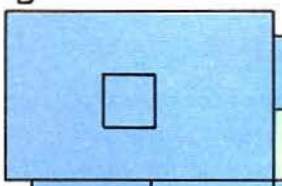
Headline for the Day: Tell stories to fit a pair of headlines (for example: $10 + 7 = 17$ and: $10 - 7 = 3$) in columns 1-11 of the *What Are My Rules?* table on page 63.

A



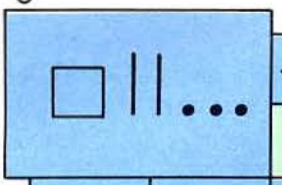
		−12	−20	+9
		30	22	51
−9	+35	−31	+18	−17
33	77	11	60	25

B



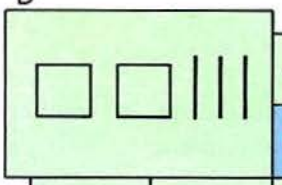
		−10	+28	−50
		90	128	50
+100	+154	−70	−25	−13
200	254	30	75	87

C



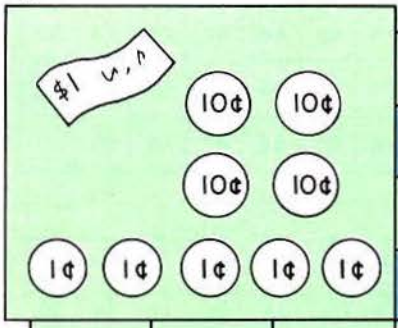
		−100	−20	−3
		23	103	120
−50	−110	−33	−93	−25
73	13	90	30	98

D



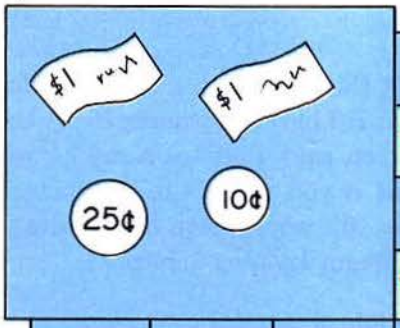
		−50	+75	−140
		180	305	90
+70	+308	−2	−24	−36
300	538	228	206	194

E



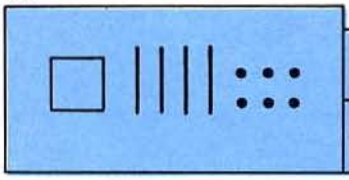
			+10¢	−45¢
			$1.55	$1.00
			−50¢	−16¢
			95¢	$1.29
+55¢	+25¢	−$1.18	−95¢	+$1.05
$2.00	$1.70	27¢	50¢	$2.50

F



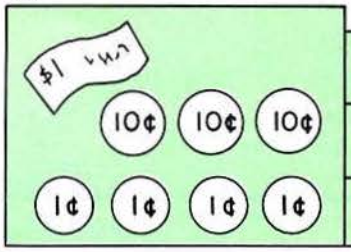
			−35¢	−$1.25
			$2.00	$1.10
			+75¢	+90¢
			$3.10	$3.25
−$2.05	−50¢	−$1.50	−57¢	$1.08
30¢	$1.85	85¢	$1.78	$1.27

G



twice as many	292	three times as many	438
half as many	73	fifty less	96

H



twice as much	$2.68	enough more to make $5.00	$3.66
half as much	67¢	nine dimes more	$2.24
a half-dollar less	84¢	nine dimes less	44¢

64 A variety of practice situations involving addition, subtraction, multiplication, division, and applications.

Tasks: *In each exercise pupils determine the number or quantity represented by the picture, and then determine the missing numbers and operations.*

Purpose: To explore a variety of situations (many of which are applications) in which to practice computational skills.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: Ditto masters.

Materials for Pupils: Scratch paper; Ditto sheets.

The Lesson:

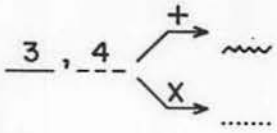
Much computation is required for this page. Some pupils may use the standard algorithms for addition and subtraction in A-F. But others will undoubtedly respond to the quasi-concrete settings and use more primitive but just as acceptable procedures, doing most of the problems without scratch paper. For example, in E and F it is very tempting to manipulate coins (mentally, that is) and thus find answers. Try it.

For exercises G and H, since pupils have not learned a standard algorithm, they must use some other means of finding the answer. For example, to compute the result of doubling 146 in the first part of G, a pupil would probably think: 200 and 80 and 12 make 292. The value of this as a foreshadowing of an algorithm is evident.

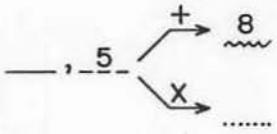
A Game with Addends-Sums, Factors-Products

This game depends on the fact that when a pair is associated with a sum and associated with a product, four numbers are involved—the components of the pair, the sum, and the product. If any two are given, can you find the other two? (Yes, unless 0 is given as one of the pair and as the product.)

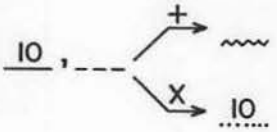
There are many ways to ask for the missing numbers. Here are examples of one way of asking:



Answer: Sum must be 7; product must be 12.



Answer: Missing component of the pair is 3; missing product must be 15.



Answer: Missing component of the pair must be 1; missing sum must be 11.

more

Headline for the Day: Make a table to record the various ways of making up 75¢, using half-dollars, quarters, dimes, and nickels. (Hint: There are 20 different ways.)

In another form, the missing numbers may be asked for in this way:

	Product	Pair	Sum	
A		3, 2		Missing product is 6; missing sum is 5.
B	16		10	Missing pair must be (2, 8) or (8, 2).
C	13	1,		Missing component of the pair must be 13; sum must be 14.
D		3,	6	Missing component of the pair must be 3; product must be 9.
E	7			Missing pair must be (1, 7) or (7, 1); sum must be 8.

(When a given product is a prime number, as in E, then the components of the pair must be that number and 1—the only factors of a prime number.)

The questions may be asked in this format:

$$\begin{bmatrix} \square + \bigcirc = \triangle \\ \square \times \bigcirc = \hexagon \end{bmatrix}$$

We follow the customary agreement: In each pair of sentences any number to be used for a certain frame must be used for all frames of that size and shape. This agreement does not prevent the same number from being used for frames of different shapes. (See "Enrichment Activity 3" in Notes on Pupil Page 5.)

	□	○	△	⬡
I	3	1		
II		4		16
III	5		8	
IV		5		5
V				

In example I we are told that the pair is (3, 1). Then the sum is recorded under the triangle in the table, and the product is recorded under the hexagon. In example II we are given one component of the pair (4) and the product (16). Since $4 \times 4 = 16$, the missing component of the pair must be 4. It follows that the sum is 8. In example III the pair is obviously (5, 3) and the missing product is 15. In example IV the product is 5 and one component of the pair is 5; therefore, the other component of the pair must be 1; the sum must be 6.

An advantage of this latter formulation of the questions is that the format can be varied:

$$\begin{bmatrix} \hexagon = \square \times \bigcirc \\ \bigcirc + \square = \triangle \end{bmatrix} \qquad \begin{bmatrix} \triangle - \bigcirc = \square \\ \bigcirc = \hexagon \div \square \end{bmatrix}$$

Drill

Here is a suggestion for a drill sheet:

	I	II	III	IV	V
A	20¢ + 14¢ − (10¢ + 6¢) = ¢ + ¢	34¢ − 16¢ = ¢	40¢ + 8¢ − (20¢ + 6¢) = ¢ + ¢	48¢ − 26¢ = ¢	40¢ − 27¢ = ¢
B	$.60 + $.17 − (.30 + .08) = $ + $	$.77 − .38 = $	$.60 + $.18 − (.40 + .09) = $ + $	$.78 − .49 = $	$.35 − .07 = $
C	$1.20 + $.09 − (.70 + .03) = $ + $	$1.29 − .73 = $	$1.50 + $.10 − (.90 + .03) = $ + $	$1.60 − .93 = $	$1.64 − .75 = $
D	$2.40 + $.15 − (1.00 + .08) = $ + $	$2.55 − 1.08 = $	$2.40 + $.07 − (.90 + .05) = $ + $	$2.47 − .95 = $	$1.68 − .82 = $
E	$5.10 + $.13 − (2.00 + .07) = $ + $	$5.23 − 2.07 = $	$4.40 + $.16 − (1.90 + .09) = $ + $	$4.56 − 1.99 = $	$7.00 − 2.63 = $

	VI	VII	VIII	IX
F	$3.00 + $.60 + $.07 − (1.00 + .50 + .03) = $ + $ + $	$3.67 − 1.53 = $	$5.00 + $1.20 + $.06 − (3.00 + .70 + .05) = $ + $ + $	$6.26 − 3.75 = $
G	$9.00 + $1.10 + $.15 − (7.00 + .30 + .09) = $ + $ + $	$10.25 − 7.39 = $	$8.00 + $.50 + $.15 − (3.00 + .10 + .06) = $ + $ + $	$8.65 − 3.16 = $

	X	XI	XII	XIII	XIV	XV
H	80¢ − 32¢ = ¢	87¢ − 38¢ = ¢	$1.25 − .70 = $	$5.35 − 1.80 = $	$7.45 3.29 = $	$6.07 3.88 = $

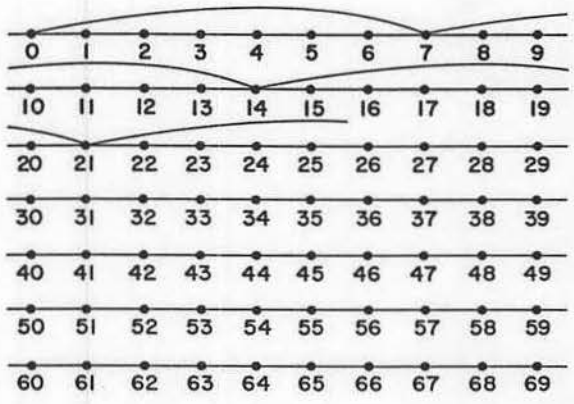
Here are some stories for headline practice:

1. Suzie had 30 bottle tops. She grouped them by 3's and said, "1 out of every 3 is green."
2. Our family ate 4 eggs for breakfast each morning for one week.
3. Sol, Pamela, and Camelia played a peanut hunt game. They each found 7 peanuts.

Also consider using as drill sheets pages 33 and 34 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

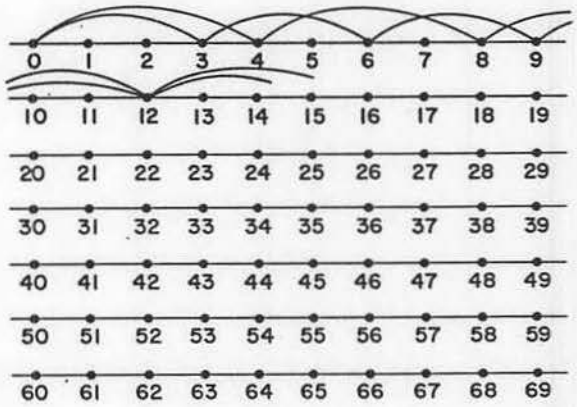
Color Multiples of 7 Blue.

0	1	2	3	4	5	6	7	8	9	10
	11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30	
31	32	33	34	35	36	37	38	39	40	
41	42	43	44	45	46	47	48	49	50	
51	52	53	54	55	56	57	58	59	60	
61	62	63	64	65	66	67	68	69	70	



Multiples of 3 (color Blue) and 4 (color Yellow)

0	1	2	3	4	5	6	7	8	9	10
	11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30	
31	32	33	34	35	36	37	38	39	40	
41	42	43	44	45	46	47	48	49	50	
51	52	53	54	55	56	57	58	59	60	
61	62	63	64	65	66	67	68	69	70	



Draw a ring around the number that would make the sentence true.

$3 \times 4 = 22 - \square$

12 32 **10** 13

A $2 \times \triangle = 9 + 7$

11 27 15 **8**

B $\square + 3 = 5 \times 3$

12 5 8 18

C $\triangle + 6 = 9 \times 3$

27 **21** 17 54

D $5 \times \bigcirc = 52 - 32$

2 **4** 6 7

E $4 \times 9 = \triangle \times 6$

4 **6** 8 13

F $(3 \times \square) + 4 = 5 \times 5$

7 3 25 9

G $\triangle \times 6 = 3 \times 8$

1 2 3 **4**

H $(2 \times 9) + 3 = (\square \times 3) + 3$

4 5 **6** 7

I $(4 \times \triangle) - 9 = (2 \times 8) - 9$

4 5 6 11

J $20 + 25 = \bigcirc \times 5$

5 7 **9** 11

K $7 \times 9 = \triangle \times 7$

16 7 10 **9**

L $\square \times 15 = (5 \times 9) - 0$

0 **3** 6 9

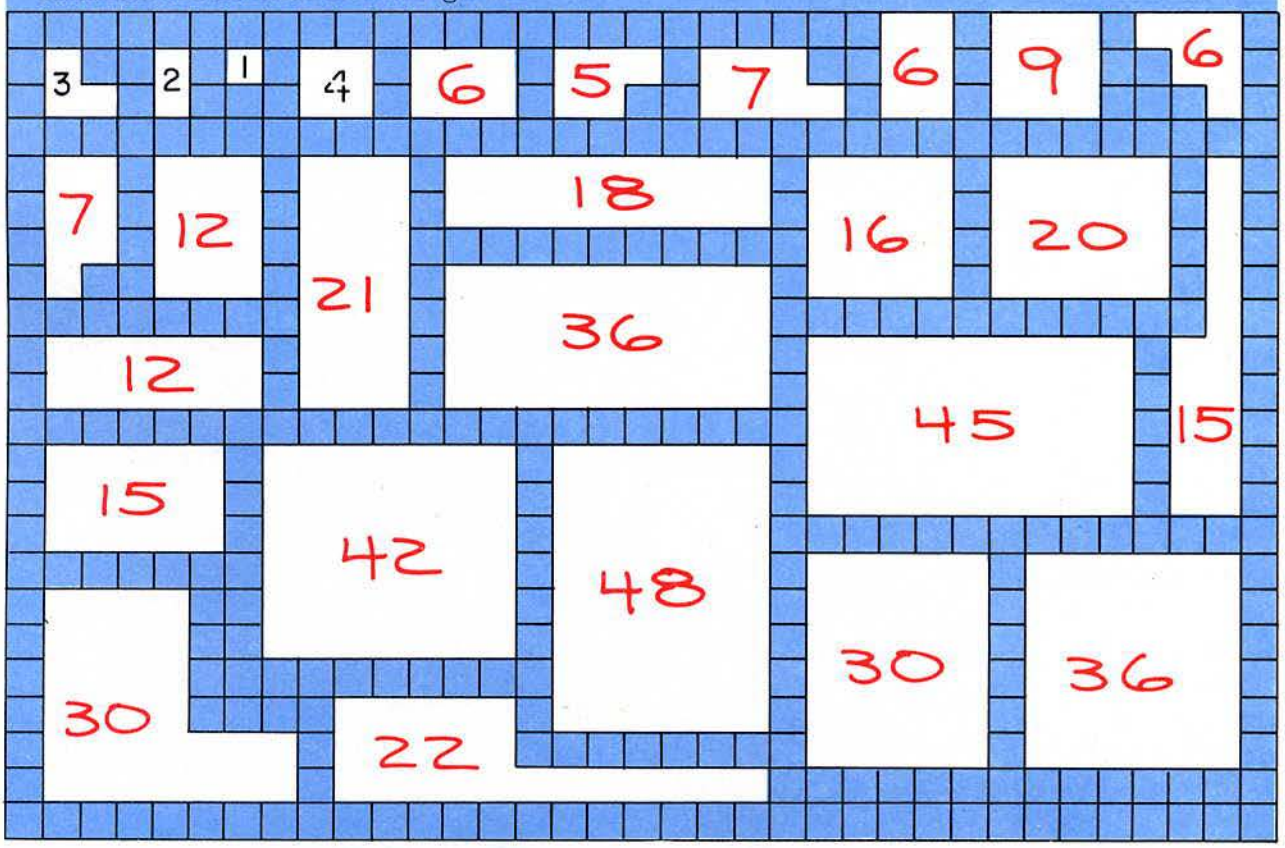
M $(\triangle \times 4) + 73 = (2 \times 10) + 73$

73 77 **5** 12

N $(3 \times 8) + 16 = 4 \times \bigcirc$

4 5 7 **10**

Indicate the number of missing tiles.



65 Test page - multiple-choice items - equations depicting addition, subtraction, multiplication, or division situations - using counting shortcuts (multiplication) to determine counts.

Tasks: *Pupils use the clues from the sample exercise to answer the test items A-N. In the lower half of the page pupils record in each white region the number of tiles it would take to fill it.*

Purpose: To assess achievement in computational skill and mastery of basic facts.

Unifying Ideas: Number and counting; addition and subtraction, multiplication and division.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets as described below.

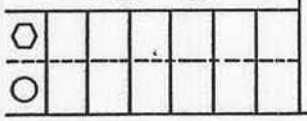
The Lesson:

We suggest that you assign 51 points to this test—28 points to A-N and 23 points to the exercises in the bottom half of pupil page 65.

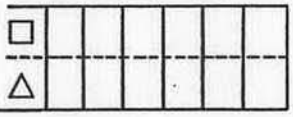
Here are some supplementary activities suitable for presentation on Ditto sheets.

Open Number Sentences

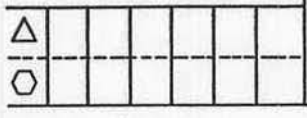
A. $3 + \hexagon + \bigcirc = 10$



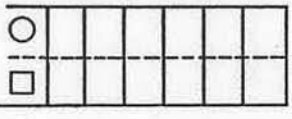
B. $10 + \square = \triangle + 10$



C. $10 + \triangle = \hexagon - 10$



D. $7 + \bigcirc - \square = 14$



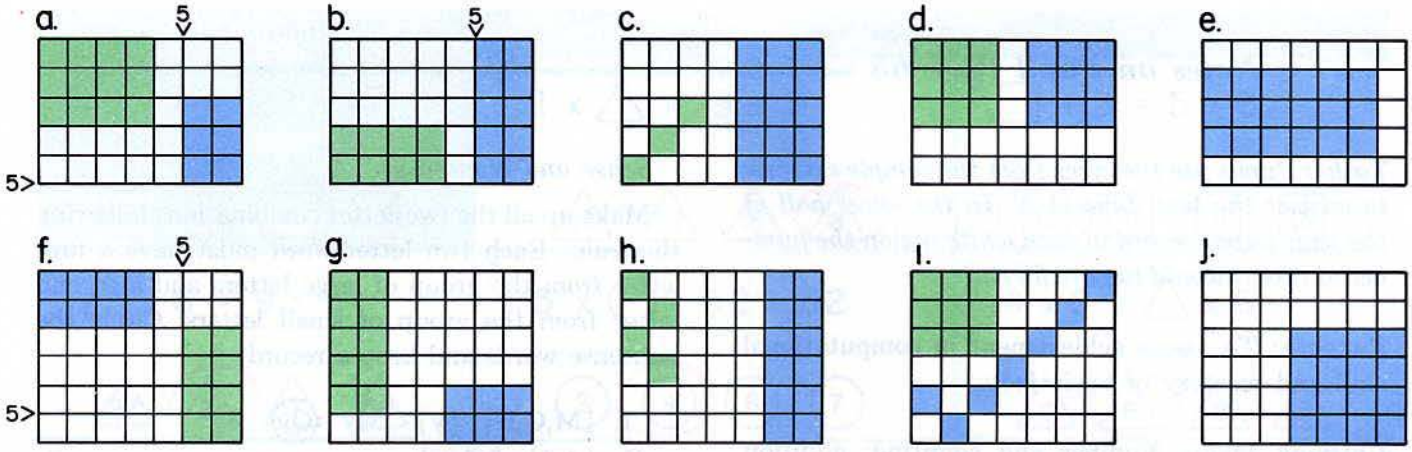
Sense and Nonsense

Make up all the two-letter combinations following this rule: Each two letter word must have a first letter from the group of large letters and a second letter from the group of small letters. Circle the nonsense words and keep a record.

I. {M,Q,B}, {y} My (Qy) By
II. {A,I}, {s,k,n}
III. {A,E,I,O}, {n,f}
IV. {M,U}, {s,p,e,y}
V. {A,B,C}, {d,e,f}
VI. {G,H,I}, {j,k,l}
VII. {I,O}, {f,n}
VIII. {N,M}, { }

	Total letters	Large letters	Small letters	Total words	Sense	Non-sense
I	4	3	1	3	2	1
II						
III						
IV						
V						
VI						
VII						
VIII	2	2	0			
	Sum	Addends or Factors		Prod-uct		

Headline for the Day: Tell a story to fit the picture on one of cards A-H on page 65.



	a	b	c	d	e	f	g	h	i	j
Total	35	35	35	35	35	42	42	42	42	42
Green	12	8	3	9	0	8	10	4	9	0
Blue	6	10	15	9	18	8	6	12	7	16
Blue *or* green	18	18	18	18	18	16	16	16	16	16
White	17	17	17	17	17	26	26	26	26	26

3-way chain reactions

K

35	−3	32
−8	−18	−15
27	−10	17

L

42	−4	38
−10	−16	−12
32	−6	26

M

35	−0	35
−8	−18	−18
27	−10	17

N

42	−7	35	−15	20	+12	32	+13	45	+0	45
−16	−16	−9	−18	−3	+5	−7	+5	−8	−18	−18
26	−0	26	−9	17	÷8	25	+12	37	−10	27

O

45	−8	37	+3	40
±0	−18	−10	+13	+10
45	−18	27	+23	50
+8	+18	+36	+23	±0
53	+10	63	−13	50

P

32	−11	21	+20	41
−6	−16	−5	+17	−3
26	−10	16	+22	38
+4	+36	+46	+86	+64
30	+32	62	+40	102

66-67 Counting and tabulation problems which encourage pupils to use shortcuts to counting - 3-way Chain Reactions involving additions and subtractions.

Tasks: *In each of exercises* a-j *pupils count or use shortcuts to counting to record in the table the number of green tiles, blue tiles, blue or green tiles, and white tiles. In the chain reaction games at the bottom pupils use the clues provided to determine the missing operations and numbers.*

Purpose: To encourage the development and use of shortcuts to counting and to perfect computational skills.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; functions and relations.

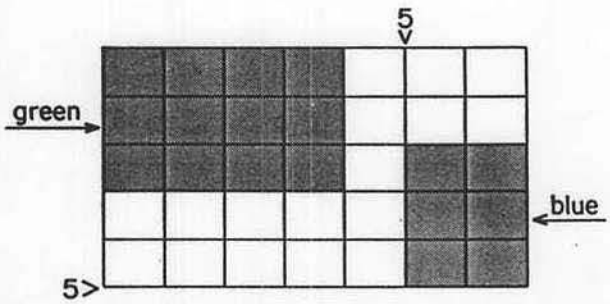
Materials for Teacher: Graph board; colored chalk.

Materials for Pupils: None.

The Lesson:

You may wish to have pupils proceed on their own on this page or to organize teams to complete the work without preliminary instructions. Perhaps you can circulate to guide pupils who may need a bit of help here and there.

If readiness work is needed, copy on the chalkboard or graph board a diagram like *a* on pupil page 66:



Ask: What can you say about this sketch? Look for answers such as:

There are 35 tiles because $5 \times 7 = 35$.

Another way to get 35 is to say that $5 \times 5 = 25$ and $5 \times 2 = 10$ and $25 + 10 = 35$.

There are 3×4, or 12, green tiles.

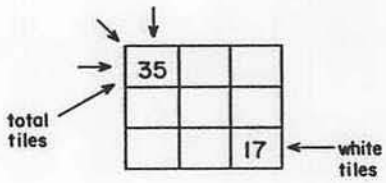
There are 3×2, or 6, blue tiles.

There are $(2 \times 5) + 1 + (2 \times 3)$, or 17, white tiles.

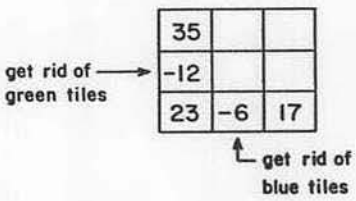
Another way to get the number of white tiles is to subtract the number of tiles which are green from the total number of tiles and then subtract the number which are blue:

$$35 - 12 - 6 = 23 - 6 = 17$$

Now begin with a chain reaction problem like this which is related to exercise *a*:



Using the diagram for exercise *a* above, let's get from 35 to 17 in two moves:



Notice that the chain reaction problems can now be interpreted by such pictures as those at the top of the page. Perhaps pupils might be asked to match the *three routes* in some of the *three-way* chain reaction problems with diagrams at the top of the page.

They should note either by examining the pictures or the table, or both, that

K is illustrated by *c*, *b*, and *e;*

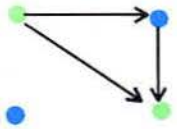
L is illustrated by *h*, *g*, and *j;*

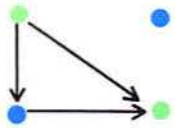
M is illustrated by *b* and *e*.

Headline for the Day: Make a table showing the various ways of making up 90¢ using half-dollars, quarters, dimes, nickels. (Hint: There are 32 different ways.)

More 3-way chain reactions.

Look for patterns.





A.

4	+1	5
+3	+6	+5
7	+3	10

B.

12	−4	8
−5	−7	−3
7	−2	5

C.

7	+2	9
−5	−4	−6
2	+1	3

D.

12	−5	7
+2	+5	+10
14	+3	17

E.

15	−2	13
+2	−10	−8
17	−12	5

F.

20	−4	16
−6	−11	−7
14	−5	9

G.

13	+10	23	−10	13
−4	+7	−3	−6	+4
9	+11	20	−3	17

H.

20	−3	17	−16	1
−7	−13	−10	+2	+18
13	−6	7	+12	19

I.

12	+10	22	+20	42
+8	+18	+8	+11	−9
20	+10	30	+3	33
+5	−10	−20	−27	−30
25	−15	10	−7	3

J.

36	−20	16	+5	21
−8	−18	+2	+5	±0
28	−10	18	+3	21
−9	−19	−9	−7	−10
19	−10	9	+2	11

K.

24	+8	32	+15	47
+9	+18	+10	+19	+4
33	+9	42	+9	51
+2	−11	−20	−21	−30
35	−13	22	−1	21

L.

3	+9	12	+12	24
+3	+18	+9	+8	−4
6	+15	21	−1	20
+17	+25	+10	+19	+20
23	+8	31	+9	40

Tasks: *Pupils use the clues provided in exercise A to complete the chain reaction games B-L.*

Purpose: To explore alternative algorithms via the vehicle of chain reaction games while simultaneously perfecting computational skills.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

Now is the time to call pupils' attention to the important message of the 3-way chain reactions games. Let's look at exercise A but delete the numbers in the tinted cells:

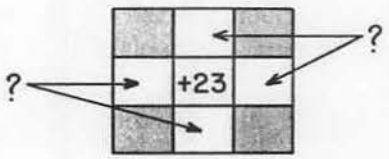
	+1	
+3	+6	+5
	+3	

If we put 7 in the upper left-hand cell, what must go in the lower right-hand cell? There are 3 ways to find out that it is 13 that must go there:

$$7 + 1 + 5 = 8 + 5 = 13$$
$$7 + 6 = 13$$
$$7 + 3 + 3 = 10 + 3 = 13$$

The first way also tells us that 8 must go in the upper right-hand cell; the second way also tells us that 10 must go in the lower left-hand cell. Repeat for other numbers in the upper left-hand cell. In each case you discover that adding 1 and then adding 5 amounts to adding 6 and that adding 6 amounts to adding 3 and then adding 3 again.

Now let's make up our own 3-way chain reaction game. Start with a 3-by-3 form and write an operation in the center cell:



What operations do we need for the middle cell in the top row and the middle cell in the right-hand column? What operations do we need for the middle cell in the left-hand column and the middle cell in the bottom row? There are as many answers as there are pairs of adding or subtracting operations which when applied "in tandem" are equivalent to adding 23. Here are just a few possibilities:

	+3	
+10	+23	+20
	+13	

	+30	
+25	+23	−7
	−2	

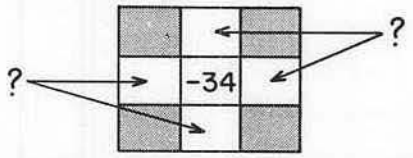
	+5	
+40	+23	+18
	−17	

more

Headline for the Day: Fill in the blanks to continue the pattern.

200, , ,188,184, , ,172,168, ,160

Similarly, if the operation for the center cell is a subtracting operation:



the pairs of operations for the pairs of middle cells must be such that when they are applied in tandem to a number written in the upper left-hand cell, the result is the same as that obtained by subtracting 34 from the number. So, the pairs of middle cells can list such pairs of operations as (−30, −4), (−4, −30), (−40, +6), and (−17, −17).

A good test of understanding is to ask pupils to pick out the one table among the following which cannot be used to make a 3-way chain reaction game:

	+1	
+15	+11	+10
	−4	

	−9	
+15	−13	−3
	−2	

	+30	
+40	+31	+1
	−9	

Clearly, the second table won't work. Although adding 15 and then subtracting 2 is equivalent to subtracting 13, subtracting 13 is not equivalent to subtracting 9 and then subtracting 3. If in the second table you start with a number in the upper left-hand cell (at least as large as 13), you will find that you get two results for the lower right-hand cell.

If you have a feeling for structure, you have probably guessed, in view of what you have noticed about addition operations and subtraction operations, that you can make 3-way chain reaction games using multiplication operations and division operations for the middle cells. Here are some examples. You complete the last one:

	×4	
×2	×12	×3
	×6	

	×10	
×100	×5	÷2
	÷20	

	÷10	
×5	÷20	

Drill

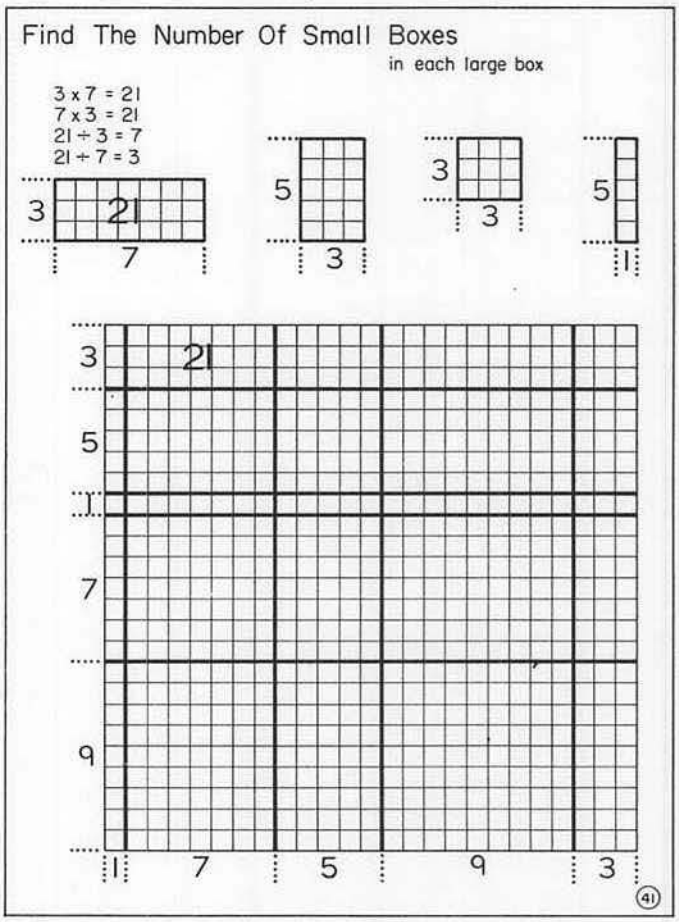
Here are some stories for headline practice:

1. At a Halloween party each of 9 children ate 5 gumdrops.
2. There were 97 boys on the playground. Fifty-seven of them were playing on the equipment.
3. Simon counted his steps to the candy store. He went 83 steps to the corner and 38 more steps to the candy store.
4. At the repair shop Nan counted 9 tricycles.

$$9 \times 3 = 27 \qquad 3 \times 9 = \square$$

5. Andrea had 18 bottle tops. She grouped them by 3's and said, "1 out of every 3 is green."
6. Jim built 3 insect cages. He placed 6 grasshoppers and 3 crickets in each cage.
7. Bonita saw an ad for a sale on basketballs. The sale price for a basketball was 79¢. She opened her piggy bank and found that she had 95¢. She figured she had enough to buy her brother a basketball for a present.
8. Sunday Margie went to visit her uncle. She rode 183 miles on the train and 32 miles by bus.

Consider using as a drill sheet page 41 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



On the next page are suggestions for two more drill sheets.

	I	II	III	IV	V	VI
A	130 + 70	250 +250	175 +725	415 +315	75 +75	375 +375
B	132 +532	429 +351	357 +157	600 +600	508 +508	1,500 + 700
C	987 +303	738 +825	427 +562	652 +651	950 + 75	840 +560
D	5,750 + 850	4,375 +1,025	3,247 + 818	980 +980	2,705 + 304	4,209 +1,608
E	3,827 + 504	1,219 +1,247	4,600 + 987	3,200 +1,894	3,207 +1,894	3,247 +1,870
F	9,462 + 809	9,462 + 869	15,500 + 500	23,126 + 7,000	45,250 +15,250	28,297 + 3,525
G	52,309 +20,203	17,510 + 8,250	8,550 + 500	6,257 +5,758	9,675 +4,675	18,123 + 9,296
H	3,428 +1,859	5,673 +2,498	16,255 + 7,907	7,648 +1,295	6,429 +8,854	14,856 + 9,558

I

Rule: Add 160

227, , , , 867, ,

	I	II	III	IV	V	VI
A	900 − 10	700 − 50	100 − 25	250 − 25	600 − 2	400 − 8
B	800 − 15	320 − 30	515 − 25	1,000 − 800	1,000 − 850	1,200 − 750
C	6,000 −2,000	4,550 −4,025	5,100 −1,050	2,050 −1,025	5,024 −2,012	6,035 −2,018
D	3,000 − 100	1,250 − 900	1,250 − 825	1,250 − 75	5,000 −1,500	2,000 − 10
E	1,000 − 150	3,000 −1,150	4,100 −1,095	7,526 −4,000	1,835 − 920	1,025 − 915
F	2,400 − 225	8,400 −3,300	9,642 −7,401	5,387 −1,385	8,710 −5,706	3,525 −3,219
G	4,853 −4,716	1,600 − 599	8,538 −8,040	1,542 − 718	6,346 − 809	6,931 −2,915
H	7,526 −6,413	8,947 −1,522	7,435 −6,218	3,600 −1,247	4,518 −2,253	1,045 − 746

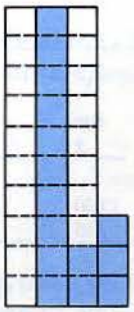
I

Rule: Subtract 120

2490, , , ,2010, , ,

When in doubt, draw a sketch at the bottom of the page and COUNT.

A

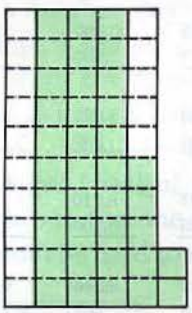


33 − 15 = 18	33 − 10 = 23	33 − 3 = 30
33 − 25 = 8	33 − 31 = 2	33 − 19 = 14

B

28 − 14 = 14	28 − 18 = 10	28 − 9 = 19
28 − 20 = 8	28 − 28 = 0	28 − 19 = 9

C



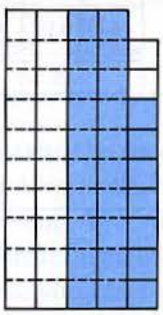
52 − 37 = 15	52 − 22 = 30	52 − 9 = 43
52 − 33 = 19	52 − 16 = 36	52 − 25 = 27

* SEE BELOW

D

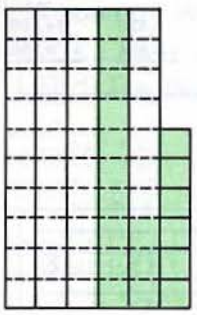
40 − 16 = 24	40 − 25 = 15	40 − 8 = 32
* SEE BELOW 40 − 29 = 11	40 − 14 = 26	40 − 37 = 3

E



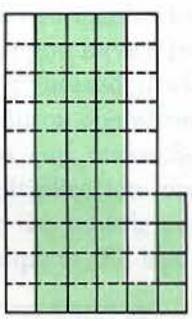
49 − 27 = 22	49 − 29 = 20	49 − 10 = 39
49 − 22 = 27	49 − 28 = 21	49 − 14 = 35

F



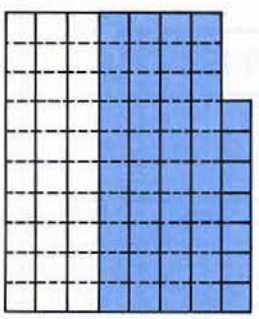
56 − 19 = 37	56 − 48 = 8	56 − 13 = 43
56 − 27 = 29	56 − 30 = 26	56 − 38 = 18

G



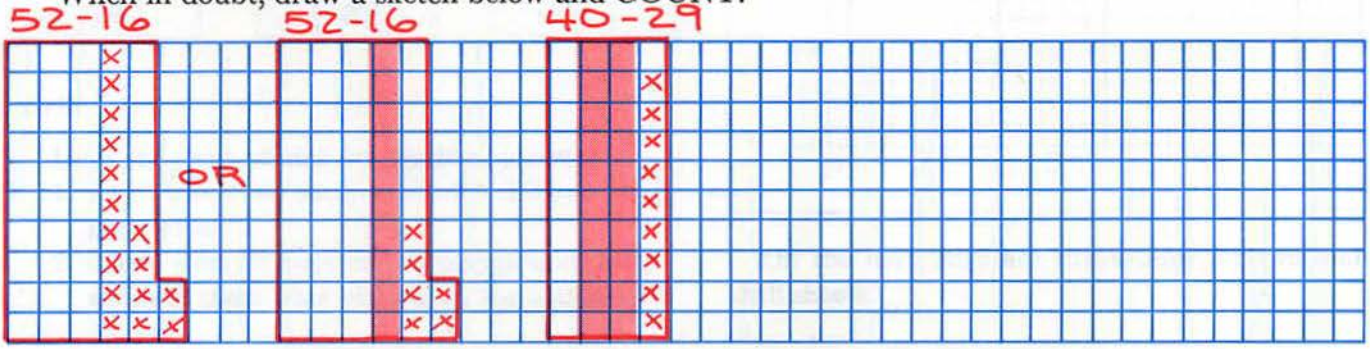
54 − 35 = 19	54 − 42 = 12	54 − 8 = 46
54 − 16 = 38	54 − 29 = 25	54 − 47 = 7

H



77 − 47 = 30	77 − 30 = 47	77 − 19 = 58
77 − 58 = 19	77 − 16 = 61	77 − 28 = 49

When in doubt, draw a sketch below and COUNT.



68-69 Using addition and subtraction representations in employing addition and subtraction algorithms to find sums, differences, and missing addends.

Tasks: *In each of exercises A-H pupils refer to the representation when necessary to compute the missing numbers as required. Note that in each exercise the common sum (top number) is the total number of small squares in the representation.*

Purpose: To continue perfecting an algorithm for column subtraction.

Unifying Ideas: Number and counting; addition and subtraction.

Materials for Teacher: Graph board; Ditto masters.

Materials for Pupils: Cuisenaire rods; graph paper; Ditto sheets as described below.

The Lesson:

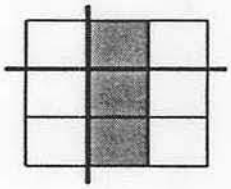
One appropriate way to regard the study of arithmetic is that it is a search for shortcuts to counting. All along the way we have continued to provide representations and examples which help pupils to understand and to discover some of these shortcuts.

On this page and the next, pupils are given subtraction problems which can be worked out with or without recourse to pictures. The key idea here and throughout the program is: *When in doubt—count!*

To begin the work on these pages, discuss with pupils the relationship between the diagrams and the examples in the tinted boxes. Let pupils use graph paper while you use the graph board to do several subtraction problems. Also, let pupils use Cuisenaire rods to do subtraction problems.

Cross Number Puzzles

Present some subtraction problems in *Cross Number Puzzle* format with the special rule that the center column is reserved for multiples of 10, including 0. The first part of the work sheet might look like the following.



Rule: Shaded boxes are reserved for multiples of 10, including 0.

1.

27	20	7
3	0	3
24	20	4

2.

22	10	12
4	0	4
18	10	8

3.

45	30	15
17	10	7
28	20	8

Here are some suggestions for problems:

39—4	88—6	72—1	31—8
45—6	27—9	47—15	88—16
75—22	95—34	46—13	58—21
42—11	86—18	22—14	83—46
31—29	104—36	121—82	

Drill

Here are some stories for headline practice:

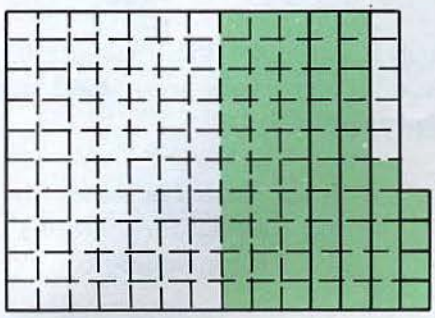
1. Marvin needs 98 cents to buy a toy. He has 37 cents.
2. Dad served a scoop of vanilla ice cream and a scoop of chocolate ice cream to each of eight children.
3. During the annual Girl Scout cookie sale, Monica's team sold 243 boxes of cookies. Beulah's team sold 473 boxes of cookies.
4. Sandy's teacher cut 36 inches of ribbon into pieces which were 4 inches long.

Headline for the Day: Fill in the blanks to continue the pattern.

200, , ,176,168, ,152,144, , ,120

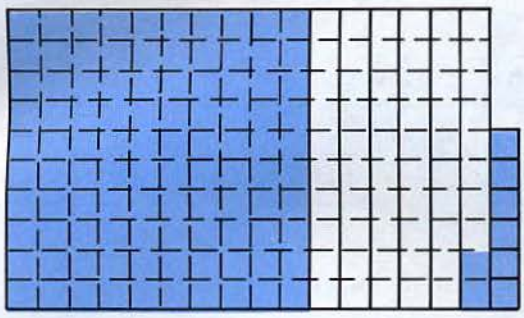
When in doubt, draw a sketch at the bottom of the page and COUNT.

I



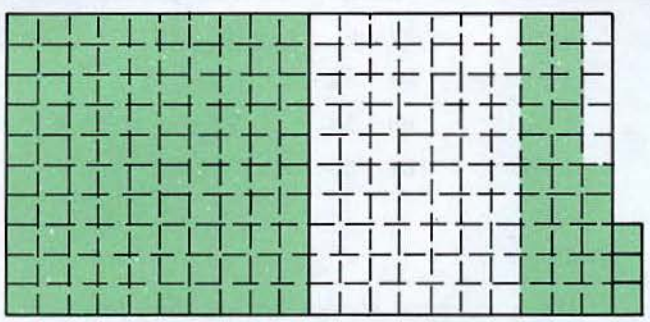
134 − 59 = 75	134 − 70 = 64	134 − 17 = 117	134 − 104 = 30	134 − 94 = 40
134 − 68 = 66	134 − 86 = 48	134 − 109 = 25	134 − 35 = 99	134 − 57 = 77

J



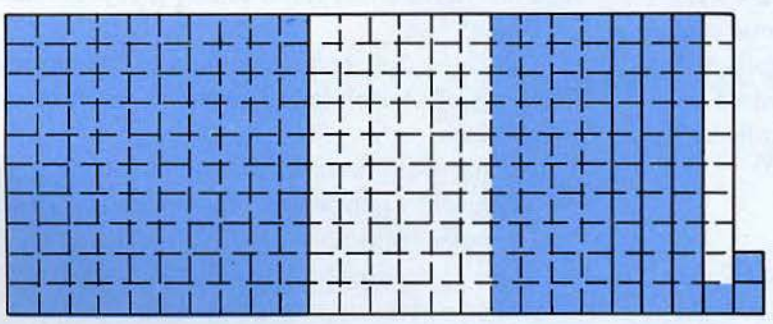
166 − 108 = 58	166 − 43 = 123	166 − 75 = 91	166 − 129 = 37	166 − 96 = 70
166 − 67 = 99	166 − 18 = 148	166 − 81 = 85	166 − 87 = 79	166 − 148 = 18

K



203 − 128 = 75	203 − 200 = 3	203 − 160 = 43	203 − 43 = 160
203 − 85 = 118	203 − 109 = 94	203 − 17 = 186	203 − 56 = 147

L



242 − 173 = 69	242 − 90 = 152	242 − 115 = 127
242 − 84 = 158	242 − 146 = 96	242 − 177 = 65

Tasks: *In each of exercises I-L pupils refer to the representation, when necessary, to compute the missing numbers.*

Purpose: To continue perfecting an algorithm for column subtraction.

Unifying Ideas: Number and counting; addition and subtraction.

Materials for Teacher: Graph board; Ditto masters.

Materials for Pupils: Graph paper; Ditto sheets as described below.

The Lesson:

We do not expect pupils to have mastered the ideas we have been working with up to now. Again and again in this book and in other levels of this program, we shall reexamine these ideas in still other ways to help pupils gain increased understanding of arithmetical processes.

More Cross Number Puzzles

As we suggested in Notes on Pupil Page 68, you may use the *Cross Number Puzzle* format with special rules to give insight into an algorithm for column subtraction.

multiples of 100 including 0

multiples of 10 including 0

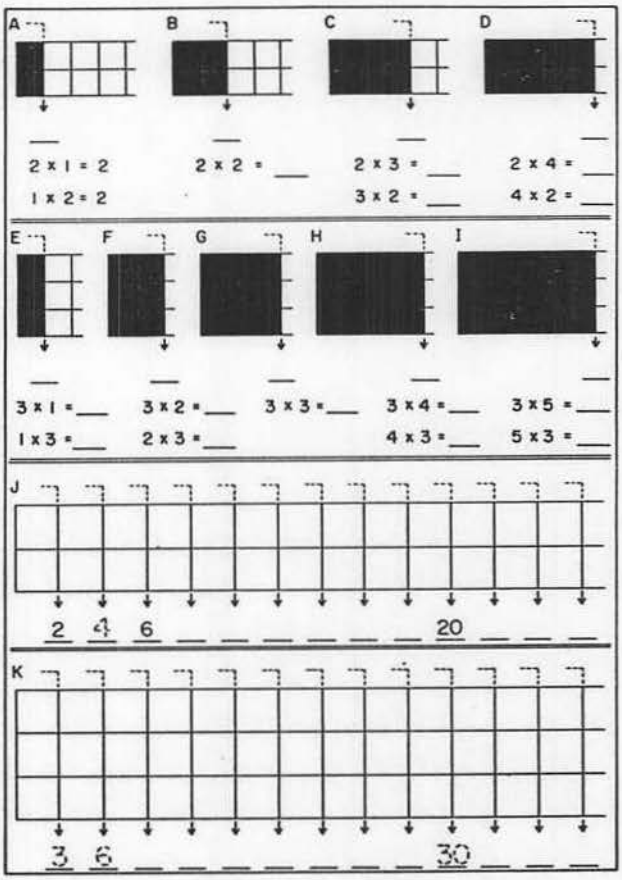
248	100	130	18
189	100	80	9
59	0	50	9

Here are some suggested subtraction problems:

378—165	284—116	933—245
350—128	817—388	234— 48

Drill

Here is a drill sheet on multiplication:



Here are some stories for headline practice:

1. Ginger's large turtle is 67 years old. Her small turtle is 28 years old.
2. Salome had a birthday party. She received 15 presents. She put the same number of presents in each of 3 piles.
3. The pupils of Northside Elementary School collected gifts for needy families. They collected 143 books and 326 toys.

Headline for the Day: Fill in the blanks to continue the pattern.

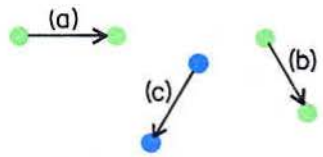
200, , ,173,164,155, , , ,119,110

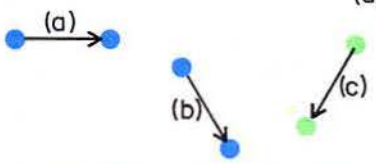
Here are some 3-way chain reactions with these ways of moving:

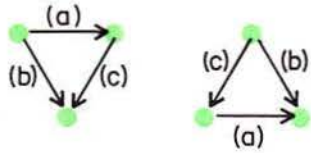
(a) left to right

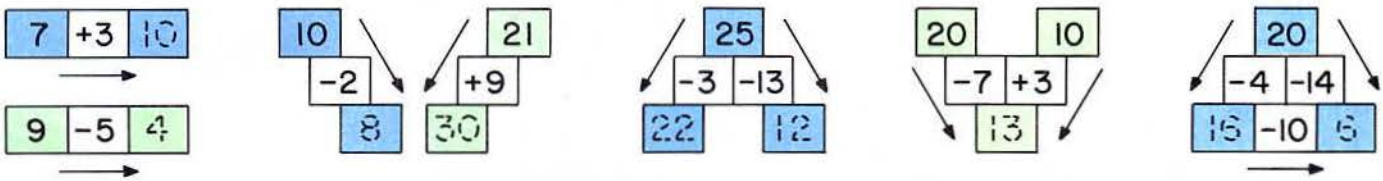
(b) down and to the right

(c) down and to the left









a. 11 -3 8

b. 9 +4 13

c. 10 -6 4

d. 7 +5 12

e. 8 +2 10

f. 11 -6 +4 5 ÷10 15

g. 7 -3 4 -6 -3 1

h. 9 ÷1 -7 10 -8 2

i. 12 -3 9 -8 -5 4

j. 4 +4 +10 8 +6 14

k. 13 -9 4 -7 +2 6

l. 18 -6 -9 12 -3 9

m. 8 -2 6 +8 +10 16

n. 7 +12 +9 19 -3 16

o. 20 +3 23 -7 -10 13 +5 +15 18 +10 28 +13 +3 31

p. 17 -7 -9 10 -2 8 +18 +20 28 -20 -19 8 +1 9

q. 21 -3 18 → -13 -10 -14 8 -4 4 +10 +17 +21 18 +7 25 +14 +7 32 →

MANY POSSIBLE ANSWERS

70-71 3-way Chain Reactions involving additions and subtractions and constructions of Chain Reactions.

Tasks: *Pupils use the clues provided by the directions and the sample examples to complete each of the chain reaction games* a-q. *In chain reaction game* q *pupils supply some of their own entries for the green boxes to extend the chain reaction beyond the given data.*

Purpose: To practice computational skills.

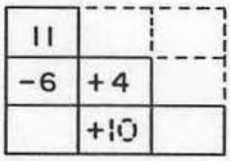
Unifying Ideas: Structure; addition and subtraction; functions and relations.

Materials for Teacher: None.

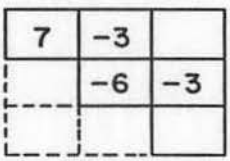
Materials for Pupils: None.

The Lesson:

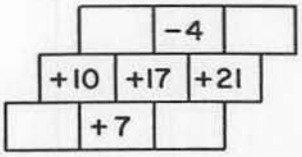
The chain reaction games on this pupil page and the next are based on minor variations of the rules used in the previous games. For example, exercise *f* is nothing more than:



and exercise *g* is simply:



The basic 3-by-3 table of the previous chain reaction games (see pupil page 67) would look like the following diagram in terms of the present rules.



Ask pupils to tell you what can be learned from certain exercises. For example, exercise *f* shows that, starting with 11, subtracting 6 and then adding 10 amounts to adding 4. Exercise *g* shows that, starting with 7, subtracting 3 and subtracting 3 again amounts to subtracting 6. Exercise *o* shows that, starting with 20, subtracting 7, then adding 5, and then adding 13 amounts to adding 3, then subtracting 10, then adding 15, and then adding 3. Etc.

Drill

Here is a suggestion for a drill sheet:

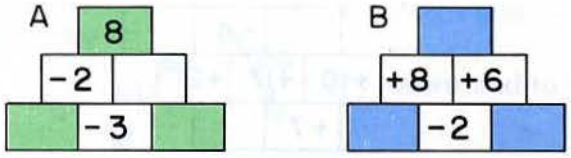
1.	$4 \times 3 = \square$	2.	$7 \times 3 = \square$
3.	$5 \times 4 = \square$	4.	$4 \times \square = 16$
5.	$2 \times \square = 6$	6.	$3 \times \square = 18$
7.	$8 \times 3 = \square$	8.	$9 \times 2 = \square$
9.	$5 \times 5 = \square$	10.	$\square \times 6 = 12$
11.	$\square \times 9 = 27$	12.	$\square \times 8 = 80$

Here are some stories for headline practice:

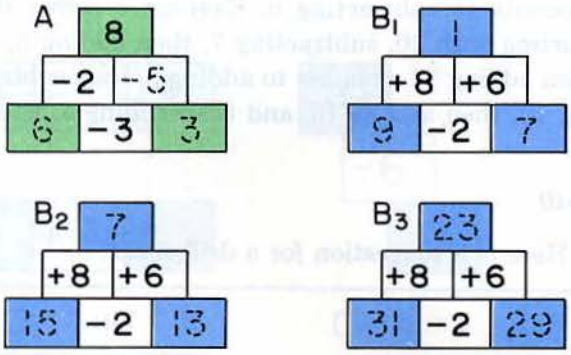
1. Sam had 24 bottle tops. He grouped them by 3's and said, "1 out of every 3 is orange."
2. Donna's mother bought 3 bananas. They weighed 1 pound and 11 ounces.
3. For Christmas, Beulah received a camera. She bought 9 rolls of film for it. She could take 10 pictures on each roll.

Headline for the Day: Jacob spent 7¢ more than Irene. Make a table to record what you can say about this situation.

Look at the chain reactions in these two examples:

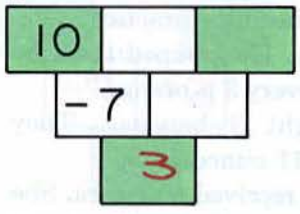


There is only one way to complete example A. But there are many ways to complete example B.

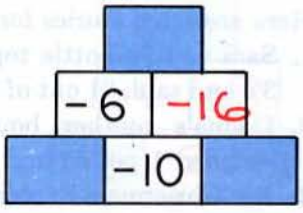


The chain reaction examples on this page are not like example A. They are like example B. Complete them in any way you like.

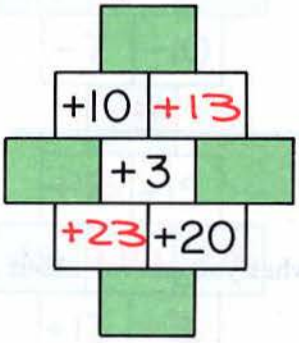
I



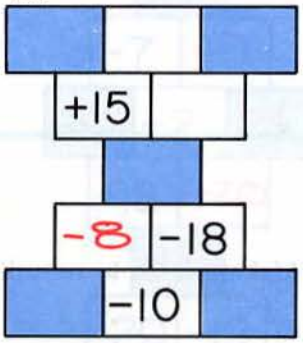
II



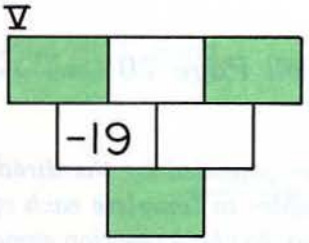
III



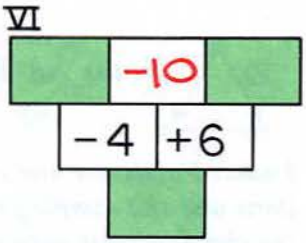
IV



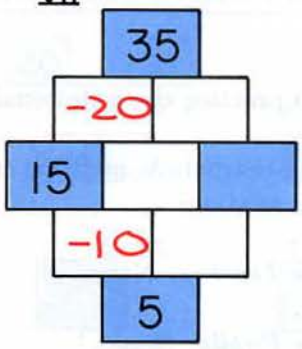
V



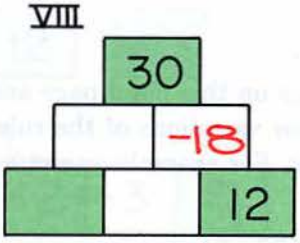
VI



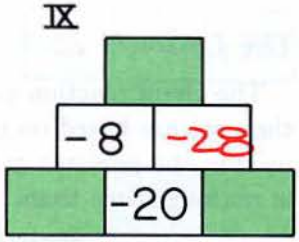
VII



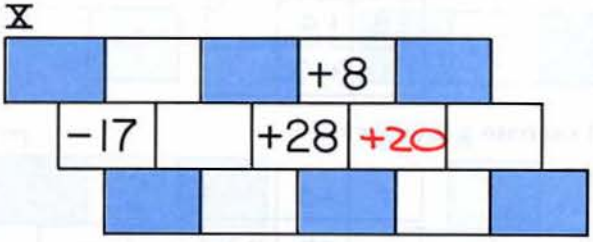
VIII



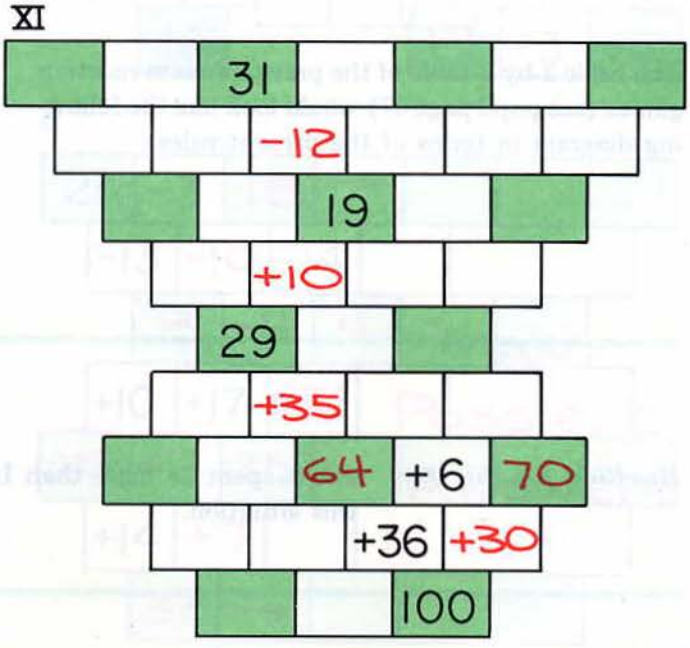
IX



X



XI



Notes on Pupil Page 71

Tasks: *In exercises I-XI pupils construct (within the limits of the given data) their own chain reaction games and carry out the necessary computations.*

Purpose: To continue perfecting computational skills.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

Materials for Teacher: Ditto masters.

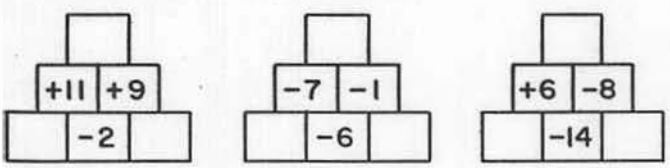
Materials for Pupils: Ditto sheets as described below.

The Lesson:

The exercises on this pupil page will reveal to pupils the pattern of the operations written in the white cells. The answers (in red) on your copy of the page show which cells have been determined by the given data. For example, in exercise II the blank white cell must be filled with a -16 because subtracting 16 is the single operation which is equivalent to subtracting 6 and then subtracting 10.

On the other hand, one does not have complete freedom in filling the nondetermined cells. For example, in exercise I the white cells must be filled by a pair of operations whose composite effect is that of subtracting 7. Some correct pairs are $(+5, -12)$, $(+7, -14)$, $(-10, +3)$, and $(+3, -10)$.

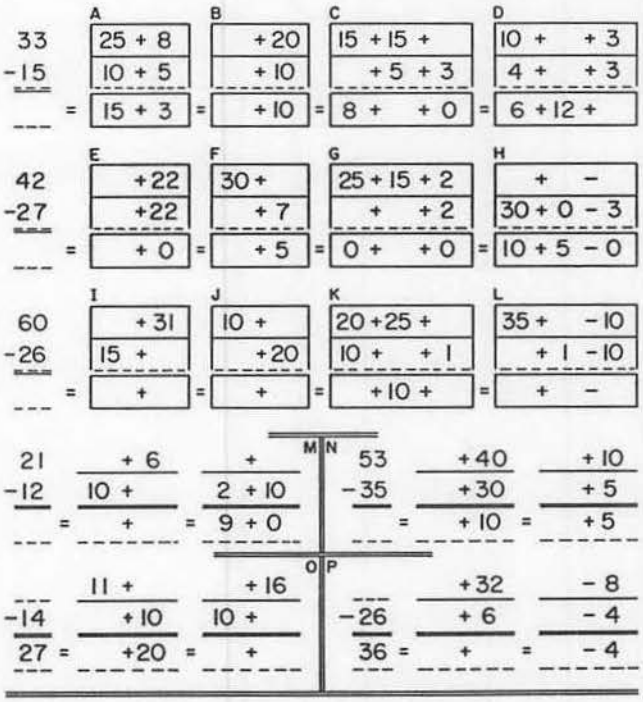
To make sure that pupils are aware of the pattern of operations ask them which of the following will *not* make a chain reaction game:



The center one will not work because the composite effect of subtracting 7 and then subtracting 6 is not that of subtracting 1.

Disguised Cross Number Puzzles

Here are some exercises which give insight into the basic principles underlying an algorithm for column subtraction. Note that they are nothing more than *Cross Number Puzzles.*



Drill

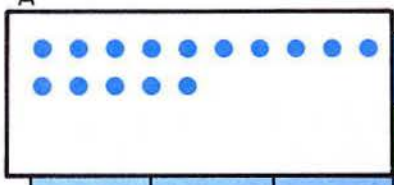
Here are some stories for headline practice:

1. André pasted some pictures in his album. On each of the first 9 pages he placed 7 pictures.
2. Jeffrey cut a 2-pound pie into 8 pieces of the same size.
3. Candy had 8 nickels and 3 dimes.

Headline for the Day: Carol spent twice as much as Denise. Make a table to record what you can say about this situation.

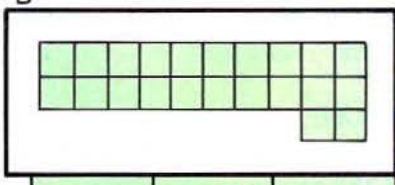
What Are My Rules?

A



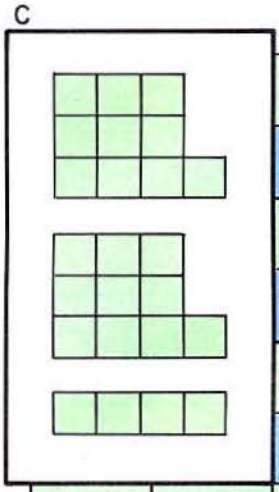
			+6	−6
			21	9
+15	+9	−9	+10	−7
30	24	6	25	8

B



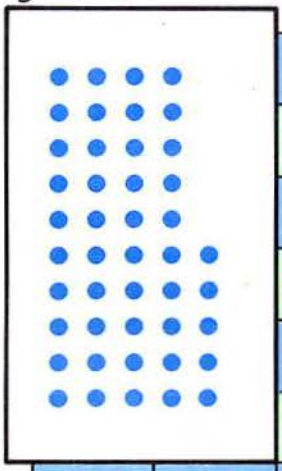
			−3	+8
			19	30
−9	+18	−16	+19	+29
13	40	6	41	51

C



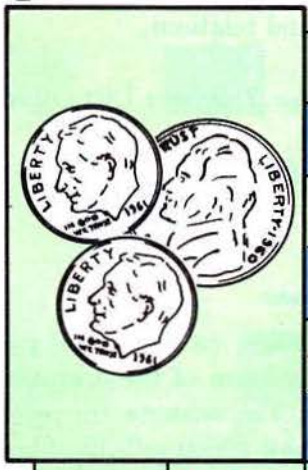
		−6	
		18	
		+17	
		41	
		−15	
		9	
+26	+7	+0	
50	31	24	

D



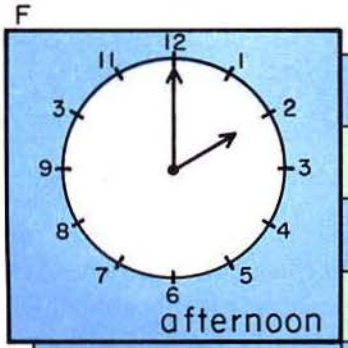
		−25	
		20	
		+8	
		53	
		−8	
		37	
+5	+30	−29	
50	75	16	

E



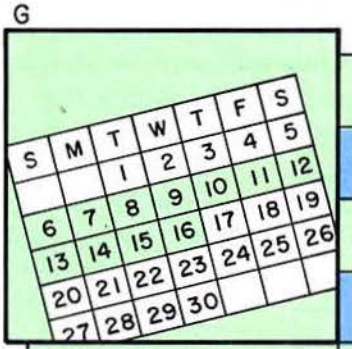
		+10¢	
		35¢	
		−7¢	
		18¢	
		+25¢	
		50¢	
+17¢	−16¢	+75¢	
42¢	9¢	$1.00	

F



time shown	
2:00 P.M.	
three hours later	
5:00 PM	
1 hour earlier	1 hr., 20 min. later
1:00 PM	3:20 PM
11 hours earlier	12 hours later
3:00 AM	2:00 AM

G



S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

weeks and days shaded green	
1 wk., 4 da.	
5 days more	
2 wks., 2 da.	
5 days less	twice as many
0 wks., 6 da.	3 wks., 1 da.
½ as many	1 week, 6 days more
0 wks., $5\frac{1}{2}$ da.	3 wks., 3 da.

H

eight more				
22	35	71	48	94
30	43	79	56	102

I

eight less				
29	35	20	110	72
21	27	12	102	64

72 A variety of practice situations involving addition, subtraction, multiplication, division, and applications.

Tasks: *In exercises A-G pupils determine the number or quantity represented by the pictures and then compute the missing numbers and operations. In exercises H and I pupils complete the tables in accordance with the rules specified above the table.*

Purpose: To test ability to compute in applied situations.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

Score 7 points for each of exercises A and B, 6 points for each of C-G, and 4 points for each of H and I. (Don't be surprised by the fact that many pupils will be able to tell you that half as many as 11 days is 5½ days but will not be able to write a 5½.)

Following are some suggestions for enrichment activities.

Extended Cross Number Puzzles

The following exercises use the *Cross Number Puzzle* format in asking pupils to find the sums for strings of consecutive even numbers or odd numbers:

I

1	3	
5	7	
9	11	

II

1	11	
3	9	
5	7	

III

2	12	
4	10	
6	8	

IV

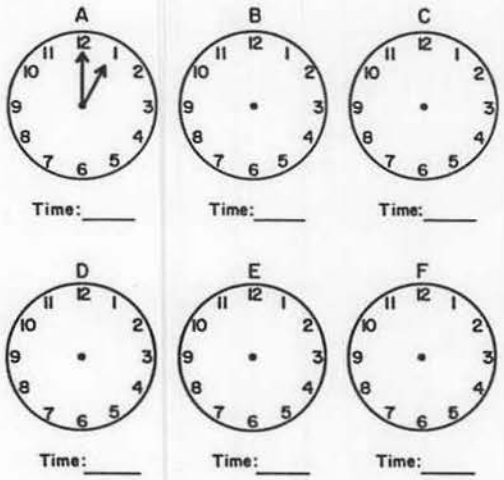
1	3	5	7	
15	13	11	9	

V

8	6	4	2	
10	12	14	16	

Clock Reading and Fractions

Draw the hands to show the time one and a half hours after one o'clock; one and a half hours after that; one and a half hours after that; etc. Write the time for each clockface.



Open Sentences

A. $13+\square=\bigcirc$

B. $32-\hexagon=\triangle$

C. $\hexagon\times\hexagon=\triangle$

D. $\bigcirc\times 1=\square$

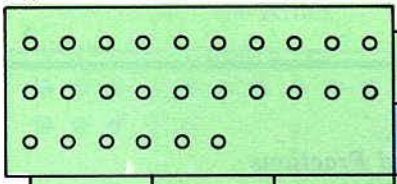
Drill

Here is a suggestion for a drill sheet:

1. $(4 \times 7)-(3 \times 7)=\square$
2. $(5 \times 6)-(6 \times 5)=\square$
3. $(7 \times 7)-(6 \times 7)=\square$
4. $(3 \times 5)-(5+5+5)=\square$
5. $(5 \times 8)-(4 \times 8)=\square$
6. $(5 \times 98)-(4 \times 98)=\square$

Headline for the Day: Tell a story to fit this fact team: $733+\square=1000$ $\square+733=1000$ $1000-\square=733$ $1000-733=\square$

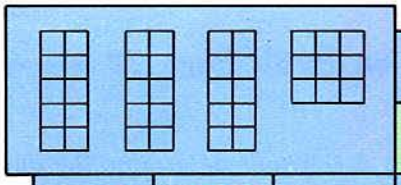
A.



+19	+65
45	91

−9	+75	−19	−17	+975
17	101	7	9	1001

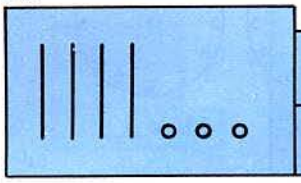
B.



+70	−21
109	18

+68	+167	+166	−15	+94
107	206	205	24	133

C.

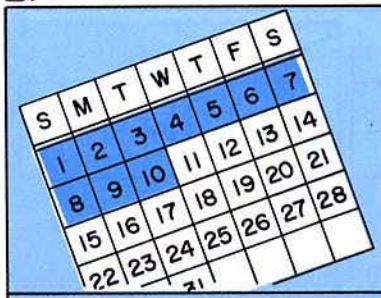


plus thirty-four	77	plus 43	86
plus 57	100	plus one hundred fourteen	157

D.

plus seventy-six cents	$1.01	minus 18¢	7¢
minus sixteen cents	9¢	plus two dollars and three cents	$2.28
plus $9	$9.25	minus twenty-five cents	0¢

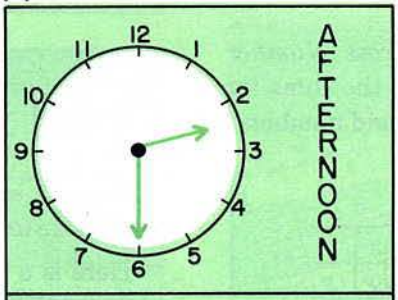
E.



S	M	T	W	T	F	S
						7
1	2	3	4	5	6	14
8	9	10	11	12	13	21
15	16	17	18	19	20	28
22	23	24	25	26	27	

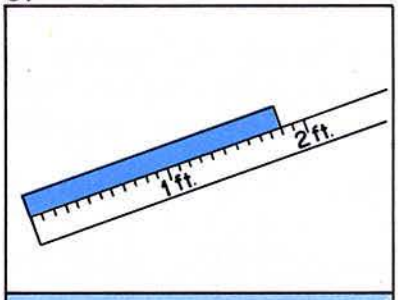
plus 1 week, 4 days
3 weeks, 0 days
plus 0 weeks, 5 days
2 weeks, 1 day
plus 5 weeks, 6 days
7 weeks, 2 days
plus 2 weeks, 6 days
4 weeks, 2 days

F.



2 hours and 15 minutes later
4:45 P.M.
1 hour and 35 minutes earlier
12:55 P.M.
3½ hours earlier
11:00 AM
3 hrs, 35 min. later
6:05 PM

G.



plus 1 foot, 2 inches
3 feet, 0 inches
minus 9 inches
1 foot, 1 inch
plus 14 inches
3 feet, 0 inches
plus 22 inches
3 feet, 8 inches

73 A variety of exercises providing practice in computational skills.

Tasks: *Pupils determine the number or quantity represented by each picture and then compute the missing numbers and operations.*

Purpose: To continue exploring a variety of situations (some of which are applications) while simultaneously perfecting computational skills.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets as described below.

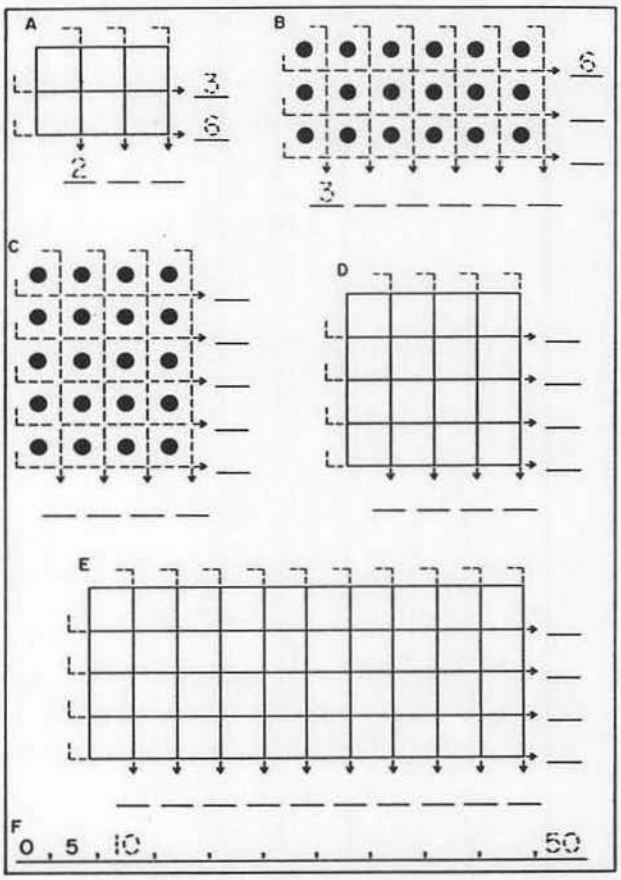
The Lesson:

Exercise F provides an opportunity to do a bit of "clock arithmetic." Notice that there are infinitely many correct answers for the third problem in F. Here are a few of them:

3 hours and 30 minutes earlier
27 hours and 30 minutes earlier
51 hours and 30 minutes earlier
75 hours and 30 minutes earlier
•
•
•
20 hours and 30 minutes later
44 hours and 30 minutes later
68 hours and 30 minutes later
92 hours and 30 minutes later
•
•
•

Drill

Here is a suggestion for a drill sheet dealing with multiplication as repeated addition:



Here are some stories for headline practice:

1. Ozzie's father is 75 inches tall. Ozzie is 56 inches tall.
2. Carlotta bought 4 pencils for 28 cents. Each pencil cost the same amount of money.
3. A box held 12 pieces of candy. Emmanuela ate half the candy.

Headline for the Day: Rhoda will be home in 6 minutes. Make a table to record what you can say about this situation.

What Are My Rules?

FRONT	BACK
9	1,2

CLUE:

4	0,4	7	1,0	8	1,1	11	1,4		
14	2,0	0	0,0	21	3,0				

a.

6	0,6

b.

13	1,6

c.

15	2,1

d.

12	1,5

e.

10	1,3

f.

17	2,3

g.

16	2,2

Please watch the signs!

h.

	14	2,0
+	7	1,0
	21	3,0

i.

	28	4,0
−	14	2,0
	14	2,0

J.

	19	2,5
+	9	1,2
	28	4,0

k.

	13	1,6
−	9	1,2
	4	0,4

l.

	4,4	
−	1,4	
	3,0	

m.

	2,5	
+	1,4	
	4,2	

n.

	4,0	
−	2,5	
	1,2	

o.

	4,5	
−	3,4	
	1,1	

p.

	0,5	
+	2,6	
	3,4	

q.

	4,2	
−	3,6	
	0,3	

r.

	4,0	
	0,3	
+	0,3	
	4,6	

s.

	1,3	
	2,3	
+	0,1	
	4,0	

t.

	2,4	
	0,4	
+	1,3	
	4,4	

u.

	1,3	
	1,4	
+	1,1	
	4,1	

v.

	2,3	
	0,5	
+	0,6	
	4,0	

w.

	1,6	
	1,4	
+	0,5	
	4,1	

74 Numeration exercises in translating from base 7 to base 10 and conversely, and addition and subtraction using either notation.

Tasks: *Pupils use the clues provided by the calendar and the completed WAMR cards at the top of the page to complete the WAMR cards in* a-k, *and to compute missing sums or addends in* l-w.

Purpose: To compute following algorithms based on a special numeration system.

Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Demonstration-size calendar; Ditto masters.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

Introduce this page either by having teams try to find the rule for WAMR Set L, or present the following activities for their consideration.

Activity 1

days	
5	0, 5
6	0, 6
7	1, 0
9	1, 2
15	2, 1

What can this table mean? Of course, the rule here considers weeks and days. Thus, "0, 5" can be read as 0 weeks and 5 days. Similarly, "1, 2" can be read as 1 week and 2 days. Seven days cannot be written "0, 7" but rather "1, 0." Use the fewest units to express the quantity.

Activity 2

Show pupils cards that belong to WAMR Set L:

front	back
8	1, 1
14	2, 0

Then show just fronts or backs and ask about what is written on the other side.

Activity 3

Period	3 days	7 days	13 days	14 days	23 days	
Weeks	0	1				3
Days	3	0				4

Activity 4

Have pupils complete expressions such as the following using the shorthand of commas to separate weeks and days.

1, 0= 7 (read as "1 week and 0 days is 7 days")
1, 3= 10
2, 2=___
3, 5= 26
___, ___ = 35
6, ___ = 49

What is the largest number of days we can express in this shorthand if we use only the numbers from 0 through 6? (6, 6—or 6 weeks and 6 days—48 days.)

more

Headline for the Day: Leroy's dad cut the piece of lumber one half foot shorter than he planned to. Make a table to keep a record of what you can say about this situation.

Activity 5

In this weeks-days shorthand, and using only the numbers 0, 1, 2, 3, 4, 5, and 6, write the following:

a. Four days	0 , 4
b. Seven days	1 , 0
c. Ten days	,
d. One week and four days	,
e. Fifteen days	,
f. One week and ten days	,
g. Twenty days	,
h. Twenty-one days	,
i. Two weeks and nine days	,

Activity 6

$$\begin{array}{r} 1,3 \\ +\ 0,5 \\ \hline 2,1 \end{array}$$

This shorthand says: one week and three days plus no weeks and five days is the same as two weeks and one day. Complete the following and explain what each means in this shorthand:

a. $\begin{array}{r} 1,3 \\ +\ 1,1 \\ \hline \end{array}$ b. $\begin{array}{r} 0,4 \\ +\ 1,3 \\ \hline \end{array}$

c. $\begin{array}{r} 1,2 \\ +\ 1,6 \\ \hline \end{array}$ d. $\begin{array}{r} 2,4 \\ +\ \\ \hline 3,2 \end{array}$

Activity 7

Here are examples of both addition and subtraction. Watch the signs.

A. $\begin{array}{r} 3,4 \\ -1,1 \\ \hline \end{array}$ B. $\begin{array}{r} 1,1 \\ +\ \\ \hline 3,4 \end{array}$ C. $\begin{array}{r} 2,3 \\ +0,3 \\ \hline \end{array}$ D. $\begin{array}{r} 3,6 \\ -\ \\ \hline 0,3 \end{array}$ E. $\begin{array}{r} 1,0 \\ -0,2 \\ \hline \end{array}$

F. $\begin{array}{r} 2,0 \\ -0,2 \\ \hline \end{array}$ G. $\begin{array}{r} 3,1 \\ -0,2 \\ \hline \end{array}$ H. $\begin{array}{r} 1,6 \\ +1,6 \\ \hline \end{array}$ I. $\begin{array}{r} -2,4 \\ \hline 1,6 \end{array}$ J. $\begin{array}{r} -1,5 \\ \hline 1,5 \end{array}$

Activity 8

$1,0 + 1,6 =$ ____ ,
$1,4 + 1,4 =$ ____ ,
$3,0 - 0,3 =$ ____ ,

Consider each of these sentences as a headline. Select one and tell an appropriate story to go with it. You might offer an example of your own, such as:

In a week I am going to my aunt's. I'll come home after I've stayed a week and 6 days. I'll be coming home 2 weeks and 6 days from now.

Or: It takes 4 days more than a week to travel by rocket to the space station, and it takes the same length of time to return. If I pilot the rocket on a fast round trip, I'll be gone only one day more than 3 weeks.

Multiplication and Division

It should be evident that an important dimension of the kind of work required on pupil page 74 is its affording of practice in multiplication and division.

As pupils express 23 days in terms of weeks and days and in the fewest units, they need to think: What is the largest number of sevens in 23 and how many are left over? This is analogous to:

$$\begin{array}{r} 3 \\ 7\overline{)23} \\ \underline{21} \\ 2 \end{array}$$ or 3 weeks, 2 days

And in the other direction, if we have 4 weeks and 5 days, we need to think:

$$(4 \times 7) + 5 = 28 + 5 = 33$$

Drill

Here are some stories for headline practice:

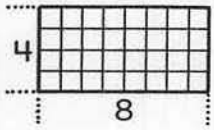
1. A girl had 90 cents in her bank. This amount of money was made up of 2 quarters and the remainder in nickels.
2. Rebecca cut a pie into 6 pieces. She ate 3 of them and gave 2 pieces to Sigmund.
3. Vicky has $26 to buy a bike. The bike costs $43.
4. Mario had a bag of 24 marbles. He gave 10 of them to Santo and 6 of them to Dominic.
5. Kitty and Carl made 6 birdhouses. They used 8 pieces of wood for each house they built.
6. Six people can sit at each table in the hotel dining room. Sue noticed there were 30 people seated at dinner.

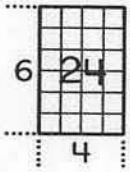
Consider using as drill sheets pages 42 and 43 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

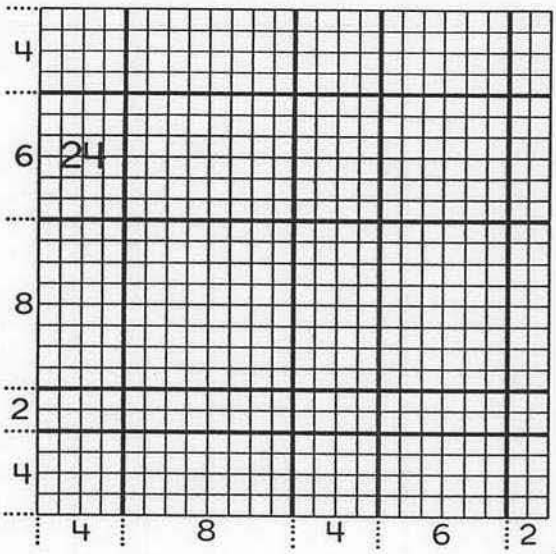
Find The Number Of Small Boxes

in each large box

$4 \times 8 = 32$
$8 \times 4 = 32$
$32 \div 4 = 8$
$32 \div 8 = 4$

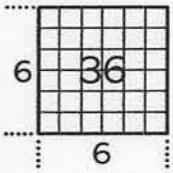


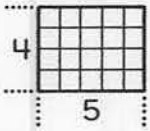


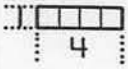


Find The Number Of Small Boxes

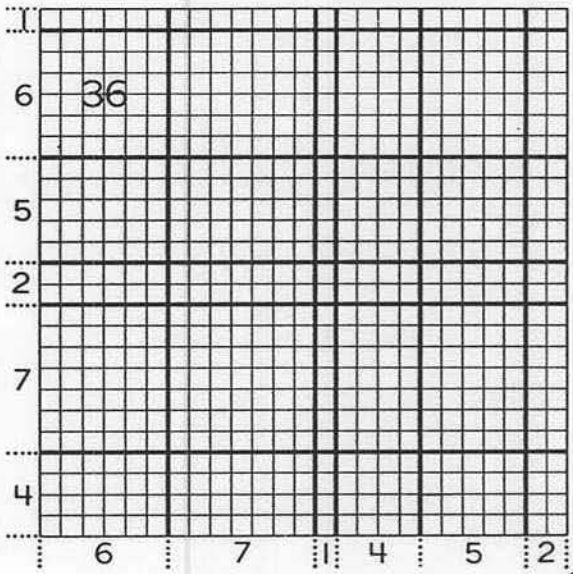
in each large box







$4 \times 5 = 20$
$5 \times 4 = 20$
$20 \div 4 = 5$
$20 \div 5 = 4$



A nickel (n.) is worth five pennies (p.).

What Are My Rules?

WAMR... Set M

	FRONT n.	FRONT p.	BACK n.	BACK p.
	0	1	0	4
	0	2	1	0
a.	0	4	1	2
b.	0	0	0	3
c.	1	0	1	3
d.	0	3	1	1
e.	1	2	2	0
f.	2	2	3	0
g.	2	1	2	4
h.	1	3	2	1

WAMR... Set N

	FRONT n.	FRONT p.	BACK n.	BACK p.
	0	2	0	4
	0	3	1	1
i	0	4	1	3
j.	1	0	2	0
k.	0	1	0	2
l.	2	2	4	4
m.	1	3	3	1
n.	1	1	2	2
o.	1	4	3	3
p.	2	3	5	1

There are sixty minutes (min.) in an hour (hr.).

Watch the signs!

Only numbers less than 60 in the white boxes.

	hr.	min.
A.	0	30
+	0	30
	1	00
B.	1	45
+	0	15
	2	00
C.	1	00
−	0	15
	0	45
D.	7	00
−	2	00
	5	00
E.	0	50
+	0	15
	1	05
F.	2	45
+	1	15
	4	00
G.	3	30
−	0	29
	3	01
H.	1	47
+	8	13
	10	00
I.	17	27
−	7	27
	10	00
J.	12	00
−	0	45
	11	15
K.	11	45
+	12	15
	24	00
L.	24	00
−	20	59
	3	01
M.	15	50
−	6	40
	9	10
N.	6	39
+	17	21
	24	00
O.	10	05
−	0	06
	9	59

75 Numeration exercises (bases 5, 60) in translation and in addition and subtraction using these notations.

Tasks: *Pupils use the clues provided in the samples to discover the rules and complete the cards in exercises* a-k. *In A-O pupils compute sums or differences as required.*

Purpose: To explore numeration systems in applications.

Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Coins (pennies and nickels); clock; several egg cartons and eggs.

Materials for Pupils: None.

The Lesson:

The rule for WAMR Set M is to express on the back of the card in the fewest nickels and pennies the amount obtained by adding 3 cents to the amount indicated on the front of the card.

The rule for WAMR Set N is that the back of the card indicates in the fewest nickels and pennies the amount that is double that indicated on the front of the card.

Use coins to help pupils with WAMR Sets M and N. Use a demonstration clock to help pupils with exercises A-O (see "Activity 4" below).

The Decimal System of Numeration

On pupil pages 76-79 we focus attention on the structure of the decimal system through exploration of such questions as:

How much less is 5 days than 2 weeks?

If I have 3 nickels and 3 pennies, how much change will I have left if I spend 4 cents?

more

Headline for the Day: Grace said the thermometer now reads 6 degrees warmer than it did this morning. Make a table to record what you can say about this situation.

Using only dimes and pennies, how many of my 5 dimes and 7 pennies will I have left if I spend 9 cents?

How much less is 18 inches than 2 feet?

If I buy 3½ pounds of ground beef and use ¾ of a pound, how much do I have left?

If I buy 3½ dozen eggs and use 9 eggs, how many have I left?

Each of these situations leads pupils to a different system of numeration. For example, in the last question:

3 dozen and 6 eggs — less 9 eggs → 2 dozen and 18 eggs — less 9 eggs

the result is 2 dozen and 9 eggs. The thinking required in this case is analogous to the thinking that is required in working with a system of numeration using the *base twelve*, or in working with our own familiar *base ten*, or decimal system.

A child who can understand that 5 days less than 2 weeks is 1 week and 2 days ought to have no trouble in understanding that 5 ones less than 2 tens is 1 ten plus 5 ones, that is, that $20 - 5 = 15$.

Such activities as the following can be used to introduce pupil page 75.

Activity 1

Complete the table using fewest coins:

Amount	4¢	5¢	6¢	10¢	15¢	18¢	
Nickels	0	1					3
Pennies	4	0					6

Activity 2

What can this table mean?

Cents	
3	0, 3
4	0, 4
5	1, 0
6	1, 1
9	1, 4
15	3, 0

The table expresses amounts of money in terms of nickels and pennies, using fewest coins. Thus "0, 3" stands for 0 nickels and 3 pennies and "2, 1" stands for 2 nickels and 1 penny.

Activity 3

Have pupils complete expressions such as the following in this shorthand:

1, 0 → 5 (1 nickel and 0 pennies is 5 cents)
1, 3 → 8
2, 2 → ___
3, 4 → ___
, → 22

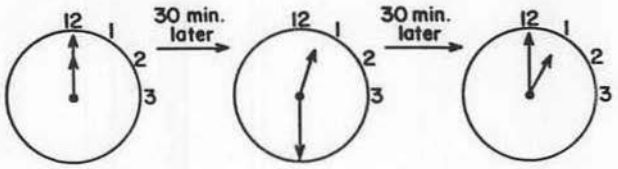
What is the largest number of cents we can express if we limit the record to nickels and pennies and to the numbers 0 through 4? (4, 4 or 4 nickels and 4 pennies—24 cents.) Such a limitation points to the need to continue the pattern so as to record more than 24¢. On pupil page 77 we shall extend this pattern for recording number of cents in fewest coins by introducing quarters. Thus it will be possible to continue recording amounts of money for a long while without using any number greater than 4.

Cents	Quarters	Nickels	Pennies
25	1	0	0
26	1	0	1
30	1	1	0
40	1	3	0
56	2	1	1
The largest number of cents possible using just 0 through 4	4	4	4

Activity 4

Before pupils attempt exercises A-O at the bottom of the page, do similar examples on the board to establish an understanding of the convention or rule that when we add minutes and get 60 or more, we partition this number into the largest possible multiple of 60 plus a number less than 60.

It is helpful to have a clock available for these exercises. Begin each example by assuming that it is 12 o'clock. Then each sum or difference can be seen more clearly. A representation of exercise A on pupil page 75 is shown at the top of the next page.



Drill

Here are some stories for headline practice:

1. A high school auditorium has 856 seats. The PTA sold 562 tickets to their annual play.
2. Daphne had $1.80 to spend at the county fair. She returned home with only 25 cents.
3. Minnie wrote 87 postcards on her first trip to New York City. On her next trip she wrote 47.

At the top of the following column is a suggestion for a drill sheet.

1. $2 \times \square = 12$	**2.** $5 \times \square = 5$		
3. $5 \times \square = 0$	**4.** $\square \times 4 = 20$		
5. $\square \times 3 = 24$	**6.** $\square \times \square = 49$		
7. $\square \times 2 = 10$	**8.** $\square \times 4 = 16$	**9.** $5 \times \square = 35$	
10. $6 \times \square = 48$	**11.** $10 \times \square = 10$		
12. $8 \times 3 = \square \times 8$	**13.** $9 \times \square = 8 \times 9$		
14. $\square \times 7 = 7 \times 6$	**15.** $17 \times 19 = \square \times 17$		

There are seven days (da.) in a week (wk.).

Use only numbers less than 7 in the white boxes.

a.

wk.	da.
1	2
+ 2	4
3	6

b.

wk.	da.
1	3
+ 0	4
2	0

c.

wk.	da.
0	2
+ 0	5
1	0

d.

0	4
+ 0	4
1	1

e.

0	6
+ 0	5
1	4

f.

wk.	da.
1	6
+ 2	0
3	6

g.

1	3
+ 1	5
3	1

h.

2	3
+ 0	4
3	0

i.

2	2
+ 1	3
3	5

j.

2	4
+ 1	4
4	1

There are twelve inches (in.) in a foot (ft.).

Use only numbers less than 12 in the white boxes.

k.

ft.	in.
0	4
+ 0	7
0	11

l.

ft.	in.
0	6
+ 0	6
1	0

m.

ft.	in.
0	10
+ 0	5
1	3

n.

1	8
+ 1	4
3	0

o.

1	4
+ 1	10
3	2

p.

ft.	in.	+	ft.	in.	ft.	in.
1	0		2	7	3	7

q.

1	2	0	11	2	1

r.

1	8	+ 2	3	3	11

s.

1	3	0	9	2	0

A dime (d.) is worth ten pennies (p.).

Use only numbers less than 10 in the white boxes.

t.

d.	p.
2	3
+ 0	5
2	8

u.

d.	p.
0	5
+ 0	5
1	0

v.

d.	p.
1	6
+ 0	7
2	3

w.

2	4
+ 6	5
8	9

x.

1	2
+ 0	8
2	0

y.

d.	p.	+	d.	p.	d.	p.
2	9		2	2	5	1

z.

1	7	+ 1	4	3	1

76 Numeration exercises (bases 7, 12, 10) in addition and subtraction using these notations.

Tasks: *Pupils use the clues provided in exercises* a, b, k, l, p, *and* t *to compute the missing numbers.*

Purpose: To explore numeration systems via applications.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: Demonstration-size calendar; foot-ruler; coins (dimes and pennies).

Materials for Pupils: None.

The Lesson:

Prepare pupils for this page by reviewing ways of adding quantities using the weeks and days notation described earlier.

Develop similar exercises for feet and inches notation, and dimes and pennies notation. Obviously, when we use dimes and pennies, we are using the familiar base ten notation.

All examples should be read in the following form to keep pupils alert to the fact that these numbers represent quantities:

$$\begin{array}{r} 0,2 \\ +0,5 \\ \hline 1,0 \end{array}$$ { 2 days plus 5 days is 1 week and 0 days

$$\begin{array}{r} 1,8 \\ +1,4 \\ \hline 3,0 \end{array}$$ { 1 foot 8 inches plus 1 foot 4 inches is 3 feet 0 inches

$2,9+2,2=5,1$ { 2 dimes 9 pennies plus 2 dimes 2 pennies is 5 dimes 1 penny

Consider the examples on pupil page 76 as headlines. Have pupils tell stories for these headlines. For the three examples above, we might expect:

It will be 2 days till we start our trip and the trip will take 5 days. We'll be back in a week.

I have a board 1 foot 8 inches long and I need another board 1 foot 4 inches long to make a 3-foot shelf.

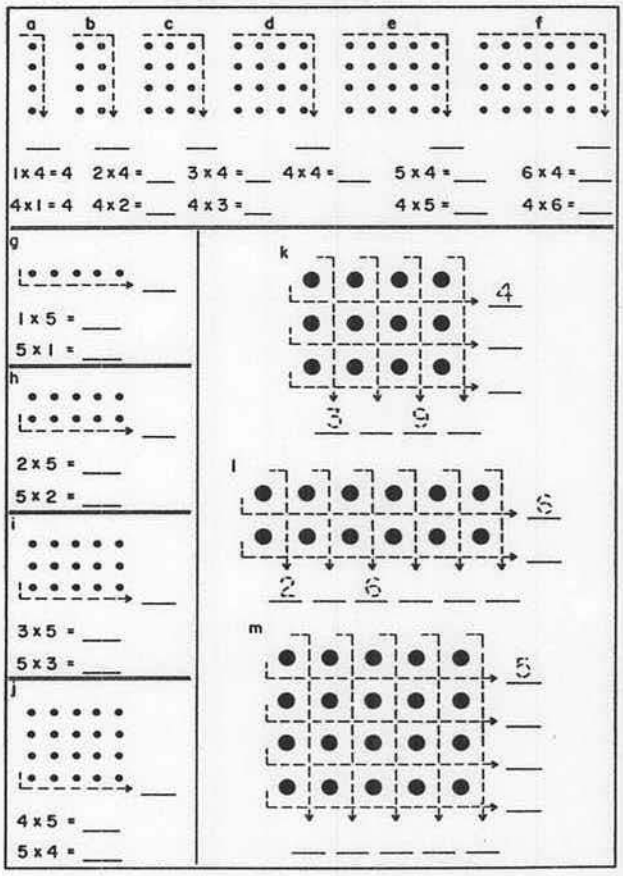
I had two dimes and nine pennies and I needed fifty-one cents. So I had to earn 22¢.

Drill

Here are some stories for headline practice.

1. Sammy, the sea lion, weighs 582 pounds. Billy, the bear, weighs 756 pounds.
2. Dante counted 36 birds at the pet shop. There were 6 birds in each cage.
3. Patty had 14 dolls. Sylvia had 18 dolls. Beatrice had 19 dolls.

Here is a suggestion for a drill sheet:



Headline for the Day: Harry said it is now 6 degrees cooler than it was at lunchtime yesterday. Make a table to record what you can say about this situation.

A quarter is worth five nickels and a nickel is worth five pennies.

Only numbers less than 5 in the white boxes.

a.

25¢	5¢	1¢
1	0	0
+ 0	2	1
1	2	1

b.

25¢	5¢	1¢
1	2	3
+ 0	2	1
1	4	4

c.

25¢	5¢	1¢
1	1	1
+ 1	1	3
2	2	4

d.

25¢	5¢	1¢
1	2	4
+ 1	3	2
3	1	1

e.

25¢	5¢	1¢
1	1	1
0	2	4
+ 0	3	3
2	2	3

f.

25¢	5¢	1¢
1	0	2
1	2	0
+ 1	1	2
3	3	4

g.

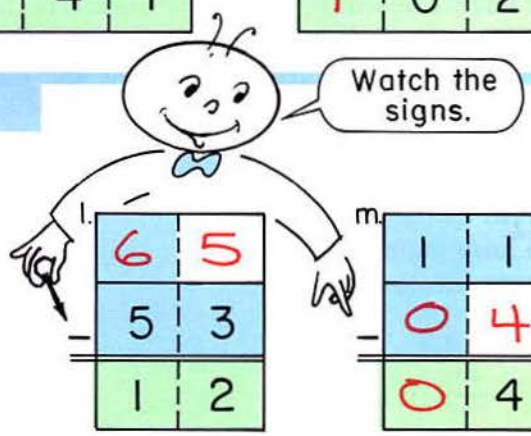
25¢	5¢	1¢
0	2	2
0	2	2
+ 0	4	2
1	4	1

h.

25¢	5¢	1¢
0	0	4
0	1	4
+ 0	2	4
1	0	2

There are seven days in a week.

Only numbers less than 7 in the white boxes.



i.

w.	d.
2	1
+ 1	3
3	4

j.

4	4
+ 2	3
7	0

k.

4	5
− 2	3
2	2

l.

6	5
− 5	3
1	2

m.

1	1
− 0	4
0	4

n.

5	5
+ 5	6
11	4

o.

5	0
− 4	6
0	1

p.

6	6
+ 7	3
14	2

q.

10	0
− 4	6
5	1

r.

10	0
− 0	6
9	1

A dime is worth ten pennies.

Only numbers less than 10 in the white boxes.

s.

d.	p.
1	5
1	4
+ 1	6
4	5

t.

0	9
1	7
+ 1	4
4	0

u.

2	7
4	6
+ 2	7
10	0

v.

1	4
2	4
+ 2	2
6	0

w.

1	
1	
+ 1	
5	5

* MANY POSSIBLE ANSWERS

77 Numeration exercises (bases 5, 7, 10) in addition and subtraction using these notations.

Tasks: *Pupils use the clues provided above each group of exercises to compute missing addends or sums, as required.*

Purpose: To continue exploring numeration systems via applications.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: Coins (quarters, nickels, pennies); calendars; Ditto master.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

The ideas and activities on this page should now be familiar. Activities suggested in earlier Notes may be adapted as needed to prepare pupils for this page.

Make sure that pupils can convert the examples on pupil page 77 in the following way:

a.

	(25¢)	(5¢)	(1¢)	
	1	0	0	= 25¢
+	0	2	1	= 11¢
	1	2	1	= 36¢

i.

	Weeks	Days	
	2	1	= 15 days
+	1	3	= 10 days
	3	4	= 25 days

Have pupils make up stories that fit the examples on this page.

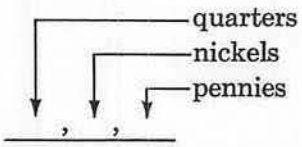
Quarters-Nickels-Pennies

In the top third of pupil page 77 we have extended the nickels-pennies representation of amounts of money to include quarters.

more

Headline for the Day: Mabel told her mother to buy a half-dozen eggs more than she bought last week. Make a table to record what you can say about this situation.

Now we have a new shorthand, or numeration system.



Pupils are, in effect, asked to indicate an amount of money by naming the least number of quarters, nickels, and pennies required to make up the amount. Here are some examples:

	25¢	5¢	1¢
31¢	1	1	1
38¢	1	2	3
14¢	0	2	4
64¢	2	2	4

etc.

An extension of this activity might include writing in sequence on the board the fewest coins required for each amount from 1¢ through $1.24. Why stop at $1.24? Consider the pattern:

	25¢	5¢	1¢
1¢	0	0	1
2¢	0	0	2
3¢	0	0	3
4¢	0	0	4
5¢	0	1	0
6¢	0	1	1
7¢	0	1	2
8¢	0	1	3
9¢	0	1	4
10¢	0	2	0

	25¢	5¢	1¢
$1.16	4	3	1
$1.17	4	3	2
$1.18	4	3	3
$1.19	4	3	4
$1.20	4	4	0
$1.21	4	4	1
$1.22	4	4	2
$1.23	4	4	3
$1.24	4	4	4
$1.25	?	?	?

Up to $1.25, all amounts can be indicated without requiring more than 4 of any one coin. Unless this system is expanded, there is no way to indicate $1.25 without using more than 4 of one of the coins.

In this system, each coin after the penny is worth five times the coin next smaller in value. So far we have used only the digits 0, 1, 2, 3, and 4.

We see a clear analogy in this example with our system of numeration. In our decimal system each group after the 1-group is ten times as large as the next smaller group. In it we use only the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Extending the Pattern

What do we do in the decimal system when we wish to write 1 more than 999? We move from 3-digit notation to 4-digit notation.

So if we were to extend our penny-nickel-quarter system, we could invent a 125¢ coin:

	125¢	25¢	5¢	1¢
$1.23	0	4	4	3
$1.24	0	4	4	4
$1.25	1	0	0	0
$1.26	1	0	0	1
$1.27	1	0	0	2

etc.

We could operate in this system without requiring more than 4 of any of the coins until we reached $6.24. To extend the pattern with complete regularity, we would then have to invent a new coin.

An Independent Activity

Prepare Ditto sheets like the following for some useful practice:

Check the row suggested by the picture.		Fewest number of coins				Amounts of money			
		25¢	5¢	1¢	Total	Quarters	Nickels	Pennies	Total
25¢ 5¢	a.	1	1	1					
	b.	1	2	3					
1¢ 5¢	c.	0	3	1					
1¢ 1¢	d.								14¢
	e.		2			50¢		4¢	
1¢ 1¢	f.	2	2	2					
	g.			0		50¢	20¢		
5¢ 5¢	h.								62¢
25¢ 25¢	i.								49¢
	j.				5	25¢		3¢	
1¢ 5¢ 1¢	k.		3	2		25¢			
	l.	2	4		8				
5¢	m.								93¢
25¢ 5¢	n.								$1.24
	o.				9		20¢	2¢	

Drill

Here are some stories for headline practice:

1. Two crickets started at 0 on the number line. One cricket jumped by 5's and the other jumped by 7's.

   $5 \times \square = 7 \times \triangle$ $\qquad$ $7 - 5 = \bigcirc$

2. Mrs. Friedman's class watched a mathematics program on TV. The 28 children arranged themselves in 4 rows. There were the same number of children in each row.
3. Danny has 58 marbles. Tony has 62 marbles.
4. Miss Anderson had 100 pencils. She gave one to each of her 37 students.
5. A farmer used boards 6 feet long to repair his fence. He used 5 of these boards.
6. Elsa had 49 crayons. She used 7 boxes to hold her crayons. Each box contained the same number of crayons.

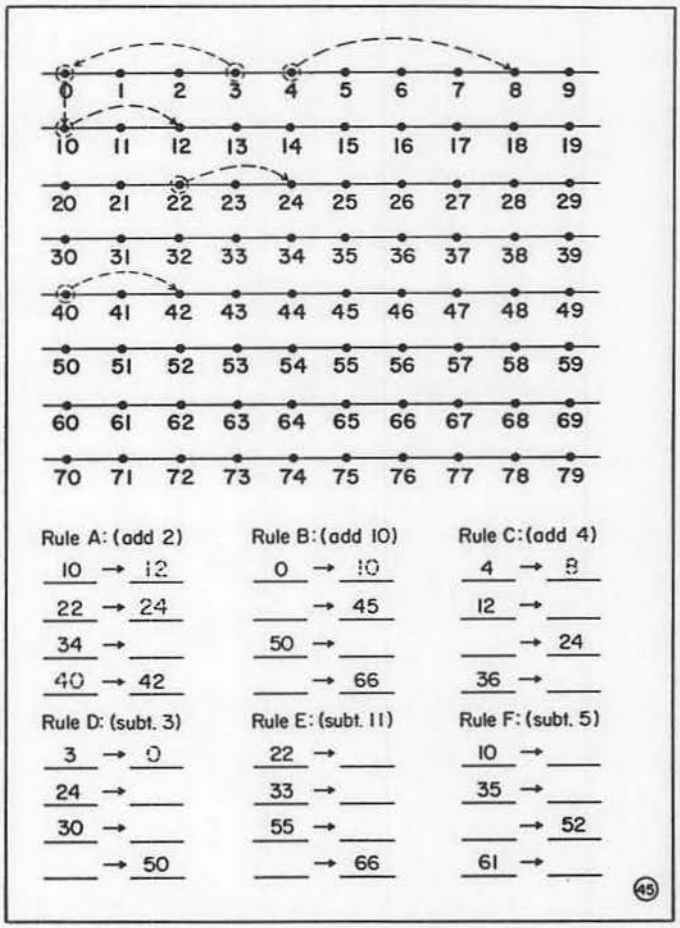
Now might be a good time for a drill sheet of multiplication tables:

×	1	2	3
6			
7			
8			
9			
10			

×	4	5	6
6			
7			
8			
9			
10			

×	7	8	9
6			
7			
8			
9			
10			

Also consider using as a drill sheet page 45 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



What Are My Rules? Look for patterns when you complete the tables.

I

Number of nickels	pennies	cents
8	0	40
7	1	36
6	2	32
5	3	28
4	4	24
4	0	20
3	1	16
2	2	12
1	3	8
0	4	4

less than 5

II

Number of feet	inches	inches
0	5	5
0	10	10
1	3	15
1	8	20
2	1	25
2	6	30
2	11	35
3	4	40
3	9	45
4	2	50

< 12

III

Number of dimes	pennies	cents
4	5	45
4	2	42
3	9	39
3	6	36
3	3	33
3	0	30
2	7	27
2	4	24
2	1	21
1	8	18

< 10

IV

Number of weeks	days	days
0	4	4
0	6	6
1	1	8
1	3	10
1	5	12
2	0	14
2	2	16
2	4	18
2	6	20
3	1	22

< 7

V

Number of pounds	ounces	ounces
0	2	2
0	7	7
0	12	12
1	1	17
1	6	22
1	11	27
2	0	32
2	5	37
2	10	42
2	15	47

< 16

VI

Number of gallons	quarts	quarts
6	3	27
6	0	24
5	1	21
4	2	18
3	3	15
3	0	12
2	1	9
1	2	6
0	3	3
0	0	0

< 4

78 Numeration exercises (bases 5, 12, 10, 7, 16, 4) in translating from each nondecimal base to base 10 and conversely.

Tasks: *Pupils use the clues provided in each table to complete the table.*

Purpose: To continue exploring numeration systems via applications.

Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Coins (dimes, nickels, pennies); foot-rulers; calendars; weights (pounds and ounces); containers (gallons, quarts).

Materials for Pupils: None.

The Lesson:

When can "3, 0" or "30" stand for the measure of 15¢, 21 days, 12 quarts, 48 ounces, 36 inches as well as of 30¢? Only by our agreement that we think of the digits in "3, 0" as follows.

I.	nickels	pennies	
	3	0	→ 15 + 0, or 15, cents
II.	weeks	days	
	3	0	→ 21 + 0, or 21, days
III.	gallons	quarts	
	3	0	→ 12 + 0, or 12, quarts
IV.	pounds	ounces	
	3	0	→ 48 + 0, or 48, ounces
V.	feet	inches	
	3	0	→ 36 + 0, or 36, inches
VI.	dimes	pennies	
	3	0	→ 30 + 0, or 30, cents

On pupil page 78 and on many of the pages to follow we ask pupils to keep a record of counting using *more*

Headline for the Day: Joyce's dog is on a diet. He lost 2 ounces today. Make a table to record what you can say about this situation.

systems of numeration other than the decimal system. The reason that "30" usually stands for the number thirty is because we usually define it that way. According to our usual definition, "30" means 3 ten-groups and 0 one-groups.

In the other numeration systems above, the definitions differ. If we keep a record of days in the form of weeks and days, we have a *base seven* system. In that system "30" or "3, 0" means 3 seven-groups and 0 one-group. For short, a seven-group is called *a week* and a one-group is called *a day*. In recording a time period, we never need to use a digit other than 0, 1, 2, 3, 4, 5, or 6 in the days place. Similarly for other units. To refer to nickels and pennies we use a *base five* system and we never need a digit other than 0, 1, 2, 3, or 4 in the pennies place. Of course, the important idea here is that of record-keeping based on agreements. This is not the time to take pupils into the abstract notion of bases.

Remember that in the real world we are constantly making conversions such as those suggested on pupil page 78. For example, we say:

50 inches is the same as *1 yard, 1 foot, and 2 inches*

90 minutes is the same as *1½ hours* or *1 hour and 30 minutes*

2 pints is the same as *1 quart*

13 eggs is the same as *1 dozen eggs and 1 egg*

35 ounces is the same as *2 pounds and 3 ounces*

In each of these conversions we must know the relationships among the units of a particular system of measurement.

As we consider these systems, our main concern is for the relationships unique to each. Special efforts must be made to keep the children always aware that arithmetic is a handy device to help us report about the real world.

To introduce the work on pupil page 78 discuss the relationships among the units used on the page. Develop a conversion table like the following on the board:

1 nickel = 5 pennies
1 foot = 12 inches
1 dime = 10 pennies
1 week = 7 days
1 pound = 16 ounces
1 gallon = 4 quarts

Proceed to conversion problems (i.e., problems in expressing a given quantity in different units) using charts such as the following:

Nickels	Pennies	Cents
0	1	1
0	2	2
0	3	3
0	4	4
1	0	5
1	1	6

Extend the chart until pupils see the pattern; then present reconstruction problems such as:

n	p	c
3		18
2	3	
	4	19

Show pupils that an amount of money, such as 19¢, can be partitioned in a number of ways, unless we have a restriction:

n	p	c
0	19	19
1	14	19
2	9	19
3	4	19

On pupil page 78 the restrictions are indicated at the bottom of the ones column in each table. There you will find a limitation such as "<5" which means that numbers used in this column must be less than 5.

Since the correct completion of each table requires that pupils determine and complete the patterns in the right-hand columns, perhaps it would be well to have them complete such *What Are My Rules?* patterns as these before attempting the page:

a.

50
45
25

b.

5
10
20
25

c. 40, 35, 30, , , , 10, 5

d. 100, 97, 94, , , ,

e. 2, , 6, , 19, , 14, ,

f. , , 8, 12, , , 24, ,16

g. 2, 8, 13, 18, , , , ,

Drill

Here are some stories for headline practice:

1. Bruno had a half-dollar. He spent 15 cents for postcards.
2. Frank had 267 toy planes. He gave 82 of them to his brother-in-law Roland.
3. Danny placed 36 books on 4 shelves. He placed the same number of books on each shelf.
4. Charlene's father bought 6 bushels of apples. Each bushel weighed 48 pounds.
5. Four boys went on a hike. They each caught 6 black and 4 yellow butterflies.
6. Florence bought 3 packages of gum. There are 5 sticks of gum in a package.

Here is a suggestion for a drill sheet:

1.	$200+50+3=\square$
2.	$300+10+7=\square$
3.	$700+4=\square$
4.	$2000+600+20+7=\square$
5.	$5000+400+5=\square$
6.	$7000+80+9=\square$
7.	$8000+3+20+700=\square$
8.	$9000+500+60+5+30=\square$
9.	$4000+200+70+5+40+2=\square$
10.	$6000+700+40+8+90+3=\square$

A

Number of nickels	Number of pennies	cents
0	2	2
0	4	4
1	1	6
1	3	8
2	0	10
2	2	12
2	4	14
3	1	16
3	3	18
	less than 5	

B

Number of weeks	Number of days	days
0	1	1
0	3	3
0	5	5
1	0	7
1	2	9
1	4	11
1	6	13
2	1	15
2	3	17
	<7	

C

Number of feet	Number of inches	inches
0	2	2
0	5	5
0	8	8
0	11	11
1	2	14
1	5	17
1	8	20
1	11	23
2	2	26
	<12	

D

Number of dimes	Number of pennies	cents
0	4	4
0	8	8
1	2	12
1	6	16
2	0	20
2	4	24
2	8	28
3	2	32
3	6	36
4	0	40
4	4	44
	<10	

E

Number of dozen	Number of eggs	eggs
0	5	5
0	10	10
1	3	15
1	8	20
2	1	25
2	6	30
2	11	35
3	4	40
3	9	45
4	2	50
4	7	55
	<12	

F

Number of hours	Number of minutes	minutes
0	10	10
0	20	20
0	30	30
0	40	40
0	50	50
1	0	60
1	10	70
1	20	80
1	30	90
1	40	100
1	50	110
	<60	

79 Numeration exercises (bases 5, 7, 12, 10, 12, 60) in translating from each nondecimal base to base 10 and conversely.

Tasks: *Pupils use the clues provided in each table to complete the table.*

Purpose: To continue exploring numeration systems via applications.

Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Coins (dimes, nickels, pennies); calendars; foot-rulers; egg cartons; clock; Ditto masters.

Materials for Pupils: Ditto sheets (see below).

The Lesson:

This is a continuation of the work started on pupil page 78.

A "base ten person" counts by 2, starting with 2, as follows:

2, 4, 6, 8, 10, 12, 14, 16, 18, . . .

But, as shown in exercise A, a base five person counts by 2, starting with 2, as follows:

2, 4, 11, 13, 20, 22, 24, 31, 33, . . .

Of course, he doesn't read "11" as "eleven." He might read it as "onenick-one." For the rest, he might read them as:

onenick-three, twonick, twonick-two,
twonick-four, threenick-one, threenick-three, . . .

An Independent Activity

Open Sentences

A. $\triangle - \bigcirc = 12$

$\triangle$	$\bigcirc$

B. $12 - \hexagon = \bigcirc$

$\hexagon$	$\bigcirc$

C. $\bigcirc = \square - 12$

$\bigcirc$	$\square$

D. $12 = \square + \square + \triangle$

$\square$	$\triangle$

E. $\triangle + \bigcirc - \triangle = 12$

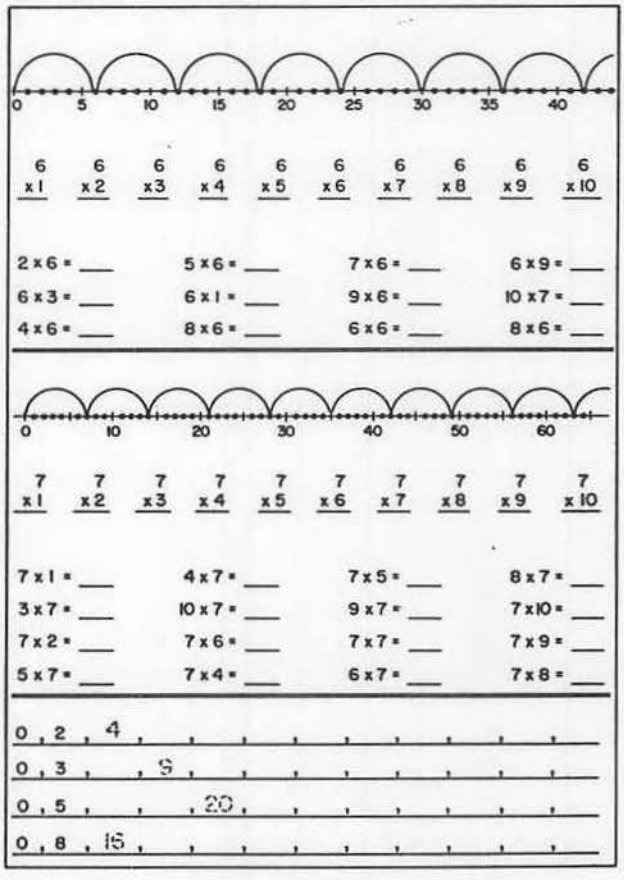
$\triangle$	$\bigcirc$

F. $\hexagon + \triangle = 12 + \hexagon$

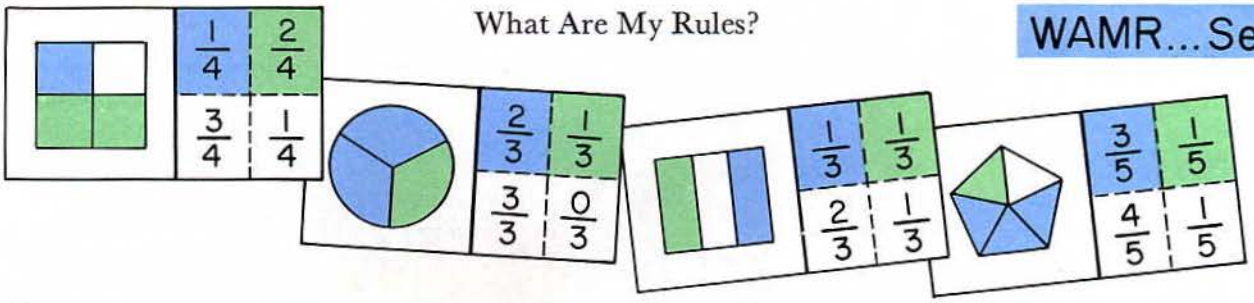
$\hexagon$	$\triangle$

Drill

Here is a suggestion for a drill sheet:



Headline for the Day: Make a table to show the various ways of making change for one dollar using half-dollars, quarters, dimes, and nickels. (Hint: There are 38 different ways.)



I

FRONT	BACK	
	2/4	1/4
	3/4	1/4

II

FRONT	BACK	
	2/5	1/5
	3/5	2/5

III

FRONT	BACK	
	1/3	1/3
	2/3	1/3

IV

	3/9	4/9
	7/9	2/9

V

	1/6	2/6
	3/6	3/6

VI

	4/8	3/8
	7/8	1/8

VII

	2/7	3/7
	5/7	2/7

VIII

	1/10	4/10
	5/10	5/10

80 Using fractional notation to record various partitions of geometric regions.

Tasks: *Pupils use the clues provided at the top of pupil page 80 to discover the four rules for WAMR Set O and apply these rules in completing exercises I-VIII.*

Purpose: To explore fraction concepts.

Unifying Ideas: Number and counting; numeration; addition and subtraction; functions and relations; geometry; measurement.

Materials for Teacher: Colored chalk; Ditto master.

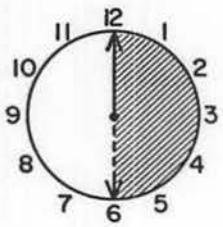
Materials for Pupils: Ditto sheets (see below).

The Lesson:

Each geometric shape on this page is divided into congruent parts (i.e., into parts of the same size and shape). Here pupils will discover that Rule A (upper left) is to write the fraction whose numerator tells the number of parts shaded blue and whose denominator tells the number of parts altogether. Rule B (upper right) deals with parts shaded green; Rule C (lower left) deals with the total number of shaded parts; and Rule D (lower right) deals with the number of unshaded parts. Naturally, the words *numerator* and *denominator* should not be used. If any talk is necessary, say *top number* and *bottom number.*

Clock Study Preparation

Draw a clockface on the board, with the minute hand pointing to 12. Ask: Where will the minute hand be in a half-hour? Have a volunteer draw an indication of the hand's new position.



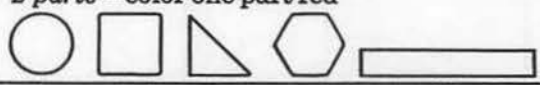
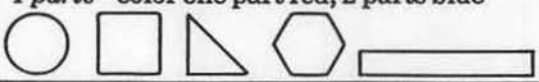
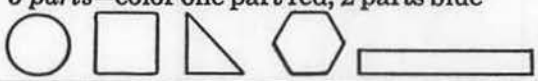
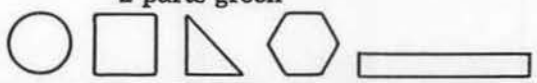
Color in the region the minute hand covered as it traveled from the first to the second position. How much of the clockface is colored? Have we divided the clockface into two parts of about the same size?

Start over with the minute hand pointing to 12. Where will the minute hand be in a quarter of an hour? Where will it be a quarter of an hour later? Indicate both positions. Where will it be half an hour from the starting time? Where will it be three quarters of an hour from the starting time? Color in the part of the face the minute hand passes over in a quarter of an hour. With chalk of another color, shade another part of the face that the hand would pass over in a half-hour.

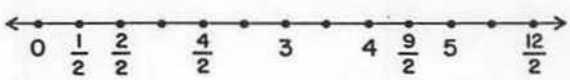
Follow a similar discussion with one third and two thirds of an hour; also with one sixth, two sixths, three sixths, four sixths, five sixths of an hour.

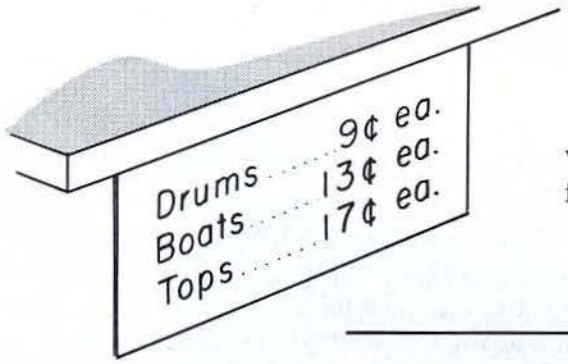
Next, draw a square on the board. Ask: Can someone draw a line that divides the square into 2 parts of about the same size? Can someone else draw a different line that divides the square into 2 such parts? Any more lines? (Some child may draw a zigzag or curved line that would have to be accepted.)

Now prepare a Ditto sheet like the following:

Divide each shape as directed. Make parts as nearly the same size as you can.
2 parts —color one part red 
4 parts—color one part red, 2 parts blue 
3 parts—color one part red, 2 parts blue 
6 parts—color one part red, 2 parts blue, 2 parts green 

Headline for the Day: Name the unnamed points on the number line.





Harry bought some toys at the toy store where there were drums and boats and tops. He bought fewer than four toys and gave the clerk a dollar.

What can we say?

Suppose that Harry bought only one toy. He could do that in three ways:

(a) 1 drum 9 ¢; and 91 ¢ change

(b) 1 boat 13 ¢; and 87 ¢ change

(c) 1 top 17 ¢; and 83 ¢ change

Suppose that he bought two toys.

	Number of drums	Number of boats	Number of tops	Cost of drums	Cost of boats	Cost of tops	Total cost	Change from $1.00
(d)	2	0	0	18¢	0¢	0¢	18¢	82¢
(e)	1	1	0	9¢	13¢	0¢	22¢	78¢
(f)	1	0	1	9¢	0¢	17¢	26¢	74¢
(g)	0	2	0	0¢	26¢	0¢	26¢	74¢
(h)	0	1	1	0¢	13¢	17¢	30¢	70¢
(i)	0	0	2	0¢	0¢	34¢	34¢	66¢

Suppose that he bought three toys.

	Number of drums	Number of boats	Number of tops	Cost of drums	Cost of boats	Cost of tops	Total cost	Change from $1.00
(j)	3	0	0	27¢	0¢	0¢	27¢	73¢
(k)	2	1	0	18¢	13¢	0¢	31¢	69¢
(l)	2	0	1	18¢	0¢	17¢	35¢	65¢
(m)	1	2	0	9¢	26¢	0¢	35¢	65¢
(n)	1	1	1	9¢	13¢	17¢	39¢	61¢
(o)	1	0	2	9¢	0¢	34¢	43¢	57¢
(p)	0	3	0	0¢	39¢	0¢	39¢	61¢
(q)	0	2	1	0¢	26¢	17¢	43¢	57¢
(r)	0	1	2	0¢	13¢	34¢	47¢	53¢
(s)	0	0	3	0¢	0¢	51¢	51¢	49¢

Now, what can you say?

81 Application problems (What Can You Say?) dealing with possible combinations of purchases and their costs.

Tasks: Pupils use the clues provided at the top of pupil page 81 to complete each table of possible purchases.

Purpose: To explore applications via *What Can We Say?* games while simultaneously perfecting computational skills.

Unifying Ideas: Addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

For pupils who are not able to complete pupil page 81 on their own, you may wish to introduce it by doing a similar problem with them. Possible conditions are:

> John went into a store to buy some ice cream, cookies, and candy. Ice cream was 15¢, cookies 7¢, and candy bars 10¢. Suppose that John had a half-dollar and bought one item. What can you say?

Answers should include:

> 1 ice cream 15¢; and ___ change
> 1 cookie 7¢; and ___ change
> 1 candy bar 10¢; and ___ change

Suppose that he bought two things. Make a table:

Number of			Cost of			
Ice cream	Cookies	Candy bars	Ice cream	Cookies	Candy bars	Total	Change
2	0	0	30¢	0¢	0¢	30¢	20¢
1	1	0	30¢	0¢	0¢	37¢	13¢
1	0	1	30¢	0¢	10¢	40¢	10¢
etc.							

more

Headline for the Day: Name the unnamed points on the number line.

In completing the charts on pupil page 81, pupils are getting experience with all the operations of arithmetic.

Remember to answer the question we posed at the top of pupil page 81: *What Can We Say?* Here are some answers:

There are exactly 3 ways of choosing 1 toy from 3 kinds of toys.

There are exactly 6 ways of choosing 2 toys from 3 kinds of toys.

There are exactly 10 ways of choosing 3 toys from 3 kinds of toys.

For the Front Runners

As a follow-up activity, pose the question: In how many ways would we be able to choose 4 toys if we added toy bats to the list? There are 35 ways to do this, and the easiest way to keep track of the possibilities is to use a systematic listing such as we have used on pupil page 81. For example, start with the purchase "4000" and then list the purchases in "decreasing" order:

4000, 3100, 3010, 3001, 2200, 2110, 2101, 2020, 2011, 2002, 1300, etc.

In effect, in making this systematic listing you are writing all the numbers that have four or fewer digits and whose digits add up to 4.

Independent Activities

Open Sentences

A. $2+\triangle=\bigcirc+10$

B. $\square+\bigcirc-3=17$

C. $2\times\bigcirc=\triangle+12$

D. $(\square\times\bigcirc)+7=\square$

E. $3\times\bigcirc=\triangle+3$

F. $\square+\bigcirc+3=17$

Sequences

Each term after the second is the sum of the two terms on its left.

I. 1, 1, , , , , , , , ,

II. 3, 3, , , , , , , , ,

III. 4, 4, , , , , , , , ,

IV. 2, 2, , , , , , , , ,

V. 6, 6, , , , , , , , ,

VI. , , , , , 8, 13, , ,

VII. , , , , 10, , 26, , ,

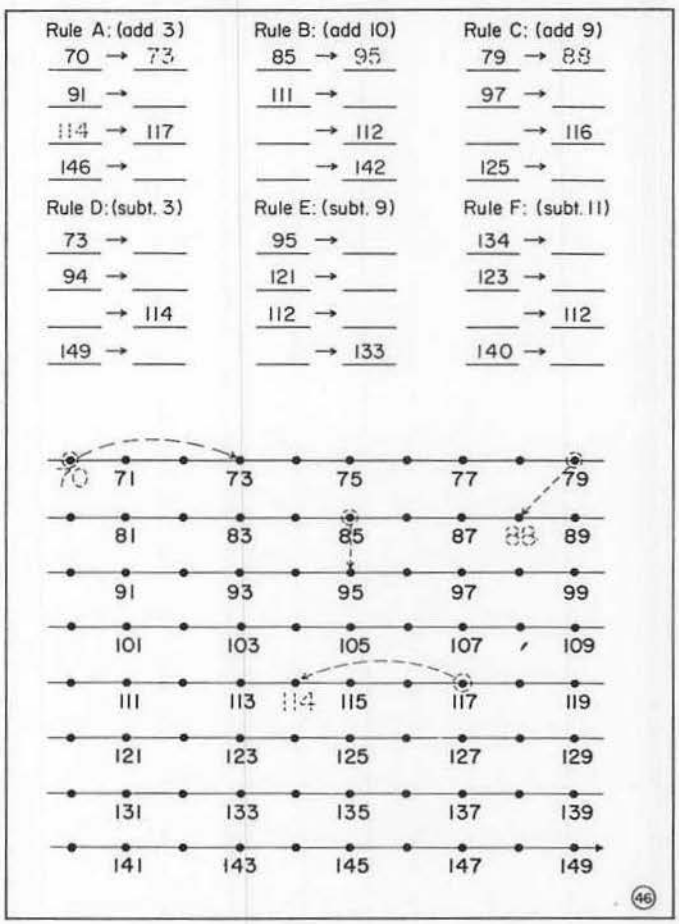
VIII. , , , 9, , 24, , , ,

IX. , 6, , 18, , , , , ,

X. , , 18, , 45, , , , ,

Drill

Consider using as a drill sheet page 46 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



At the top of the next page is another suggestion for a drill sheet.

1.	$5 \times 4 = \square$	**2.**	$2 \times 9 = \square$
3.	$3 \times 6 = \square$	**4.**	$4 \times 4 = \square$
5.	$6 \times 3 = \square$	**6.**	$3 \times 9 = \square$
7.	$4 \times 7 = \square$	**8.**	$6 \times 4 = \square$
9.	$5 \times 4 = \square$	**10.**	$7 \times 9 = \square$
11.	$8 \times 5 = \square$	**12.**	$6 \times 7 = \square$
13.	$9 \times 2 = \square$	**14.**	$6 \times 5 = \square$
15.	$9 \times 8 = \square$	**16.**	$6 \times 9 = \square$
17.	$8 \times 7 = \square$	**18.**	$8 \times 6 = \square$
19.	$35 \div 5 = \square$	**20.**	$18 \div 6 = \square$
21.	$20 \div 4 = \square$	**22.**	$18 \div 2 = \square$
23.	$14 \div 7 = \square$	**24.**	$12 \div 4 = \square$
25.	$9 \div 9 = \square$	**26.**	$56 \div 7 = \square$
27.	$42 \div 6 = \square$	**28.**	$15 \div 5 = \square$
29.	$40 \div 8 = \square$	**30.**	$24 \div 8 = \square$
31.	$63 \div 7 = \square$	**32.**	$24 \div 4 = \square$
33.	$20 \div 5 = \square$	**34.**	$45 \div 9 = \square$
35.	$30 \div 5 = \square$	**36.**	$35 \div 7 = \square$

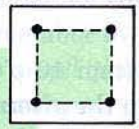
Here are some stories for headline practice:

1. Mr. Fiddler arranged the chairs in his classroom so that there were 6 rows of chairs with 4 chairs in each row.
2. We visited Uncle Joe's house from 10 o'clock in the morning until 4 o'clock in the afternoon.
3. Noren put 4 flowers in each vase. She had 28 flowers.
4. Flavio had 6 rows of stamps in his stamp book. There were 5 stamps in each row.
5. Herman started at 0 on the number line and made jumps of 7 to the right. He made 9 such jumps.
6. Dick went to the circus. He noticed 4 clowns. Each clown held 8 balloons.
7. Grace and Antoinette walked from Grace's house 22 blocks to the movies. On the way home Antoinette stopped at her house which was only 7 blocks from the movies.

A . . . on these two pages is AREA. Here is one unit of area:

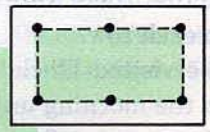
P . . . on these pages is PERIMETER. Here is one unit of perimeter: or:

a



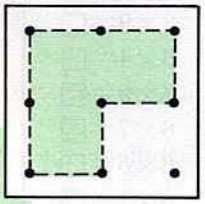
A 1

b



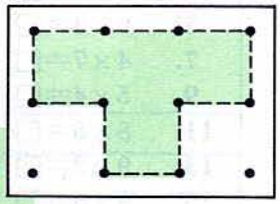
A 2

c



A 3

d

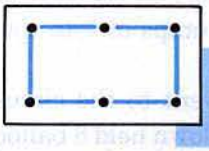


A 4

e

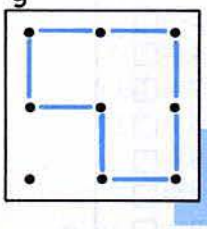
P 4

f



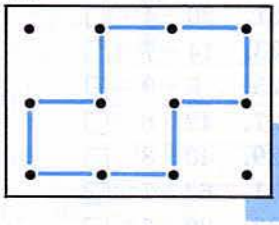
P 6

g



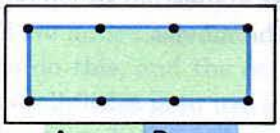
P 8

h



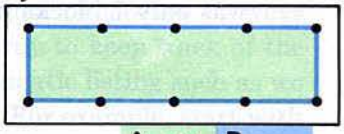
P 10

i



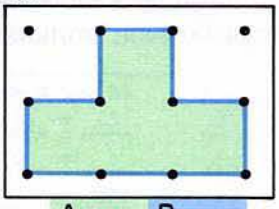
A 3 P 8

j



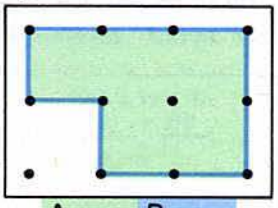
A 4 P 10

k



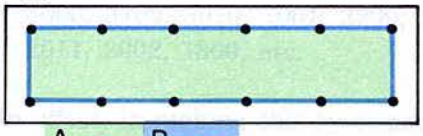
A 4 P 10

l



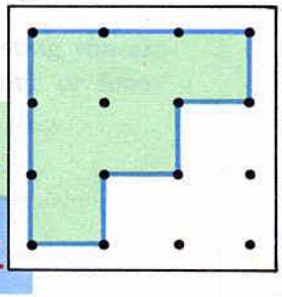
A 5 P 10

m



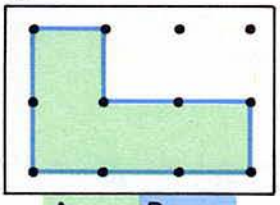
A 5 P 12

n



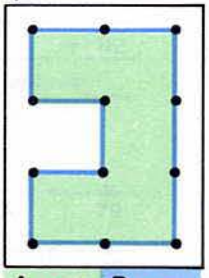
A 6 P 12

o



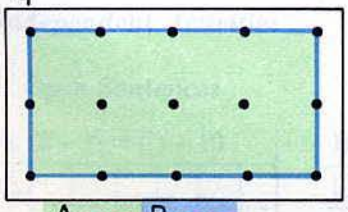
A 4 P 10

p



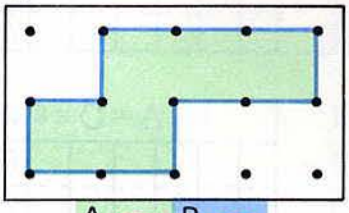
A 5 P 12

q



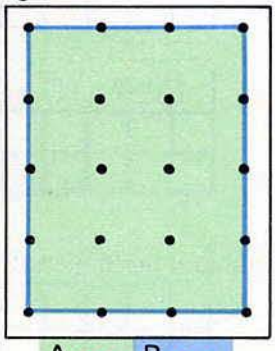
A 8 P 12

r



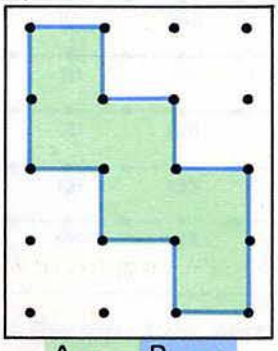
A 5 P 12

s



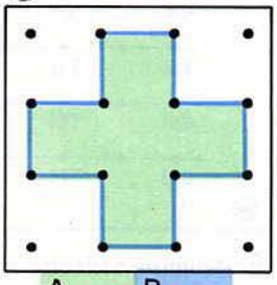
A 12 P 14

t



A 6 P 14

u



A 5 P 12

82 Computation exercises dealing with areas and perimeters of various rectilinear figures.

Tasks: Pupils use the clues provided at the top of pupil page 82 to record in the tinted box adjacent to each representation the area or the perimeter of each figure as required.

Purpose: To explore area and perimeter concepts.

Unifying Ideas: Number and counting; geometry; measurement.

Materials for Teacher: Graph Board; Counting Board; rubber bands.

Materials for Pupils: Geo-boards; rubber bands; graph paper.

The Lesson:

On pupil pages 82 and 83 pupils are exploring the ideas of area and perimeter. You will find graph paper and Geo-boards to be very helpful aids in the explorations. (See pages 80-86 of the booklet *Games and Enrichment Activities.*)

Using Geo-boards and rubber bands, pupils can create figures similar to those on pupil page 82:

Figure	Area	Perimeter
A	1	4
B	2	6
C	2	6

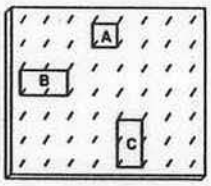


Figure	Area	Perimeter
D	3	8
E	3	8

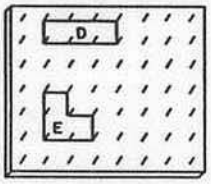
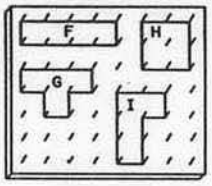
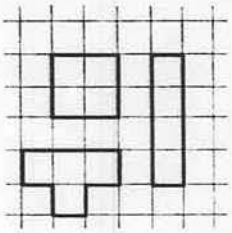


Figure	Area	Perimeter
F	4	
G	4	
H		



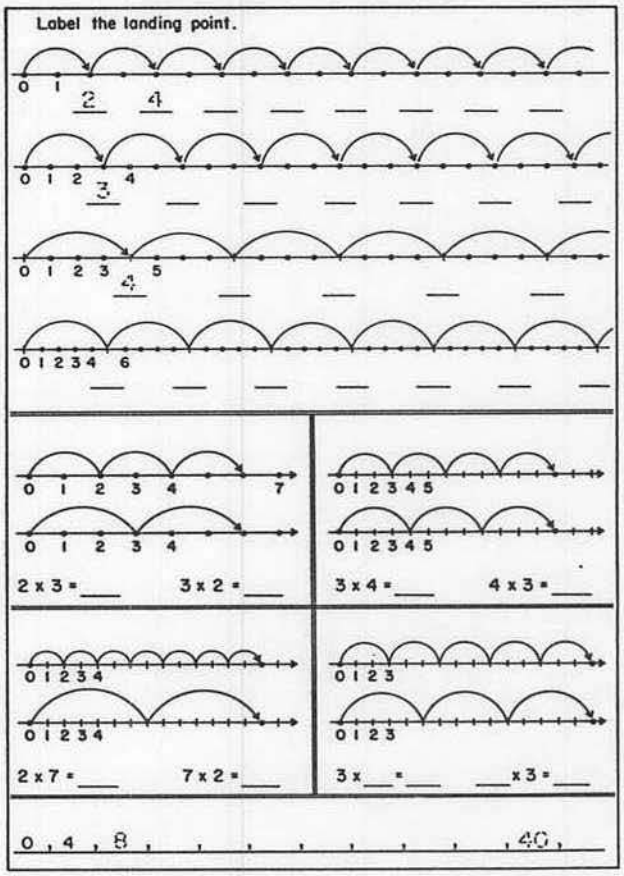
Similar investigations can be made for figures of a variety of shapes having areas of 5 units, 6 units, etc.

Pupils can use graph paper to explore the same relationships. Here are some figures on graph paper. Each has area 4 but the perimeters differ:



Drill

Here is a suggestion for a drill sheet:

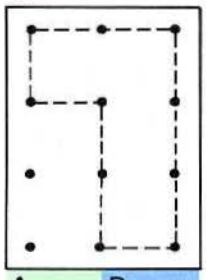


Headline for the Day: Please draw a figure with area 8 and perimeter 12.

Please draw sketches with AREAS and PERIMETERS as indicated.

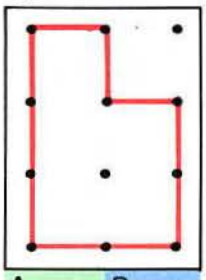
* MANY POSSIBLE ANSWERS THROUGHOUT.

I



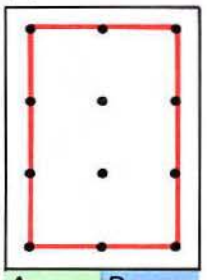
A 4 P 10

II



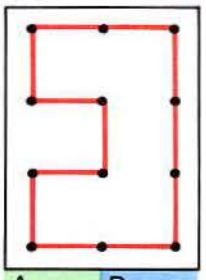
A 5 P 10

III



A 6 P 10

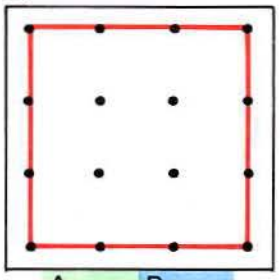
IV



A 5 P 12

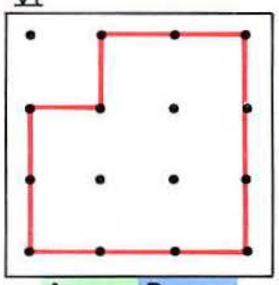
(a puzzle)

V



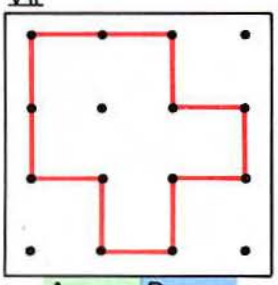
A 9 P 12

VI



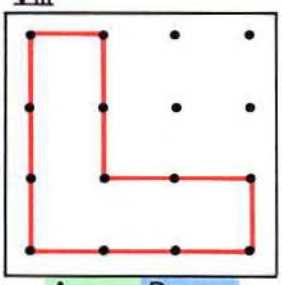
A 8 P 12

VII



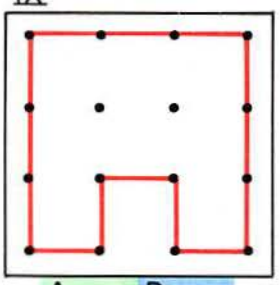
A 6 P 12

VIII



A 5 P 12

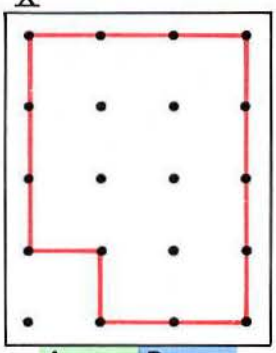
IX



A 8 P 14

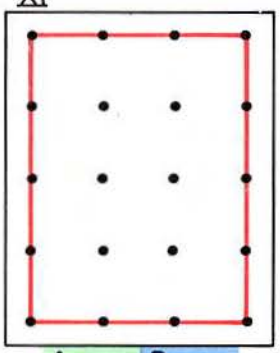
(a puzzle)

X



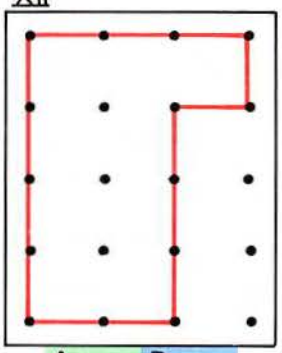
A 11 P 14

XI



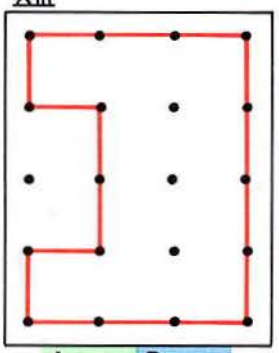
A 12 P 14

XII



A 9 P 14

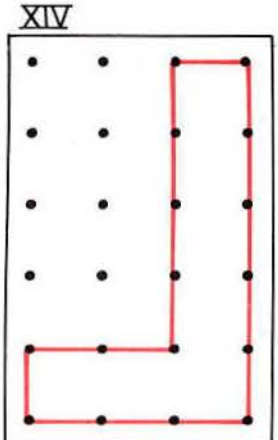
XIII



A 10 P 16

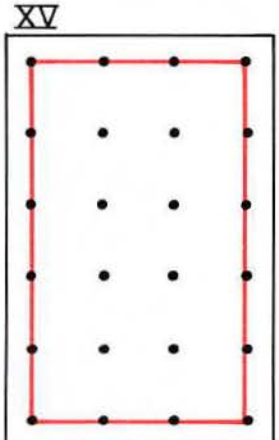
(a puzzle)

XIV



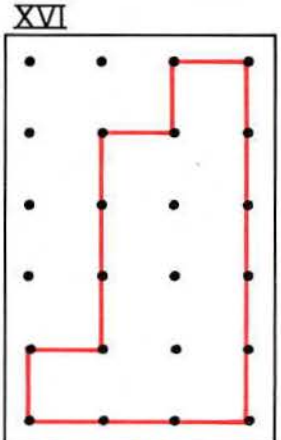
A 7 P 16

XV



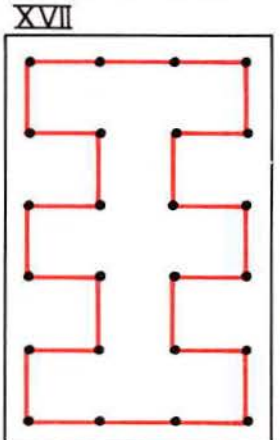
A 15 P 16

XVI



A 10 P 16

XVII



A 11 P 24

(a puzzle)

83 Construction exercises dealing with areas and perimeters of various rectilinear figures.

ks: *Pupils draw figures with the areas and perimeters indicated in the tinted boxes.*

Purpose: To continue exploring concepts of area and perimeter.

Unifying Ideas: Number and counting; geometry; measurement.

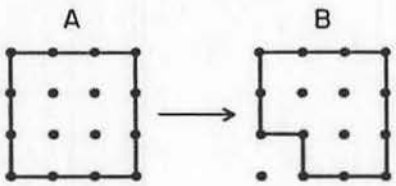
Materials for Teacher: Graph board; Counting Board; rubber bands.

Materials for Pupils: Geo-boards; rubber bands; graph paper.

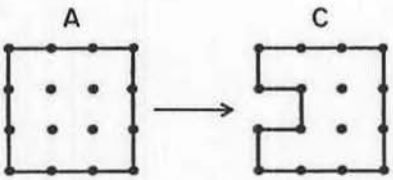
The Lesson:

On this page pupils are given areas and perimeters and are asked to draw figures to match the conditions. As on the previous page, pupils should be able to work with Geo-boards and graph paper. This page is a good one for pupil teams to work on. Do not expect each pupil to do all of the exercises.

Some pupils are likely to make interesting dis-eries. Here are two examples of such discoveries:



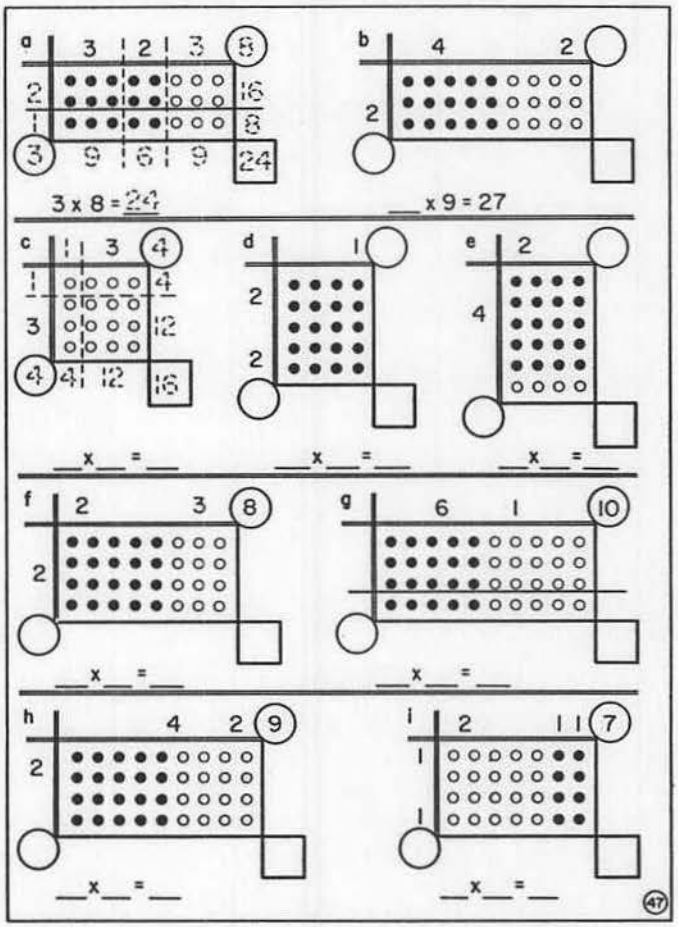
In going from A to B, you reduce the area by 1 but keep the perimeter the same.



In going from A to C, you reduce the area by 1 but increase the perimeter by 2.

Drill

Consider using as a drill sheet page 47 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Here are some stories for headline practice:

1. Lillie had 32 playing cards. She put them in packs of 4 each.
2. Dahlia had 5 rabbits. Each rabbit ate 2 bunches of carrots. In each bunch there were 4 carrots.

   $5 \times 2 \times 4 = \square$ $\quad$ $4 \times 2 \times 5 = 40$
3. Tippi needed 32 candies for her party. She bought packages of candy which contained 8 candies each.

Headline for the Day: Please draw a sketch with area 14 and perimeter 16.

Show the largest amount with an A in the blue block; the next largest with a B; the next largest with a C, etc. Write the differences in each example.

I

Nine weeks ..... A

Sixty days C

The months of April and May together ..... B

Difference between

A and B: 2 da.

B and C: 1 da.

A and C: 3 da.

II

One yard, two feet, and eleven inches ... C

Six feet ..... B

Two yards and one inch ..... A

Difference between

A and B: 1 in.

B and C: 1 in.

A and C: 2 in.

III

Forty-nine ounces C

Two pounds and eighteen ounces ... B

Three pounds ..... D

One pound and thirty-five ounces .... A

Difference between

A and B: 1 oz. B and D: 2 oz.

B and C: 1 oz. C and D: 1 oz.

A and C: 2 oz. A and D: 3 oz.

IV

Fifty-five pints C

Seven gallons ..... B

Six gallons and nine pints ..... A

Four gallons, five quarts, and twelve pints .... D

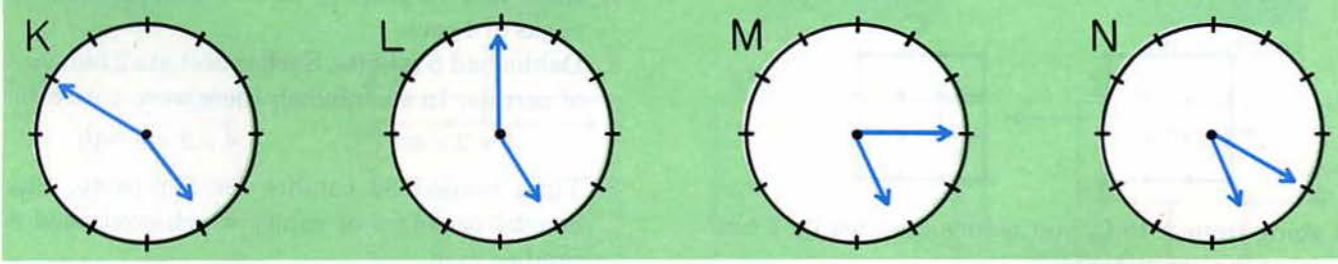
Difference between

A and B: 1 pt. B and D: 2 pts.

B and C: 1 pt. C and D: 1 pt.

A and C: 2 pts. A and D: 3 pts.

V Sally was waiting for her favorite TV program to begin at 5:45. She looked at the clock four different times before the program started:



How long was it before the program started in each case?

K: 55 min. to wait

L: 45 min. to wait

M: 30 min. to wait

N: 25 min. to wait

How long was it between the times she looked at the clock?

K and L: 10 min. K and M: 25 min.

K and N: 30 min. L and M: 15 min.

L and N: 20 min.

M and N: 5 min.

84 Numeration exercises (multi-base) in ranking magnitudes, and finding differences between magnitudes.

Tasks: *In each of exercises I-IV pupils indicate the largest amount by writing an A in the blue block following the amount; they indicate the next largest amount by writing a B, etc. They then record in the blanks the differences as required. In exercise V pupils record the time intervals and differences.*

Purpose: To explore numeration systems via applications while simultaneously perfecting computational skills.

Unifying Ideas: Numeration; addition and subtraction; measurement.

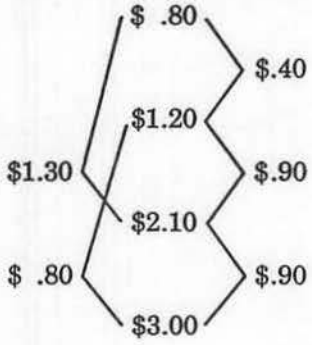
Materials for Teacher: Calendar; yardstick; foot-ruler; weights (pounds and ounces); milk containers (gallons, quarts, and pints); clock.

Materials for Pupils: None.

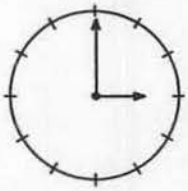
The Lesson:

If pupils need to be specially prepared for this page, you can do so through activities such as the following.

1. Arrange amounts, lengths, and numbers in order from smallest to largest.
   a. \$2.10, \$3.00, \$.80, \$1.20
   b. 1 inch, 1 foot, 5 feet, 1 yard, 10 inches
   c. 220, 130, 210, 75, 325
2. Compare and find the differences between successive or alternate amounts, distances, etc.



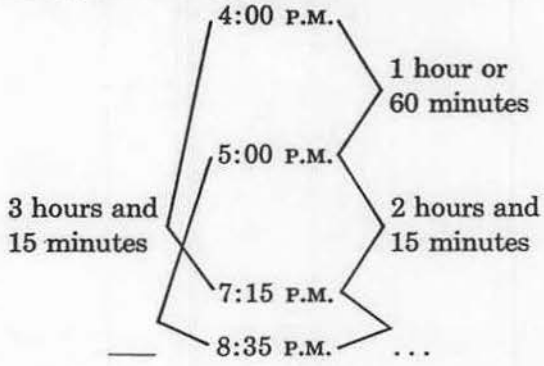
3. Readiness for the time problem should include actual manipulations of a clockface, keeping records as you go along.



time now ___
3 hours later ___
1½ hours later ___
15 minutes later ___

time now ___
3 hours earlier ___
1½ hours earlier ___
15 minutes earlier ___

4. Compare time differences:



Drill

Here is a suggestion for a drill sheet:

1.	$(2\times 5)+(3\times 5)=\bigcirc$
2.	$(4\times 2)+(5\times 2)=\square\times 2=\bigcirc$
3.	$(7\times 4)+(3\times 4)=\square\times 4=\bigcirc$
4.	$(8\times 6)+(1\times 6)=9\times\triangle=\bigcirc$
5.	$(3\times 7)+(2\times 7)=\square\times 7=\bigcirc$
6.	$(5\times 9)+(5\times 9)=\bigcirc$
7.	$(6\times 8)+(4\times 8)=\bigcirc$
8.	$(7\times 7)+(3\times 7)=\bigcirc$
9.	$(4\times 10)+(6\times 10)=\bigcirc$
10.	$12\times 7=(10\times 7)+(\square\times 7)=\bigcirc$

Headline for the Day: Fill the blanks to continue the number pattern.

350,339,328, , , , , , ,251,240

Numo the Magician

At a school show, Numo and his sister, Numa, put on a great act. At first, people thought it was magic.

First, someone took Numo out in the hall so that he couldn't see what was going on.

Then Numa asked someone to write a number on the board — one of the whole numbers — 1 thru 32. She would then cover the number with a heavy piece of cardboard and have someone call Numo back into the room.

After a look at Numa, Numo would stare at the cardboard.

"I'll put my x-ray eyes to work," Numo would say. "Through the card I can see the number 17 on the board."

Everyone said, "Wow! He was right!" Numa lifted the card for everyone to see — 17. Mothers and fathers were as surprised as the children.

After Numo did the trick many times without missing, everyone called out, "Tell us how you do it!"

Numo looked at Numa. Should he tell them? After thinking about it for a second, she said, "Let's do it."

Numo Gives the Secret Away

"Well," Numo began, "I'll tell you the secret, if you promise not to tell anyone else."

Everyone called out, "We promise."

"Well, it's very simple. Numa tells me with the fingers on her right hand what the secret number is."

Numo and Numa sat down.

Everyone was quiet for a minute.

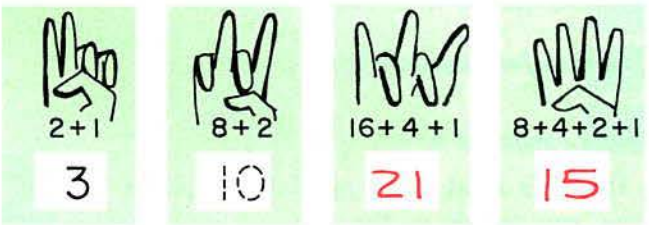
Then someone called out, "How can she do that?"

Numo stood up. "You see, most people can only count to 5 with the fingers of one hand. Numa and I can count to 32 with the same 5 fingers."

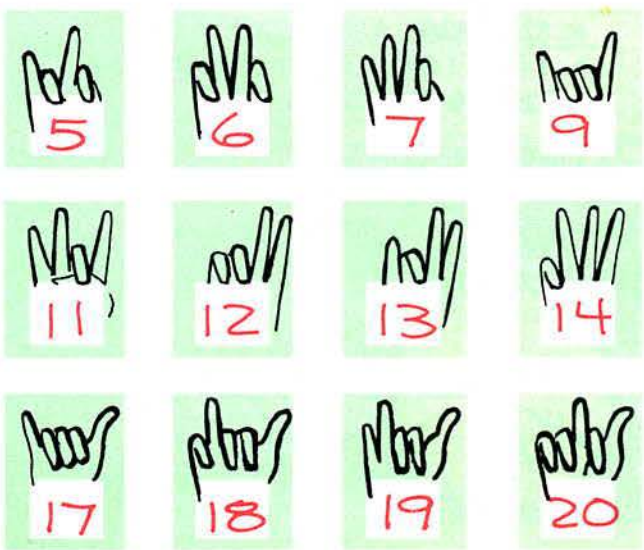
"How can you do it?"

"We do it this way. We let the little finger stand for 1, the ring finger for 2, the middle finger for 4, the forefinger for 8, and the thumb for 16." He drew a picture on the board.

"For other numbers, we use two or more fingers."



"Now you try it. What do these fingers stand for?"



Then Numo asked three questions:

1. "Can you find a whole number between 1 and 32 we can't make on the fingers of one hand?
2. "Can we make any number in more than one way?
3. "What do no fingers up stand for?" 32

85 Introducing a representation (hand and fingers) for a base 2 numeration game.

Tasks: *Pupils write numbers as required in the white boxes shown in the right-hand column.*

Purpose: To make translations from one numeration system to another via representations.

Unifying Ideas: Numeration.

Materials for Teacher: Strips of squares (1, 2, 4, 8, 16); milk containers (gallons, half-gallons, quarts, pints, half-pints).

Materials for Pupils: Cuisenaire rods.

The Lesson:

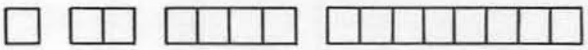
In this story of Numo and Numa, we introduce another numeration pattern—the binary pattern—one used in some computers.

This is an excellent page for pupils to work on in teams. Have them study the page and complete the examples. Then have boy-and-girl pairs demonstrate that they know Numo's secret as Numo and Numa did.

Here are two related activities which extend and reinforce the ideas on this page. They may be introduced first or they may follow the pupil page.

1-, 2-, 4-, 8-strips

We have 4 strips of cardboard that suggest 1, 2, 4, and 8:



How many numbers can be represented by these strips singly or in combination?

How many can we represent by using them one at a time?

How many by using 2 strips at a time?

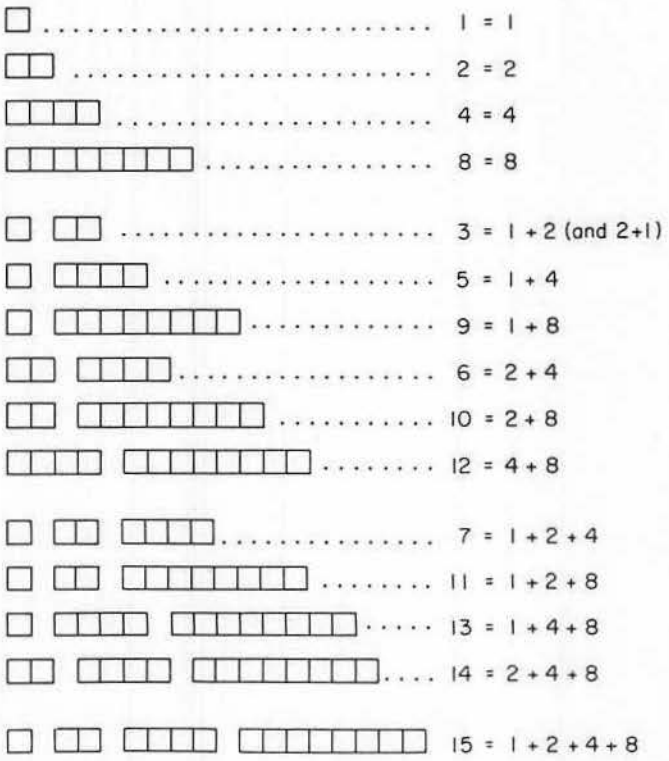
How many by using 3 strips at a time?

How many by using 4 strips at a time?

more

Headline for the Day: Write a number sentence that fits: In 1984 I will be ___ years old.

Keep a record of these results. Write an appropriate mathematical sentence for each result:



Does this list show representations for all the numbers 1 through 15?

Can we represent any of these numbers in more than one way—that is, with combinations of different strips? (If □ □□ and □□ □ are not considered different combinations the answer is No.)

Suppose that we wished to add another strip so that we could represent as many as possible of the consecutive numbers that are larger than 15: 16, 17, 18, 19, 20, 21, etc. What size strip should we choose? (An extra strip representing 16 would enable us to represent all numbers through 31.)

You may wish to point out that each of the three larger strips represents a number twice as large as the next smaller strip.

$2 = 1 \times 2$ $\qquad$ $8 = 4 \times 2$

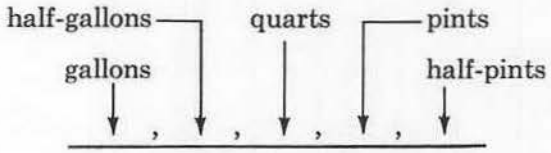
$4 = 2 \times 2$ $\qquad$ $16 = 8 \times 2$

Fewest Containers

Consider a system with which we are all familiar:

2 half-pints is the same as 1 pint
2 pints is the same as 1 quart
2 quarts is the same as 1 half-gallon
2 half-gallons is the same as 1 gallon

For this system we invent a 5-place shorthand so that we need not keep writing the names of the units:



In this shorthand we could write 1 gallon, 3 quarts, and 3 half-pints in this way:

1, 0, 3, 0, 3

But if we wish to indicate the *fewest* containers in which that amount of milk can be purchased, we note that 1 half-gallon and 1 quart is the same as 3 quarts, and 1 pint and 1 half-pint is the same as 3 half-pints. So we could purchase the total amount in one container of each size; we can indicate this by:

1, 1, 1, 1, 1

Suppose that there were 11 children in a class. Billy is sent out to buy 11 half-pints of milk in as few containers as possible. It would help him to know that:

There is 1 half-pint in a 1 half-pint container.

There are 2 half-pints in a 1 pint container.

There are 4 half-pints in a 1 quart container.

There are 8 half-pints in a 1 half-gallon container.

There are 16 half-pints in a 1 gallon container.

He would select 1 half-gallon, 1 pint, and 1 half-pint, and we could report this by writing:

0, 1, 0, 1, 1 (11 half-pints)

Suppose that there were 14 children in another class. Sam is sent to buy 14 half-pints in the fewest containers. In the shorthand we are using, he would select

0, 1, 1, 1, 0 (14 half-pints),

or 0 gallons, 1 half-gallon, 1 quart, 1 pint, and 0 half-pints.

But it is a waste of time to send two boys out to buy 11 and 14 half-pints of milk. If Billy went alone, how could he get the 11 + 14, or 25, half-pints of milk in the fewest containers? We can summarize the whole problem as follows.

Number of half-pints in the container	16 Gal.	8 Half-gal.	4 Quarts	2 Pints	1 Half-pints	
	0	1	0	1	1	11 half-pints
	0	1	1	1	0	14 half-pints
						25 half-pints

We see that he could purchase 2 half-gallons, 1 quart, 2 pints, and 1 half-pint. But that is not the fewest containers. He could get 1 quart instead of 2 pints. He might try all the following combinations:

0,	2,	1,	2,	1
0,	2,	2,	0,	1
0,	3,	0,	0,	1
1,	1,	0,	0,	1

and find that all he needed was 3 containers—1 gallon, 1 half-gallon, 0 quarts, 0 pints, and 1 half-pint.

These results can be condensed in the following shorthand:

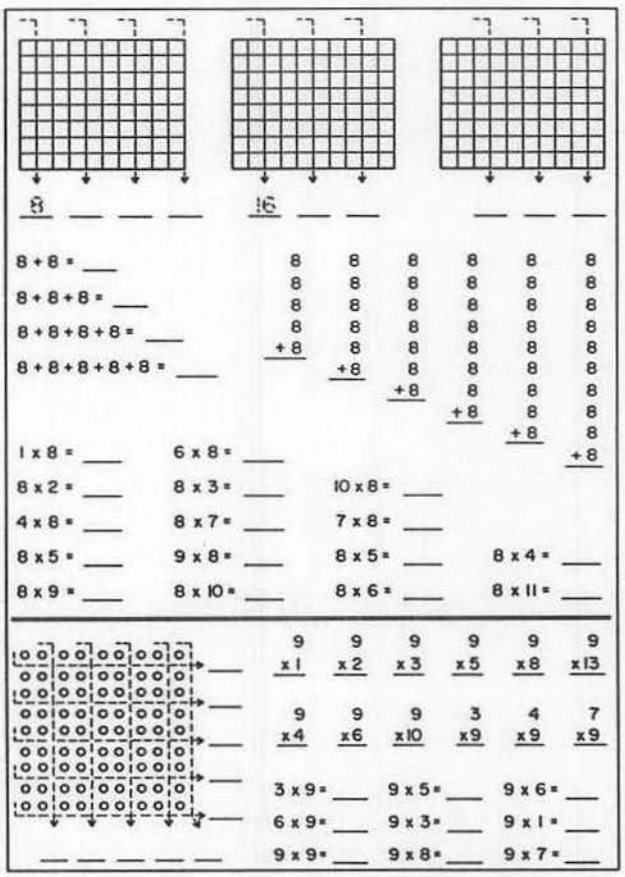
	0,	1,	0,	1,	1	(11 half-pints)
+	0,	1,	1,	1,	0	(14 half-pints)
	1,	1,	0,	0,	1	(25 half-pints)

Drill

Here are some stories for headline practice:

1. For the PTA program the school chorus used 3 rows of chairs. Eighteen children sat in each row.
2. Eddie baked a pie that weighed 30 ounces. He cut his pie into 3 pieces of the same size.
3. Connie had 3 boxes of crayons. There were 6 crayons in each box.
4. The Cub Scouts went on a picnic. There were 35 scouts in the pack. They rode in 7 cars. There were the same number of scouts in each car.
5. Solomon had \$7.03 in the Bank. He withdrew \$1.15.
6. Matt gave each of his friends 5 gumdrops. He had 35 gumdrops.

Here is a suggestion for a drill sheet:



T . . . on these two pages is the number of TILES this size: □

E . . . on these pages is the distance around EDGE in units this size: — or: |

	I	II	III	IV	V	VI
a	T 1 E 4	T 2 E 6	T 3 E 8	T 4 E 10	T 5 E 12	T 6 E 14
b	T 2 E 6	T 4 E 8	T 6 E 10	T 8 E 12	T 10 E 14	T 12 E 16
c	T 3 E 8	T 6 E 10	T 9 E 12	T 12 E 14	T 15 E 16	T 18 E 18
d	T 4 E 10	T 8 E 12	T 12 E 14	T 16 E 16	T 20 E 18	T 24 E 20
e	T 5 E 12	T 10 E 14	T 15 E 16	T 20 E 18	T 25 E 20	T 30 E 22

Can you imagine the next row of sketches?

f	T 6 E 14	T 12 E 16	T 18 E 18	T 24 E 20	T 30 E 22	T 36 E 24

86 Exercises in areas and perimeters of various rectangular arrangements of square tiles.

Tasks: *Pupils record on the card beneath each representation the number of tiles (area) and the distance around the edge (perimeter). For the bottom row of representations pupils record their predictions of what the areas and perimeters would be if the representations were completed.*

Purpose: To explore the concepts of area and perimeter.

Unifying Ideas: Number and counting; geometry; measurement.

Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

On this page we continue the study of areas and perimeters in a familiar setting. In the first place all of the sketches on the page are rectangular. Pupils will recall that such rectangular arrays are representations of the multiplication and division facts. Moreover, the arrangement of rectangular arrays parallels that of a two-dimensional multiplication table:

	I	II	III	IV	V	VI
a	1	2				
b	2	4				
c	3	6				
d						
e						
f						

Perhaps it would be well to have pupils attempt to complete a 6-by-6 multiplication table on the board. When pupils have completed the record of tiles (area) and edges (perimeter) on pupil page 86, ask them if they see the relationship of the table on the board to the page they were working on.

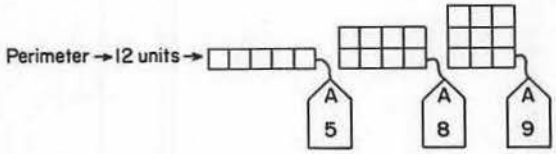
Help pupils notice that on pupil page 86 rectangles

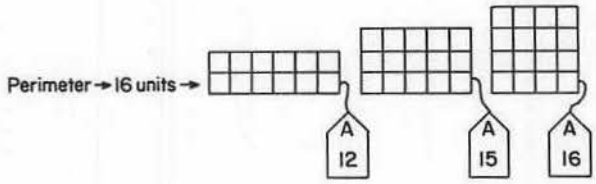
more

Headline for the Day: Tell a story to fit this fact team: $1000 + 500 = \square$ $\quad 1000 = \square - 500$

$500 + 1000 = \square$ $\quad 500 = \square - 1000$

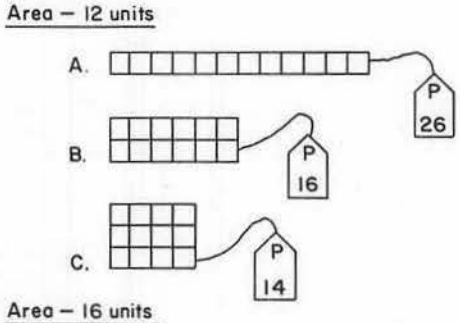
with the same perimeter are arranged along diagonals. Let's consider all the rectangles that have a certain perimeter. Of these, what is the shape of the rectangle that has the largest area?

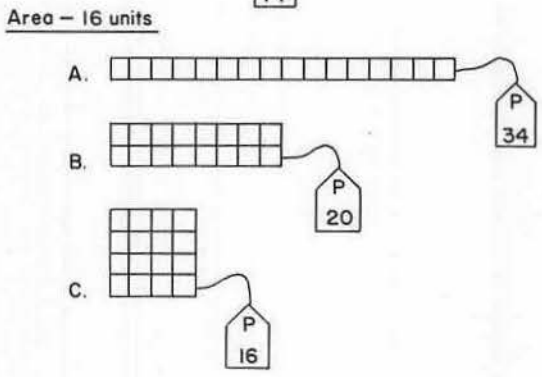




Notice that the square has the largest area for a given perimeter.

The exploration may begin from another direction. Given all rectangles with a certain area, what is the shape of the one with the smallest perimeter? Let's explore the areas 12 and 16:





As we "approach" the square among rectangles with the same area, what happens to the perimeter?

How can we enclose a rectangular region with area 24 units with the shortest length of fence? Possible rectangular regions are suggested by the open sentence:

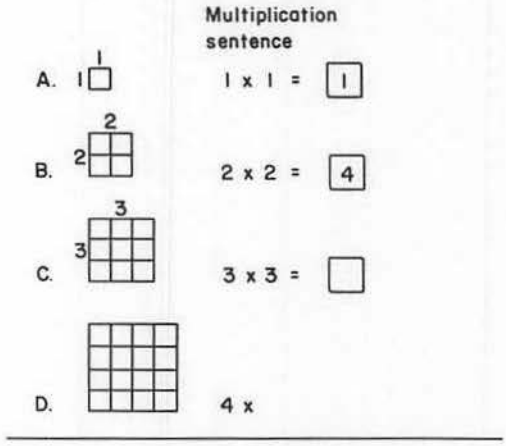
$$\bigcirc \times \triangle = 24$$
$$1 \times 24 = 24$$
$$2 \times 12 = 24$$
$$3 \times 8 = 24$$
$$4 \times 6 = 24$$

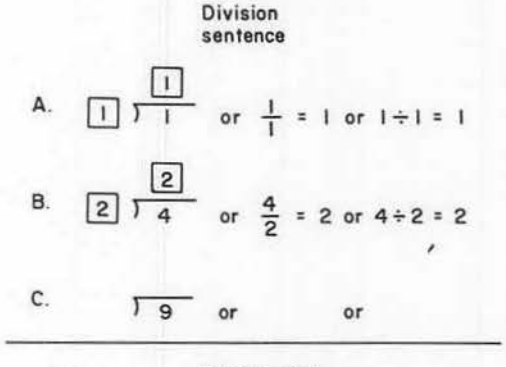
Clearly the 4-by-6 rectangle will require the shortest length of fence. Notice that it is the nonsquare rectangle which is nearest to a square in shape.

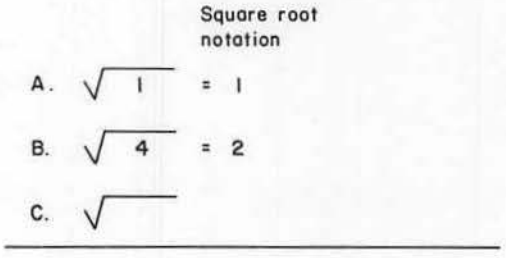
Squares and Square Roots

Suppose that we know that a square has an area of 144 area units. How long is each side? One way to introduce this problem is to review the multiplication problems involving a pair of equal factors.

Begin by having pupils draw squares on graph paper, keeping a record of the results:







Now let's come back to the original problem which might be posed in any one of these ways:

$\square \times \square = 144$ $\qquad$ $\square \overline{)144}$ $\qquad$ $\sqrt{144} = \square$

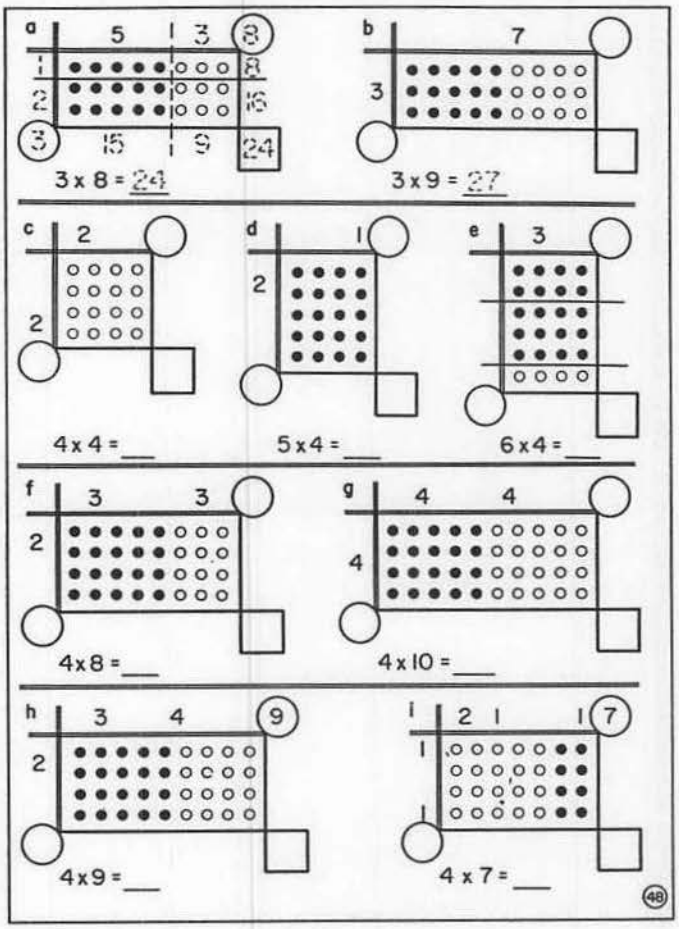
Pupils individually or in teams might attempt this problem using graph paper or counters or Cuisenaire rods. Some may approach the problem by extending the record given above.

Drill

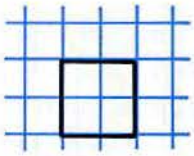
Here are some stories for headline practice:

1. Leslie saw 18 cookies on a cookie plate. She put them into bags. Each bag contained 6 cookies.
2. Edith baked a peach pie which weighed 24 ounces. She cut her pie into 6 pieces of the same size.
3. A boys' camp will be open for 10 weeks this summer. Each group of boys will stay for 2 weeks.

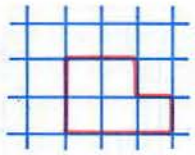
Consider using as a drill sheet page 48 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C (see "*Cross Number Puzzles* Again" in Notes on Pupil Page 94).



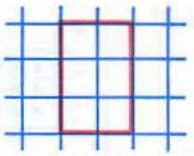
Problem: Show an arrangement of the number of tiles given with the *shortest* distance around the edge.



a. T 4 E 8

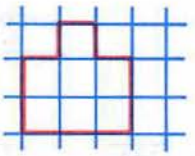


b. T 5 E 10

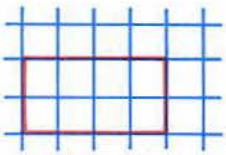


c. T 6 E 10

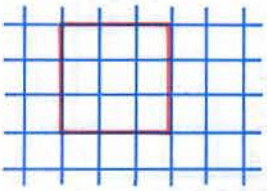
note



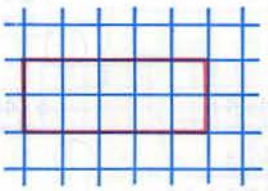
d. T 7 E 12



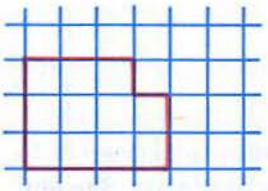
e. T 8 E 12



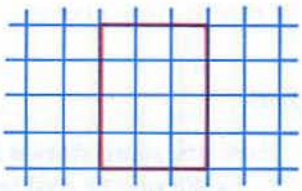
f. T 9 E 12



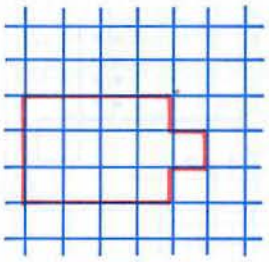
g. T 10 E 14



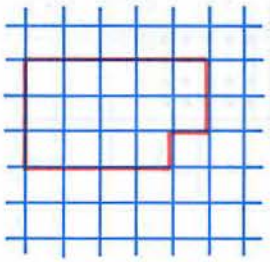
h. T 11 E 14



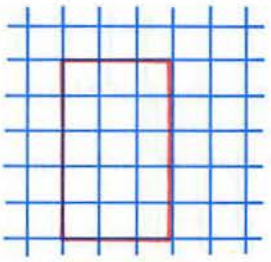
i. T 12 E 14



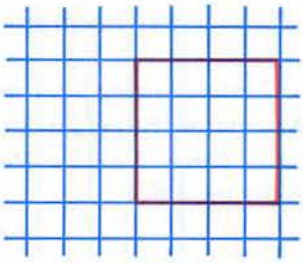
j. T 13 E 16



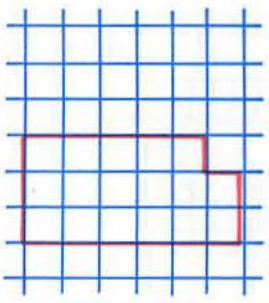
k. T 14 E 16



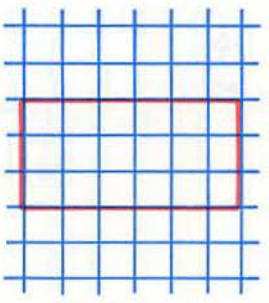
l. T 15 E 16



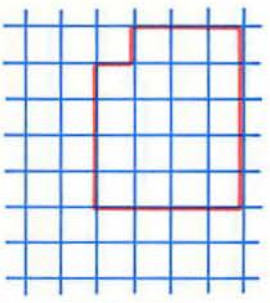
m. T 16 E 16



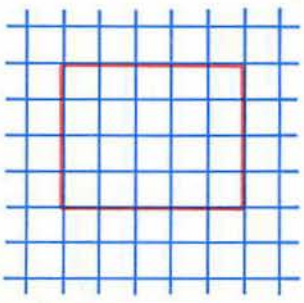
n. T 17 E 18



o. T 18 E 18



p. T 19 E 18



q. T 20 E 18

What's My Rule?

0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55

87 Exercises in constructing rectilinear arrangements of specified numbers of tiles so that each figure has a minimum perimeter.

Tasks: *In each exercise pupils construct a representation of the number of tiles specified (area) such that the distance around the edge (perimeter) is a minimum. In the game at the bottom of the page pupils complete the number pattern from the clues provided.*

Purpose: To explore the concepts of area and perimeter.

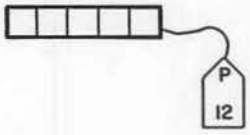
Unifying Ideas: Functions and relations; geometry; measurement.

Materials for Teacher: Graph board; Counting Board; rubber bands.

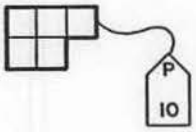
Materials for Pupils: Geo-boards; rubber bands; graph paper.

The Lesson:

The Geo-board and graph paper will help pupils understand what's going on in the exercises on pupil page 87. Pupils are given the number of tiles or units of area to be bounded and are told to draw that boundary which has the shortest perimeter. It must be made clear that the boundary does not have to be rectangular. Thus, for 5 tiles we could have a rectangular boundary:



but this boundary has a greater perimeter than:



Note that exercises *a* through *q* present areas of 4 through 20 tiles in order.

It will be useful to introduce this page by noting several examples such as the one above. Another

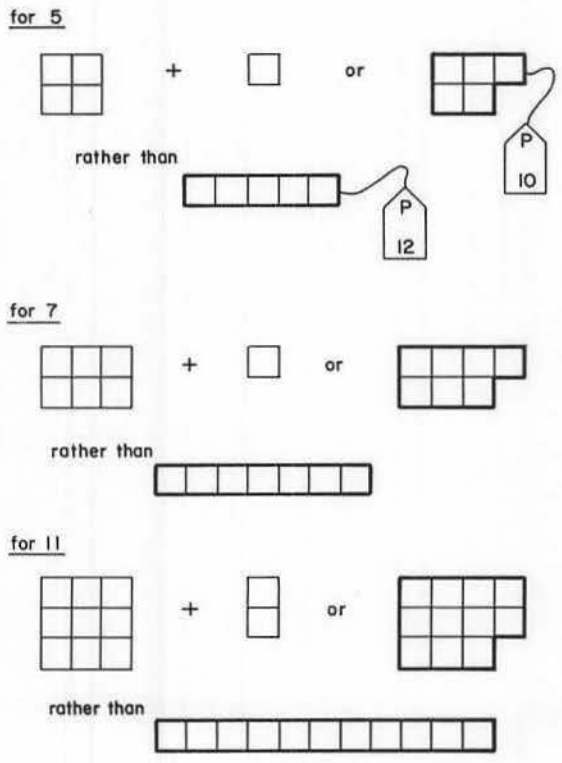
more

Headline for the Day: Tell a story to fit this fact team: $433 - 122 = \square$ $\square + 122 = 433$ $122 + \square = 433$ $433 - \square = 122$

helpful readiness activity for this page includes a study of the factors of each number, 4 to 20:

___ × ___ = 4	2 × 2 = 4; 1 × 4 = 4
___ × ___ = 5	1 × 5 = 5
___ × ___ = 6	3 × 2 = 6; 1 × 6 = 6
___ × ___ = 7	1 × 7 = 7
___ × ___ = 8	4 × 2 = 8; 1 × 8 = 8
___ × ___ = 9	3 × 3 = 9; 1 × 9 = 9
___ × ___ = 10	5 × 2 = 10; 1 × 10 = 10
___ × ___ = 11	1 × 11 = 11
___ × ___ = 12	3 × 4 = 12; 2 × 6 = 12; 1 × 12 = 12
___ × ___ = 13	1 × 13 = 13
___ × ___ = 14	2 × 7 = 14; 1 × 14 = 14
etc.	

We have noticed that the square has the smallest perimeter for a given area. Conversely, the rectangle with two of its edges one unit in length results in the largest perimeter for a given area. Looking at the results above we notice some numbers which can be factored only into a product of 1 and the number. For these numbers (called *prime numbers*) we can decrease the perimeter by building nonrectangular boundaries:



As indicated on the answer key, some numbers of tiles which are *composite numbers* can be arranged in more than one way to get the shortest perimeter.

Drill

Here is a suggestion for a drill sheet that will push pupils into noticing commutativity:

Complete the addition and multiplication tables. Look for shortcuts.

+	15	37	42	58
15	30	52		73
37		74	79	
42	57		84	
58		95	100	116

×	15	37	42	58
15	225	555		870
37		1369	1554	
42	630		1764	
58		2146	2436	3364

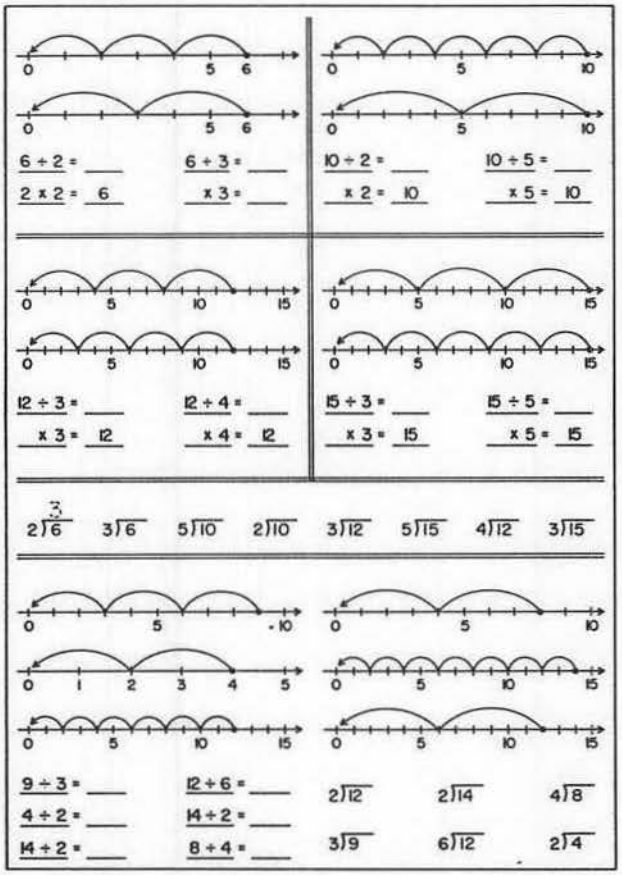
Here are some stories for headline practice:

1. Joanna's teacher arranged the 27 children in her class into teams. Each team had 3 members.
2. Frances baked a cherry pie. She cut her pie into 8 pieces of the same size. Each piece weighed 3 ounces.
3. Howard's dad told him he could have a third of the candy in the candy box. The box contained 36 pieces.
4. The grocer weighed a piece of cheese. It weighed 15 ounces. Mrs. Wronski needed a pound.

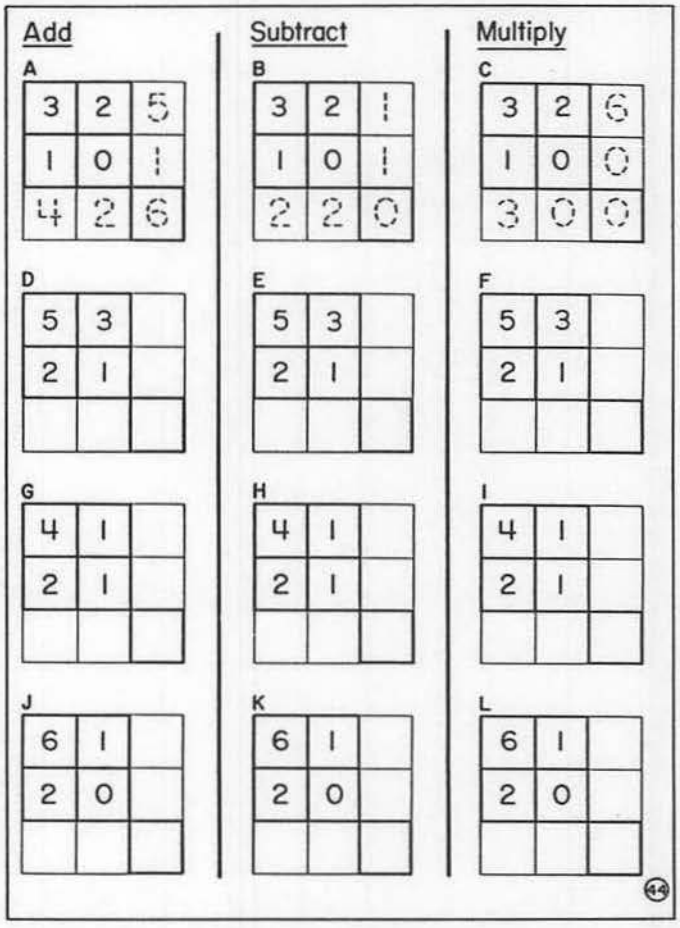
$$16 - 15 = \square \qquad 15 + 1 = 16$$

5. Johnnie had 33 ounces of cough medicine left in his bottle. The bottle held 40 ounces when it was full.
6. Mrs. Smathers had 577 bricks to make a patio. She had 29 when the patio was finished.

Here is a suggestion for a drill sheet which relates multiplication and division notation:



Another suggestion for drill is page 44 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



$18 + \square = 35 - 10$

(1) (2) (3) (4)

7 | 9 | 15 | 4

Mark correct answer. (1) (2) (3) (4) ● ○ ⊗ ○

Mark the furthest from the correct answer.

8 ← Difference between the answers you marked.

1 $\square + 19 = 14 + 17$

(1) 10 (2) 15 (3) 12 (4) 20

1 2 3 4 ○ ○ ● ⊗ 8

2 $7 + 18 = \triangle + 7$

(1) 25 (2) 35 (3) 78 (4) 18

1 2 3 4 ○ ○ ⊗ ● 60

3 $21 + \square = 49 + 21$

(1) 12 (2) 49 (3) 94 (4) 70

1 2 3 4 ○ ● ⊗ ○ 45

4 $\triangle + 40 = 70 - 13$

(1) 17 (2) 13 (3) 30 (4) 110

1 2 3 4 ● ○ ○ ⊗ 93

5 $39 + \square = 63 + 39$

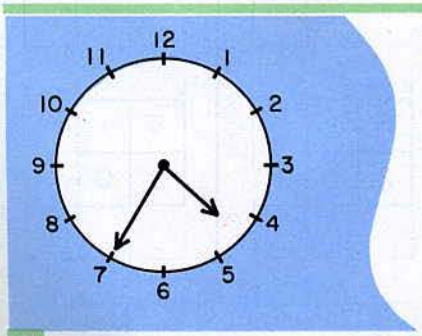
(1) 57 (2) 36 (3) 93 (4) 63

1 2 3 4 ○ ○ ⊗ ● 30

6 $128 + 45 = 45 + \triangle$

(1) 45 (2) 128 (3) 145 (4) 90

1 2 3 4 ⊗ ● ○ ○ 83



7 The time shown on the clock is:

(1) 7:20 (3) 7:35

(2) 4:35 (4) 5:35

1 2 3 4 ○ ● ⊗ ○ 3 hrs., 0 min.

8 The time in 1 hour and a half will be:

(1) 5:05 (3) 7:00

(2) 5:35 (4) 6:05

1 2 3 4 ⊗ ○ ○ ● 1 hrs., 0 min.

9 How long is it until the clock will next read 12:00?

(1) 7 hrs., 25 minutes

(2) 8 hrs., 35 minutes

(3) 12 hrs., 0 minutes

(4) 4 hrs., 35 minutes

1 2 3 4 ● ○ ⊗ ○ 4 hrs., 35 min.

10 What time will the clock show 12 hours later?

(1) 3:45

(2) 4:35

(3) 5:34

(4) 4:47

1 2 3 4 ○ ● ⊗ ○ 0 hrs., 59 min.

11 What time will the clock show in 43 minutes?

(1) 5:43

(2) 5:18

(3) 4:43

(4) 6:34

1 2 3 4 ○ ● ○ ⊗ 1 hrs., 16 min.

12 About what time will it be when the hands are together next?

(1) 5:35

(2) 12:00

(3) 6:00

(4) 5:27

1 2 3 4 ○ ⊗ ○ ● 6 hrs., 33 min.

88 Test page - multiple-choice items - solving equations, reading and computing various time statements.

Tasks: Pupils complete each test item in accordance with the directions and the sample provided at the top of the page.

Purpose: To test understanding of some of the basic concepts and mastery of some of the basic computational skills up to this point in the course; to test competence in working with time measurements.

Unifying Ideas: Structure; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

Score 3 points per item for a total of 36 points.

When pupils complete the test, ask them to tell you which questions among the first six were the easiest. Hopefully, all of them will have recognized the direct application of commutativity in exercises 2, 3, 5, and 6. Keep in mind that exercises 9, 11, and 12 are fairly difficult.

An Independent Activity

Present these exercises on a Ditto sheet. For each of the four relation signs and for each incomplete sentence, the pupil is to indicate whether that sign when written in the shaded box would convert the sentence into a true one or a false one.

= is equal to	< is less than
≠ is not equal to	> is greater than

			=	≠	<	>
3 + 7	□	20−10	T	F	F	F
100	□	25+25				
20+20	□	10+30				
463−1	□	364+1				
50−20	□	20+10				
1+0	□	1−0				
13+17	□	17+13				
29+92	□	93+29				
25+99	□	100+25				
75+725	□	724+76				
57	□	57 + 99				
275+145	□	140+280				
700+602	□	99				
101+701	□	99+703				

Drill

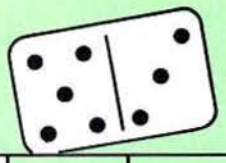
Here are some stories for headline practice:

1. Minnie had 56 bulbs to plant. She put 4 into each pot.
2. Wanda baked a blueberry pie. She cut it into 9 pieces. George, Jeffrey, and Gail each ate a piece of pie.
3. When Zeke and Harry travel the 148 miles to visit their aunt, they will then go on for 74 more miles to visit some friends.

Headline for the Day: Fill the blanks to continue the pattern.

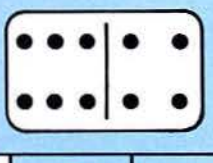
0, 3, 8, 15, 24, , , , , 99

A.



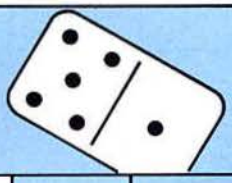
		x 2	x 3	x 5
		16	24	40
+ 9	+ 12	x 4	x 6	x 10
17	20	32	48	80

B.



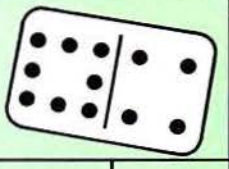
		x 2	x 3	x 5
		20	30	50
+ 8	+ 17	x 6	x 9	x 15
18	27	60	90	150

C.



		x 3	x 2	x 5
		18	12	30
+ 17	+ 45	x 6	x 4	x 10
23	51	36	24	60

D.



		x 2	x 3	x 5
		24	36	60
x 20	x 30	x 4	x 6	x 10
240	360	48	72	120

E.

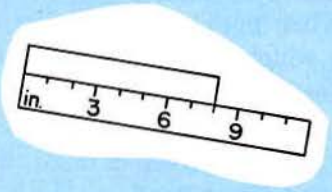
twice as much
2 nickels, 4 pennies
three times as much (use fewest coins)
4 n., 1 p.
four times as much (use fewest coins)
5 n., 3 p.
9¢ more (use fewest coins)
3 n., 1 p.
13¢ more (use fewest coins)
4 n., 0 p.

F.

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	

twice as many days as those shaded green
2 weeks, 6 days
three times as many days as those shaded green
4 wk., 2 da.
four times as many days as those shaded green
5 wk., 5 da.
9 days more than those shaded green
2 wk., 5 da.
13 days more than those shaded green
3 wk., 2 da.

G.



a one foot ruler

length of block
0 feet, 8 inches
twice as long
1 ft., 4 in.
three times as long
2 ft., 0 in.
half as long
0 ft., 4 in.
four times as long
2 ft., 8 in.

89 A variety of exercises dealing with multiplication and division using various numeration systems (bases 10, 5, 7, 12).

Tasks: *Pupils determine the number or quantity represented by the picture in each exercise, and then carry out the required computations.*

Purpose: To review and perfect computational skills, and to develop an awareness of the use of the distributive principle for multiplication over addition.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

In pupil pages 89-117 we take pupils further into algorithms for multiplication. When they have completed the work, they should be able to do examples and solve problems like the following:

$8 \times 10 =$ $\quad$ $28 \times 47 =$
$18 \times 10 =$ $\quad$ $4 \times 665 =$
$17 \times 12 =$

Tom earns 18 dollars a week. How much does he earn in 10 weeks?

The room is 28 feet long and 47 feet wide. How many square feet of linoleum will cover the floor?

If an airplane travels at the speed of 665 miles an hour for 4 hours, how far will it have gone from its starting point?

On this first page of the sequence of pupil pages there are three things we should like to note:

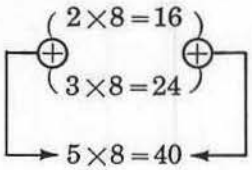
1. As in earlier pages of this design, pupils are asked to look at a picture of things to count or measure and then to multiply the count or quantity by a number. Such an approach identifies arithmetic closely with the real world, an important concern in teaching arithmetic.
2. In exercises E-G at the bottom of the page, pupils record the results of their addition or multiplication in the fewest units, providing further practice in division with remainders. ("20 days → 2 weeks, 6 days" can be represented by the form: $7\overline{)20}\ \begin{matrix}2\ \text{R6}\end{matrix}$.)

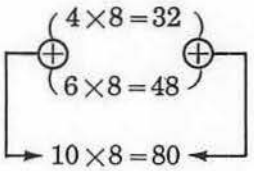
more

Headline for the Day: Erica was an hour and fifteen minutes late for her dental examination. Make a table to record what you can say about this situation.

3. Each of the four sequences of examples at the top of the page focuses attention on a very important piece of mathematical structure—*the distributive principle for multiplication over addition.* (Of course, we don't intend to use this language with pupils.)

In exercise A we have two applications of the principle:





Each of B, C, and D contains a pair of applications of the principle.

Moreover, the applications of the principle in each pair are related to each other. For example, in exercise A each of the numbers 4, 6, and 10 is the double of each of the numbers 2, 3, and 5, respectively; the products 32, 48, and 80 are the doubles of the products 16, 24, and 40, respectively. This follows from the fact that multiplication is associative.

$$8\times 4 = 8\times(2\times 2) = (8\times 2)\times 2$$
$$8\times 6 = 8\times(3\times 2) = (8\times 3)\times 2$$
$$8\times 10 = 8\times(5\times 2) = (8\times 5)\times 2$$

Of course, pupils may also use the distributive principle in computing the individual products in exercises A-D. For example, the domino in A shows 5 + 3 dots. If we double this, we get 10 + 6; if we triple it, we get 15 + 9, etc.

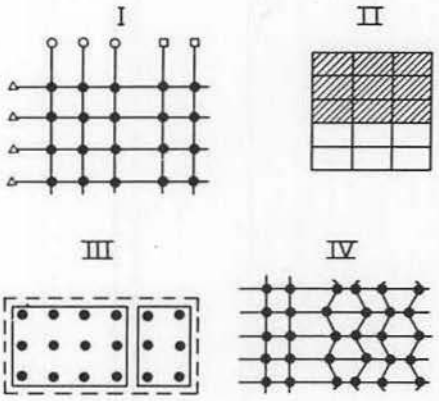
Supplementary Activities

Mastery of the Basic Facts of Multiplication and Division

This seems a good time to check mastery of the basic facts of multiplication and division and to provide needed review and drill to make the mastery of these facts more automatic. Follow the same procedures as suggested in connection with the mastery of addition and subtraction facts in Notes on Pupil Page 2.

Sentences and Diagrams

Draw several diagrams on the board and ask the children what mathematical sentences each suggests.



Mathematical sentences suggested by diagram I would include:

$4 \times 5 = \_\_\_$	$12 + 8 = \_\_\_$
$5 \times 4 = \_\_\_$	$8 + 12 = \_\_\_$
$(4\times 3)+(4\times 2) = \_\_\_$	$20 - 8 = \_\_\_$
$20 \div 3 = \_\_\_$	$20 - 12 = \_\_\_$
$4 = 20 \div \_\_\_$	$(12+8)\div 4 = \_\_\_$

Similar sentences are suggested by each of the other diagrams.

After a sufficient number of sentences for each diagram have been noted, reverse the procedure by asking pupils to find the diagram that is suggested by a mathematical statement. Which of the diagrams (shown above) is suggested by:

$3\times 5 = \_\_\_$	$5 = \_\_\_ \div 6$
$18 \div 3 = \_\_\_$	$\_\_\_ - 6 = 12$
$(2\times 3)+(3\times 3) = \_\_\_$	$10 + 20 = \_\_\_$
$\_\_\_ - (3\times 4) = 2\times 3$	$5\times 6 = 5\times(2+\_\_\_)$
$6\times 3 = 6 + \_\_\_$	$20 - (3\times \_\_\_) = 2\times 4$

Have pupils volunteer to make up sentences suggested by the diagrams. The rest of the class must find the diagram which suggests a given sentence.

Drill

Here are some stories for headline practice:

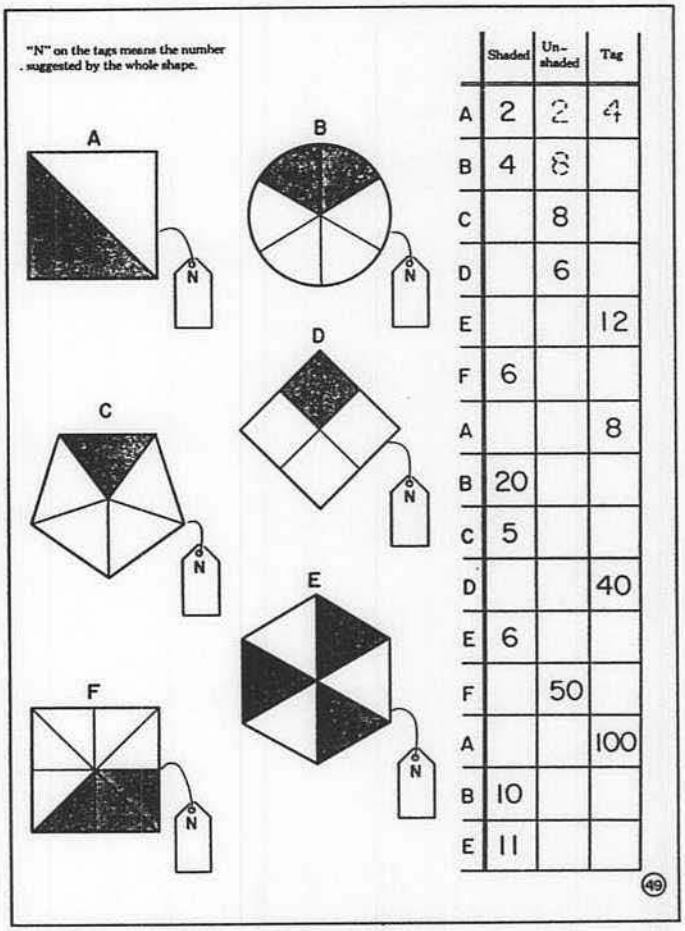
1. Mrs. Calas ordered 52 pints of milk for the picnic. The milkman gave her the order in quart containers.
2. Russell had 45 pieces of gum wrapped in packages. Each package held 5 pieces of gum.

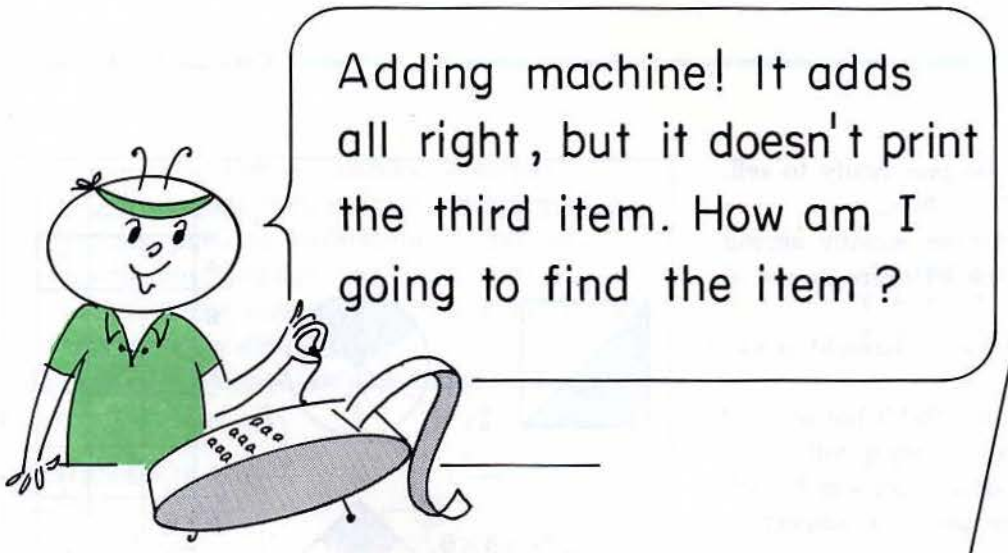
3. Leo's grandfather had 200 pigs ready to sell. One day a man bought 79 of them.
4. Four girls shared 65 marbles equally among themselves. They gave the leftovers to one of their neighbors.
5. Gerard attended school for 12 straight weeks without an absence.
6. Lola stayed at her grandmother's house for 9 days. Each day she drank 3 pints of milk.

Here is a suggestion for a drill sheet which compares addition and multiplication with respect to associativity:

1. $(3+2)+4=\square$ $3+(2+4)=\bigcirc$	**2.** $(3\times2)\times4=\square$ $3\times(2\times4)=\bigcirc$
3. $(2+2)+5=\square$ $2+(2+5)=\bigcirc$	**4.** $(2\times2)\times5=\square$ $2\times(2\times5)=\bigcirc$
5. $(2+3)+5=\square$ $2+(3+5)=\bigcirc$	**6.** $(2\times3)\times5=\square$ $2\times(3\times5)=\bigcirc$
7. $(7+6)+0=\square$ $7+(6+0)=\bigcirc$	**8.** $(7\times6)\times0=\square$ $7\times(6\times0)=\bigcirc$

Consider using as a drill sheet page 49 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown in the next column.





A.

$ .25
.36
.38
.24
1.00

T $ 2.23

B.

$ 125.00
15.93
207.75
18.09

T $ 366.77

C.

$ 302.85
48.67
88.55
100.00
17.28

T $ 557.35

D.

$ 1.08
.56
1.47
.83
2.12
.75
1.56
.50
1.10
.12
.07

T $ 10.16

E.

$ 355.75
25.00
200.00
34.83
19.66

T $ 635.24

F.

$ .30
1.00
12.34
10.50
2.25
3.75
.90
3.07
6.10
1.04
30.00
5.68

T $ 76.93

G.

$ 2,350.00
750.85
1,525.25
487.99

T $ 5,114.09

H.

$ 10.25
5.62
10.00
37.48
.56
1.27

T $ 65.18

90 Exercises requiring the computation of missing addends.

Tasks: *Pupils determine the missing addend for each section of adding-machine tape.*

Purpose: To perfect computational skills.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: None.

Materials for Pupils: Scratch paper.

The Lesson:

The check-out tapes with a missing item are not mysterious. They simply present long and involved computations. Even the most direct approach requires much figuring, and in any one of the steps involved it is possible to commit an error.

This activity demonstrates the fact that a small error anywhere along the way will result in an incorrect answer unless, by accident, a pupil makes a second compensating error. Furthermore such little slips along the way are very hard to find. So, the most economical method is to have pupils work with an emphasis on accuracy.

There is a payoff in this kind of problem. Most children experience a feeling of accomplishment when they have carried through a long chain of operations without a single error.

Readiness Activities

You may wish to prepare for pupil page 90 with the following activities.

Helen bought a present for each of her three cousins. Altogether, she spent $1.25. She remembered that the model for Bill cost 35¢ and the book for Martha cost 45¢, but she couldn't remember what she paid for the game for Roger. Can you help her out?

$$\begin{array}{r} \$\ .35 \\ .45 \\ \underline{\quad .\ ?} \\ \$1.25 \end{array}$$

Was it more or less than the cost of the model? . . . more or less than the book? (The game cost 45¢.)

Here is the scorepad for Roger's game. After each turn, the four players recorded their total scores, and added the totals. This was fortunate, because Helen's brother erased all her scores. Who won each game?

	Games					
	1st	2nd	3rd	4th	5th	6th
Roger	15	18	30	28	25	32
Helen						
Martha	20	22	30	13	29	45
Bill	15	14	35	29	21	37
Totals	75	75	120	100	110	158

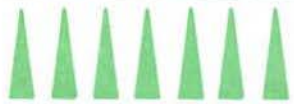
(Helen's scores were 25, 21, 25, 30, 35, and 44.)

In these introductory problems, do not suggest any rule such as: add the given numbers and subtract the sum from the total to get the missing number. Each pupil should be encouraged to try the problems any way he likes. It may well be by trial and error—a perfectly legitimate, time-wasting procedure. Or, he may perform a series of subtractions: $75 - 15 = 60$; $60 - 20 = 40$; $40 - 15 = 25$; therefore Helen's score in the first game must have been 25.

There are many ways to get to a correct answer. It takes time for children to develop a sense which guides them toward the most economical method of reaching a correct answer. No set of rules is inclusive enough to meet all situations, and every set of rules is dangerous, because there is no way to be sure that the pupil will select the right rule and apply it correctly. Instead of memorizing all the rules, a pupil must develop an insight into problems which arise, and learn to sense the shortest road to the goal. Repeatedly, he will err in his judgment and only by hindsight find that he could have saved himself much effort if he had chosen a different attack. Deciding upon a method of attack is an art, which can be hindered by rules.

Headline for the Day: Hulda's mother told her she had another half-hour to play before supper was ready. Make a table to record what you can say about this situation.

A.



a row of 7 blocks

twice as many	14	three times as many	21
5 rows of blocks	35	ten times as many	70

B.

a row of 14 blocks

half as many	7	twice as many	28
ten times as many	140	three times as many	42
		five times as many	70

C.

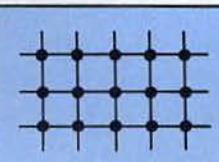
A A A A A B B B B B B B B B

a row with 5 A's and 9 B's

The total number of letters is 14

		three times as many A's	15
		three times as many B's	27
		three times as many letters	42
five times as many letters	70	ten times as many letters	140

D.



		x 2	x 3	x 5
		30	45	75
− 8	+ 9	x 4	x 6	x 10
7	24	60	90	150

E.

11.
(shorthand for twenty-one)

		x 2	x 3	x 5
		42	63	105
+ 39	+ 79	x 4	x 6	x 10
60	100	84	126	210

F.

3 times as much	48¢	6 times as much	96¢
2 times as much	32¢	4 times as much	64¢
5 times as much	80¢	10 times as much	$1.60
7 times as much	$1.12	14 times as much	$2.24
half as much	8¢	one dollar more	$1.16

91-98 Introduction and development of an algorithm for multiplication based on the distributive principle of multiplication over addition.

Tasks: *Pupils determine the number or quantity represented by the picture in each exercise and then carry out the required computations.*

Purpose: To review and perfect computational skills.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; measurement.

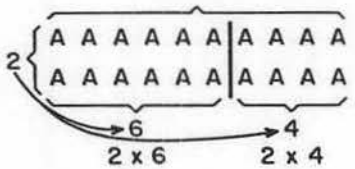
Materials for Teacher: None.

Materials for Pupils: None.

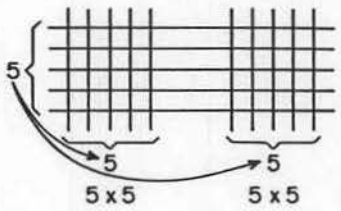
The Lesson:

If preparation for pupil page 91 is necessary, place on the board some examples like those on the page. Have pupils suggest various approaches to getting the answers.

Develop examples on the board like the following ones, eliciting class discussion for labeling and recording:



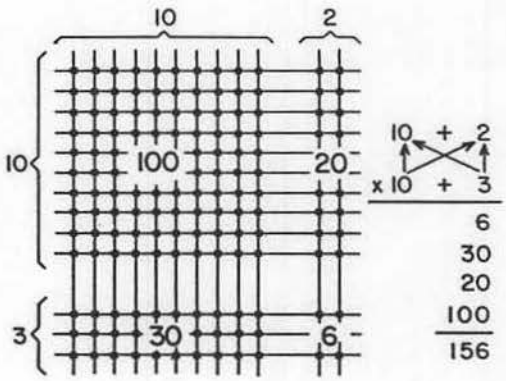
$2 \times (6+4) = (2 \times 6) + (2 \times 4)$
$= 12 + 8$
$= 20$



$5 \times (5+5) = (5 \times 5) + (5 \times 5)$
$= 25 + 25$
$= 50$

These examples of the distributive principle for multiplication over addition will help pupils see how two numbers can be multiplied by partitioning one of the factors. Such partitioning is, of course, the basis for algorithms for multiplication in which numbers are partitioned into multiples of ten and numbers less than ten:

$12 \times 13 = \square$



more

Headline for the Day: Fenton has twice as many marbles as Estes. Ferris has three times as many marbles as Estes. Errol has as many marbles as Fenton and Ferris have together. Make a table to record what you can say about this situation.

The exercises on pupil page 91 are loaded with applications of the distributive principle. Do not allow these applications to go unnoticed by the pupils. For example, exercises D and E involve the distributive principle in the same way it was involved in the exercises on pupil page 89. Here are some other occurrences of the principle:

In A notice that twice as many together with three times as many amount to five times as many.

$$(2 \times 7) + (3 \times 7) = (2 + 3) \times 7 = 5 \times 7$$

In B notice that the sum of the answers (28 and 42) to the first two problems in the right-hand column is the answer (70) to the third problem in that column. We can explain this by noting that

$$\begin{aligned} 28 &= 2 \times 14 \\ \text{and } 42 &= 3 \times 14. \\ \text{So, } 28 + 42 &= (2 \times 14) + (3 \times 14) \\ &= (2 + 3) \times 14 \\ &= 5 \times 14 \\ &= 70. \end{aligned}$$

Notice also in B that the 14 blocks are partitioned by shape into $(10 + 4)$ blocks. So, for example, $2 \times (10 + 4) = (2 \times 10) + (2 \times 4) = 20 + 8 = 28$. This partitioning is precisely the one used in an algorithm for short multiplication.

In C notice that $15 + 27 = 42$. We can explain this as follows:

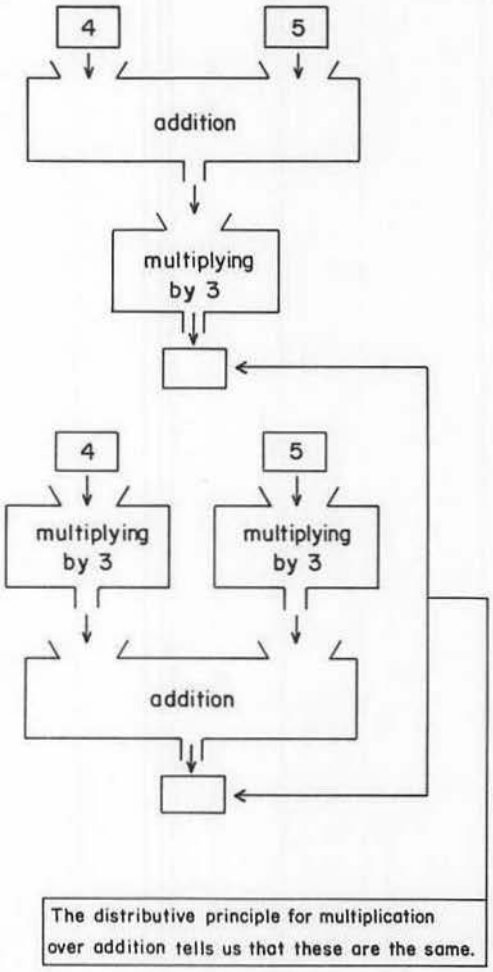
$$\begin{aligned} 15 &= 3 \times 5 \\ 27 &= 3 \times 9 \\ 15 + 27 &= (3 \times 5) + (3 \times 9) \\ &= 3 \times (5 + 9) \\ &= 3 \times 14 \\ &= 42 \end{aligned}$$

Notice also in A, B, and C the applications of the fact that ten times as many amounts to twice as much as five times as many. This follows from the fact that 10 is 2×5 and that multiplication is associative:

$$\begin{aligned} &\text{(A)} \quad 10 \times 7 = (2 \times 5) \times 7 = 2 \times (5 \times 7) \\ &\text{(B, C)} \quad 10 \times 14 = (2 \times 5) \times 14 = 2 \times (5 \times 14) \end{aligned}$$

Multiplying by a Number Distributes over Addition

The distributive principle for multiplication over addition tells you that, given the sum of a pair of numbers, the result of multiplying the sum by a third number is the sum of the results of multiplying each of the numbers of the given pair by the third. For short, one says that multiplying by a number distributes over addition. For example, consider *multiplying by 3* and *the addition of 4 and 5*. In machine notation:



We can say in ordinary notation what the distributive principle tells us in this case by writing either of the following equations:

$$\begin{aligned} (4 + 5) \times 3 &= (4 \times 3) + (5 \times 3) \\ 3 \times (4 + 5) &= (3 \times 4) + (3 \times 5) \end{aligned}$$

The first equation says that $(4 + 5)$ *multiplied by 3* is the sum of (4 *multiplied by 3*) and (5 *multiplied by 3*). The second equation says that *3 times* $(4 + 5)$ is the sum of (*3 times* 4) and (*3 times* 5). The operation *multiplying by 3* is precisely the same thing as the *3 times* operation.

If we think of multiplying $(4 + 5)$, or 4, or 5 by 3

as using (4 + 5), or 4, or 5, respectively, as an addend 3 times then both equations are shorthand for the statement:

$$(4+5)+(4+5)+(4+5)=(4+4+4)+(5+5+5)$$

which is nothing more than a consequence of the rearrangement principle for addition. Thus, the fact that multiplying by a number (whole number) distributes over addition is a consequence of the facts that multiplying by a number is repeated addition and that changing the order and the grouping of a string of addends does not affect the sum.

It is also the case that dividing by a number distributes over addition. Using the ÷ as a division sign, we can express the fact that *dividing by 2* distributes over *the addition of 8 and 6* by writing the equation:

$$(8+6)\div 2=(8\div 2)+(6\div 2)$$

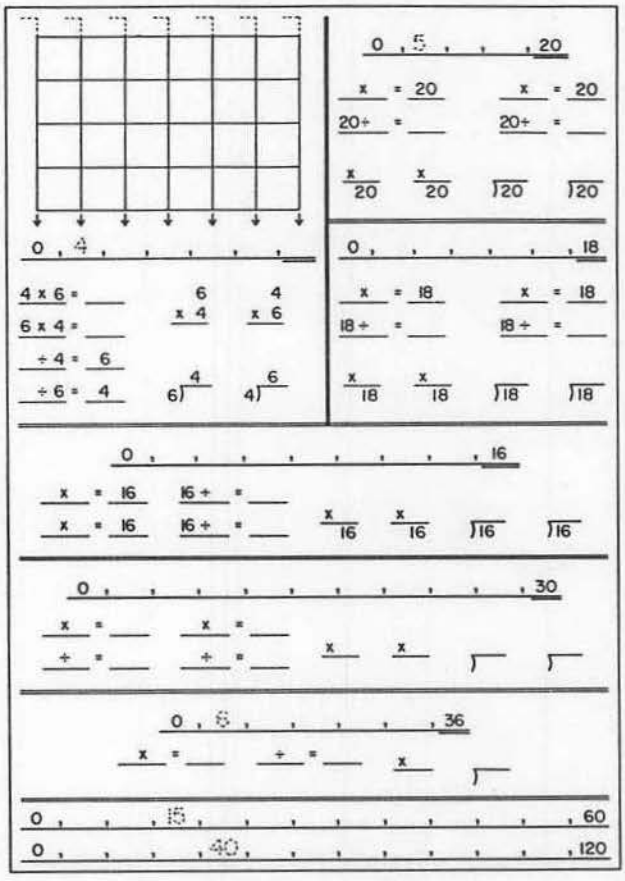
Unlike the case of multiplication in which we are free to indicate, say, *multiplying by 2* by writing a 2 × on the left or a × 2 on the right of the indicated sum, we are restricted in division to writing the ÷ 2 on the right: $2 \div 14 = 2 \div (8 + 6) \neq (2 \div 8) + (2 \div 6) = 7 \div 12$. On the other hand, we could tell the same story by writing:

$$2\overline{)8+6} = 2\overline{)8} + 2\overline{)6}$$

since we have agreed to use "$2\overline{)8+6}$" and "$2\overline{)8}$" and "$2\overline{)6}$" as number symbols.

Drill

Here is a suggestion for a drill sheet.

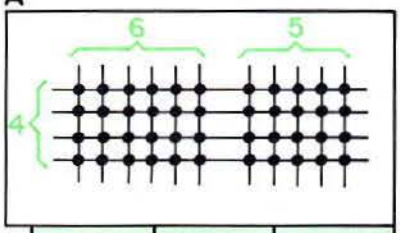


Here are some stories for headline practice:

1. For 5 weeks in a row Bob earned $1.60 per week on his paper route.
2. Manfred spelled 18 different words correctly on 5 spelling quizzes.
3. Jocko the farmhand peeled and sliced 63 apples in 30 minutes. He stopped work after one hour.

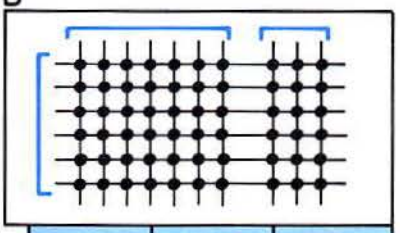
$63+63=\square$ $63\times 2=126$

A



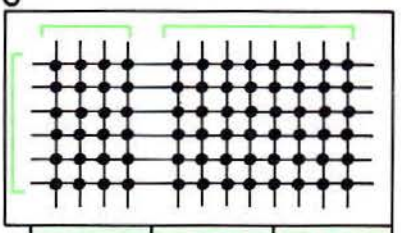
6	5	11
x 4	x 4	x 4
24	20	44

B



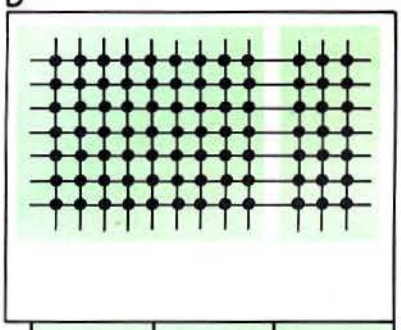
7	3	10
x 6	x 6	x 6
42	18	60

C



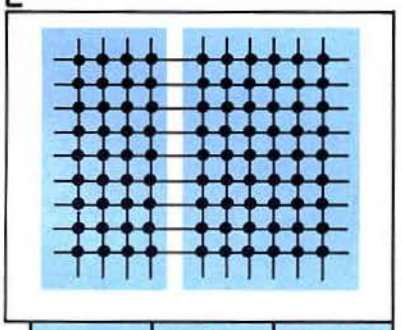
4	8	12
x 6	x 6	x 6
24	48	72

D



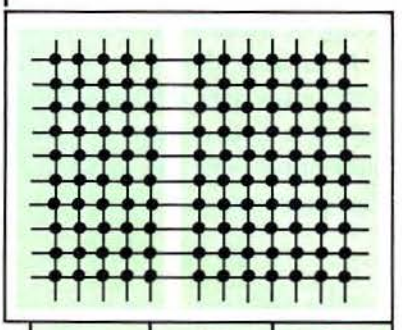
9	3	12
x 7	x 7	x 7
63	21	84

E



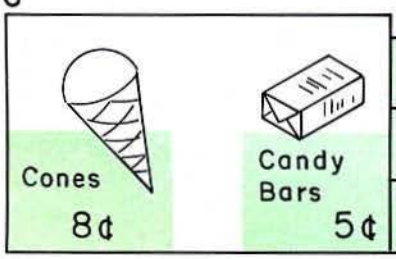
4	6	10
x 9	x 9	x 9
36	54	90

F



5	7	12
x 10	x 10	x 10
50	70	120

G



cost of 5 cones	40¢	15 cones	$1.20
cost of 5 bars	25¢	15 bars	$ .75
cost of 5 of each	65¢	15 of each	$1.95

H

8 half-dollars	$4.00	12 half-dollars	$6.00
8 quarters	$2.00	12 quarters	$3.00
8 dimes	80¢	12 dimes	$1.20
8 nickels	40¢	12 nickels	60¢
8 of each	$7.20	12 of each	$10.80

Notes on Pupil Page 92

Tasks: *Pupils are to complete the vertically arrayed multiplication examples in A-F using the clues provided in the pictures. In G and H pupils compute costs and amounts.*

Purpose: To prepare for an algorithm for multiplying a two-digit number by a one-digit number.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

We are now in sight of an algorithm for multiplying a two-digit number by a one-digit number. The underlying mathematical principle is the distributive principle for multiplication over addition. And, of course, this principle rests on the fact that when a rectangular array is partitioned into two arrays (with a common dimension), the count for the original array is the sum of the counts for the component arrays.

Let's suppose that our problem is to find the product of 11 by 4. We can find this product by counting the crossing points generated by drawing 4 horizontal lines and 11 vertical lines each of which crosses each of the horizontal lines (see exercise A). To determine this count by one-by-one counting is too tedious. So one partitions the array into two arrays whose counts are readily determined; ideally, the counts for the component arrays have been memorized. Exercise A shows one procedure for finding 11×4:

Problem:

$$\begin{array}{r} 11 \\ \times\ 4 \\ \hline ? \end{array}$$

Step 1:

$$\begin{array}{rrrr} 6+ & 5= & 11 & \text{(partitioning)} \\ \times 4 & \times 4 & \times\ 4 & \\ \hline & & ? & \end{array}$$

Step 2:

$$\begin{array}{rrrr} 6+ & 5= & 11 & \text{(determining the} \\ \times 4 & \times 4 & \times\ 4 & \text{counts for the} \\ \hline 24 & 20 & ? & \text{component arrays)} \end{array}$$

Step 3:

$$\begin{array}{rrrr} 6+ & 5= & 11 & \text{(adding the counts)} \\ \times 4 & \times 4 & \times\ 4 & \\ \hline \underbrace{24+\ 20}_{44} & & ? & \end{array}$$

Step 4:

$$\begin{array}{rrrr} 6+ & 5= & 11 & \text{(invoking the} \\ \times 4 & \times 4 & \times\ 4 & \text{distributive principle} \\ \hline \underbrace{24+\ 20}_{44}= & & & \text{for multiplication} \\ & & & \text{over addition)} \end{array}$$

One is completely free in selecting a partition for 11. The pupil needs to understand that *every* partition of 11 will lead to component arrays with counts whose sum is 11×4. Here are the records of several partitions:

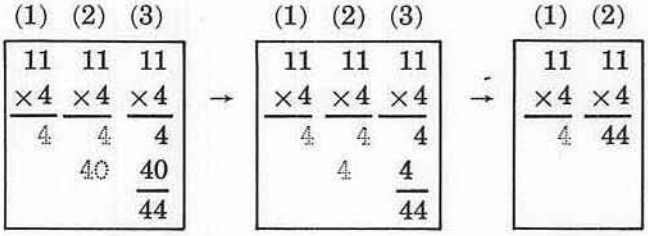
$$\begin{array}{rrrrr} \underbrace{10+1} & \underbrace{9+2} & \underbrace{8+3} & \underbrace{7+4} & \underbrace{6+5} \\ 11 & 11 & 11 & 11 & 11 \\ \times\ 4 & \times\ 4 & \times\ 4 & \times\ 4 & \times\ 4 \\ \hline 40 & 36 & 32 & 28 & 24 \\ 4 & 8 & 12 & 16 & 20 \\ \hline 44 & 44 & 44 & 44 & 44 \end{array}$$

$$\begin{array}{rrrrr} \underbrace{5+6} & \underbrace{4+7} & \underbrace{3+8} & \underbrace{2+9} & \underbrace{1+10} \\ 11 & 11 & 11 & 11 & 11 \\ \times\ 4 & \times\ 4 & \times\ 4 & \times\ 4 & \times\ 4 \\ \hline 20 & 16 & 12 & 8 & 4 \\ 24 & 28 & 32 & 36 & 40 \\ \hline 44 & 44 & 44 & 44 & 44 \end{array}$$

Notice that decimal partitioning ($11 = 10 + 1$ or $11 = 1 + 10$) leads to a record which can be readily abbreviated. Since under decimal partitioning the count of one of the compound arrays must be a multiple of 10, positional notation allows us to record it

Headline for the Day: Hortense said that she had the following coins in her bank: half-dollars, quarters, dimes, nickels, and pennies. Make a table to record what you can say about this situation.

without writing the terminal 0-digit. Note how the second and third of the following three algorithms take advantage of this idea:



(The third box shows the two steps of what is sometimes called an algorithm for short multiplication.)

Although we are specifically aiming for an algorithm that makes use of positional notation in the product of a 2-digit number by a 1-digit number, we must not lose sight of the fact that any such algorithm is based on a general mathematical principle that has no connection with the form in which a number is expressed. For example, suppose that in exercise D the pupil does not recall the product of 9 by 7. He could count the crossing points, one by one, in the component array on the left. But this might be too tedious. Instead, that array could be partitioned into arrays whose counts the pupil did recall. Here are some possibilities:

$$9\times7=(5+4)\times7=(5\times7)+(4\times7)$$
$$=35+28=63$$
$$9\times7=(6+3)\times7=(6\times7)+(3\times7)$$
$$=42+21=63$$
$$9\times7=(3+3+3)\times7=(3\times7)+(3\times7)+(3\times7)$$
$$=21+21+21=63$$

In other words, in finding 9 × 7 a pupil could treat 9 as if it were a 2-digit number (first and second possibilities) or as if it were a 3-digit number (third possibility).

A pupil who understands the distributive principle for multiplication over addition and can add is able to handle many multiplication problems without using a standard algorithm. For example, here is how such a pupil might find the product of 43 by 7:

$$10\times7=\ 70$$
$$70$$
$$20\times7=140$$
$$140$$
$$40\times7=280$$
$$3\times7=\ 21$$
$$43\times7=301$$

An interesting exercise which will help pupils appreciate the usefulness of the distributive principle for multiplication over addition is to give them a multiplication fact and ask them to use it to find another one. For example, tell pupils that $38 \times 7 = 266$ and ask them to use this fact in finding the product of 43 by 7. Since 43 is 38 + 5 it follows that $43 \times 7 = (38 \times 7) + (5 \times 7)$. Thus, 43×7 is 266 + 35, or for short, 301.

Notice that the practical problems in exercises G and H involve applications of the distributive principle for multiplication over addition. The total cost of 5 cones and 5 candy bars is, of course, the sum of the cost of 5 cones and the cost of 5 candy bars. Another way to find the total cost of 5 of each is to find the total cost of 1 of each and multiply this by 5. So, one can find the total cost either by finding the sum of (8×5) and (5×5) or by finding the product of $(8 + 5)$ by 5.

A way of getting pupils to appreciate what is happening in an exercise like G is to pose them further problems such as the following:

I

Cost of 5 cars	30¢
Cost of 5 trains	?
Cost of 5 of each	90¢

II

Cost of 7 hats	$7.00
Cost of 7 bows	?
Cost of 7 of each	$8.05

III

Cost of 3 pencils	12¢
Cost of 6 pens	60¢
Cost of 6 of each	?

IV

Cost of 4 pencils	29¢
Cost of 8 pens	60¢
Cost of 4 of each	?

(In order to do problems III and IV one must assume a uniform rate for each item.)

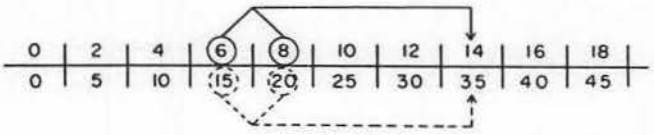
Multiplication Tables

Here is an interesting enrichment activity which focuses attention on the distributive principle for multiplication over addition. List in a table with two rows the products in the multiplication-by-2 table and the products in the multiplication-by-5 table:

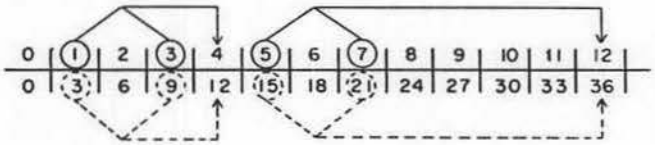
0	2	4	6	8	10	12	14	16	18
0	5	10	15	20	25	30	35	40	45

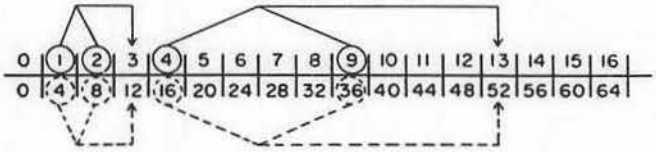
Select two numbers in one row as addends, and draw a line from them to their sum further down

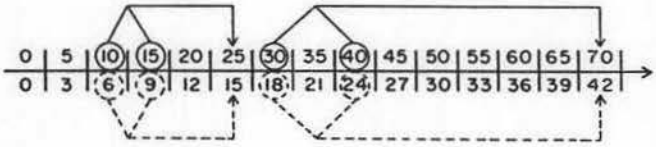
the row. Their sum can always be found if the row of products is long enough (a fact which may be surprising in itself). Next consider as addends the two numbers in the second row opposite those chosen as addends in the first row. Find the sum for these down the row. Again, the sum can be found if the row is long enough. And, it will be opposite the sum of the two corresponding addends in the other row. For example:



Now that's a surprising result. Perhaps it is an accident. Let's try other examples:







This may appear, at first, to be nothing but a trick, or an amusement. Actually, we are focusing attention on a very important piece of mathematical structure. When the sentences:

$$10+15=25 \qquad 6+9=15$$

are transformed into:

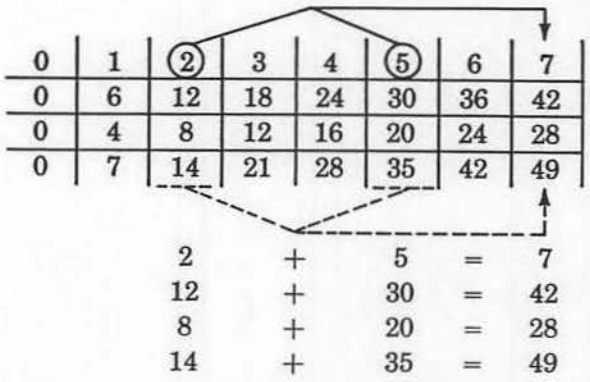
$$10+15=(2\times5)+(3\times5)=(2+3)\times5=25$$

and:

$$6+9=(2\times3)+(3\times3)=(2+3)\times3=15$$

the reason for the surprising result becomes more apparent.

In fact, we can take any number of rows out of the multiplication table, and see the surprise grow in size.

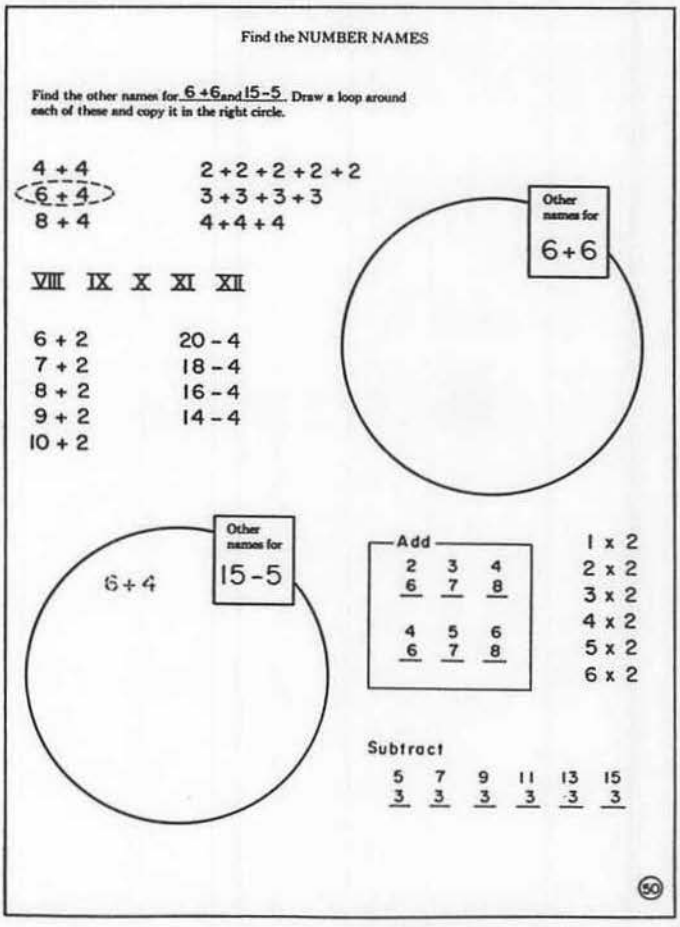


$2+5=7$ or $(2\times1)+(5\times1)=(2+5)\times1=7\times1=7$
$12+30=42$ or $(2\times6)+(5\times6)=(2+5)\times6=7\times6=42$
$8+20=28$ or $(2\times4)+(5\times4)=(2+5)\times4=7\times4=28$
$14+35=49$ or $(2\times7)+(5\times7)=(2+5)\times7=7\times7=49$

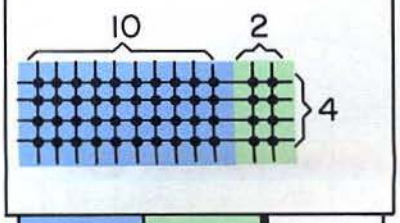
We have more than a trick, we have a fundamental law of arithmetic at work: the distributive principle for multiplication over addition.

Drill

Consider using as a drill sheet page 50 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:

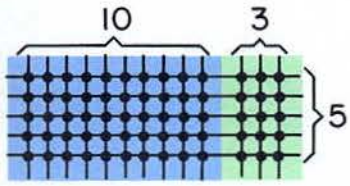


I. 12 x 4 = 48



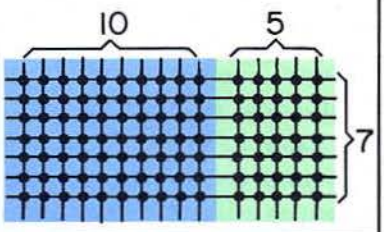
10 x 4	2 x 4	12 x 4
40	8	48

II. 13 x 5 = 65



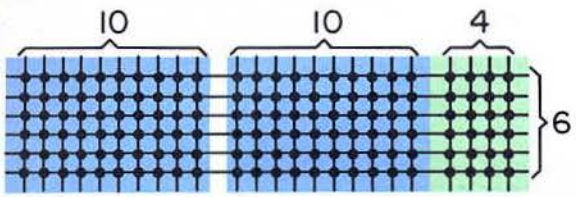
10 x 5	3 x 5	13 x 5
50	15	65

III. 15 x 7 = 105



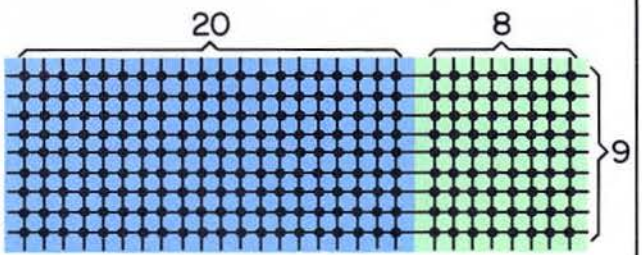
10 x 7	5 x 7	15 x 7
70	35	105

IV. 24 x 6 = 144



10 x 6	10 x 6	20 x 6	4 x 6	24 x 6
60	60	120	24	144

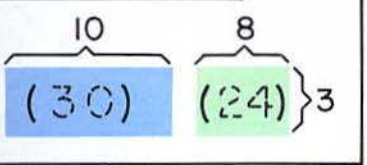
V. 28 x 9 = 252



20 x 9	8 x 9	28 x 9
180	72	252

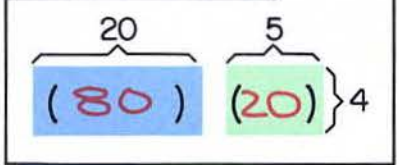
Let's begin imagining the lines and crossing points.

VI. 18 x 3 = 54



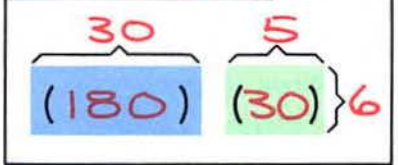
10 x 3	8 x 3	18 x 3
30	24	54

VII. 25 x 4 = 100



20 x 4	5 x 4	25 x 4
80	20	100

VIII. 35 x 6 = 210



30 x 6	5 x 6	35 x 6
180	30	210

IX.

3 x ____	4	2	6	8	14	22	36	58	94	152	246	398
	12	6	18	24	42	66	108	174	282	456	738	1194

Notes on Pupil Page 93

Tasks: *In each of exercises I-VIII pupils use the representation as a clue to guide them in carrying out the computations required to complete the sentence displayed at the top. In exercise IX pupils use the clues provided to complete the table.*

Purpose: To further extend understanding of multiplication and to prepare for an algorithm for multiplying a two-digit number by a one-digit number by stressing decimal partitioning.

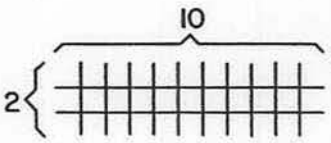
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

Materials for Teacher: Graph board; Ditto master.

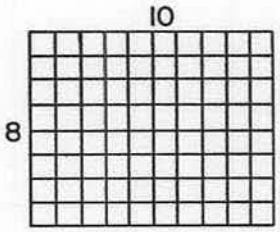
Materials for Pupils: Graph paper; Ditto sheets as described below.

The Lesson:

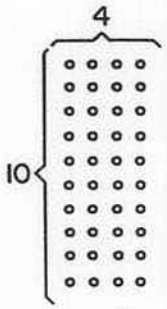
Begin the study of this page by sketching various arrays on the board with 10 as one dimension and eliciting the fact team, as shown in the following column.



$2 \times 10 = 20$ $\quad 20 \div 10 = 2$
$10 \times 2 = 20$ $\quad 20 \div 2 = 10$



$8 \times 10 = 80$ $\quad 80 \div 8 = 10$
$10 \times 8 = 80$ $\quad 80 \div 10 = 8$



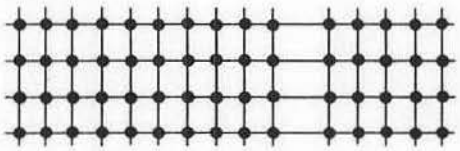
$$\begin{array}{r} 10 \\ \times\ 4 \\ \hline 40 \end{array} \qquad 4\overline{)40}\ \text{10}$$

$$\begin{array}{r} 4 \\ \times 10 \\ \hline 40 \end{array} \qquad 10\overline{)40}\ \text{4}$$

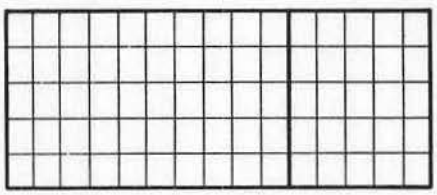
more

Headline for the Day: Susan planted 8 rows of tulips with 12 tulips in each row. Tessie planted 2 rows of tulips with a dozen tulips in each row.

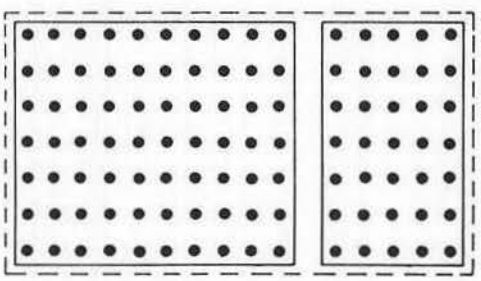
Now sketch different types of arrays in which one dimension is 10 and discuss why decimal partitioning is a shortcut in counting the array.



$4 \times 15 = 4 \times (10+5)$
$= (4 \times 10) + (4 \times 5)$
$= 40 + 20$
$= 60$



$5 \times 15 = 5 \times (10+5)$
$=$
$=$
$=$



$7 \times 15 = 7 \times (10+5)$
$=$
$=$
$=$

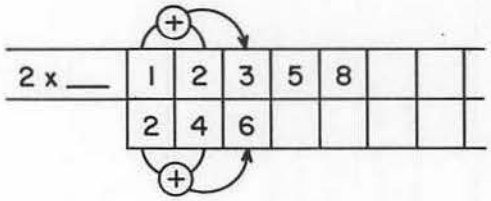
You may wish to provide some experience at this time with writing the results of multiplication in the following way:

$$12 \times 4 = (10 \times 4) + (2 \times 4)$$
$$= 40 + 8$$
$$= 48$$

Before pupils do the work in exercise IX, you may wish to have them try to discover the pattern in the top row of numbers. Do this by playing *What's My Rule?*

Begin by writing four or five numbers of the pattern on the board. Encourage pupils to extend the pattern. The secret, of course, is that each number after the second is the sum of the two numbers that precede it. This makes it possible, then, to get a number in the bottom row either by multiplying the corresponding number in the top row by 3 *or* by adding the two numbers which precede it. Here we have another application of the distributive principle.

If necessary, do a similar but easier example on the board:



Prepare pupils for the addition step in the algorithm for short multiplication by giving them the following exercises on a Ditto sheet. Compare these exercises with what is required by the exercises on pupil page 93.

(a) $50 + 15 = (50 + \square) + 5 = \bigcirc + 5 = \triangle$

(b) $70 + 35 = (70 + \square) + 5 = \bigcirc + 5 = \triangle$

(c) $120 + 24 = (120 + \square) + 4 = \bigcirc + 4 = \triangle$

(d) $180 + 72 = (180 + \square) + 2 = \bigcirc + 2 = \triangle$

(e) $30 + 24 = \triangle$ (f) $180 + 30 = \triangle$

(g) $20 + 24 = \triangle$ (h) $60 + 42 = \triangle$

(i) $90 + 21 = \triangle$ (j) $180 + 12 = \triangle$

(k) $1280 + 35 = \triangle$ (l) $240 + 48 = \triangle$

Drill

Here are some stories for headline practice:

1. Rufus measured his living room. He used a yardstick. He found that the room was 18 feet long.
2. Dottie counted 12 books on her shelf. She took half of the books to read.

   $12 - 6 = \square$ $12 \div 2 = \square$
3. Michele saw a football game. The home team scored 13 points in the first quarter, 14 points in the second quarter, 12 points in the third

quarter, and no points in the fourth quarter. The visiting team had the same score as the home team.

4. Polly bought 3 dozen eggs at 62 cents per dozen.
5. Barnaby had a snapshot album. He said there were 12 pages with 8 pictures on each page. He used 4 gummed corners for each snapshot.

Here is a suggestion for a drill sheet which is directly related to the material of pupil page 93:

1. $19 \times 10 = (10+9) \times 10$
$= (10 \times 10) + (9 \times 10) = \bigcirc$

2. $13 \times 10 = \bigcirc$ **3.** $15 \times 10 = \bigcirc$

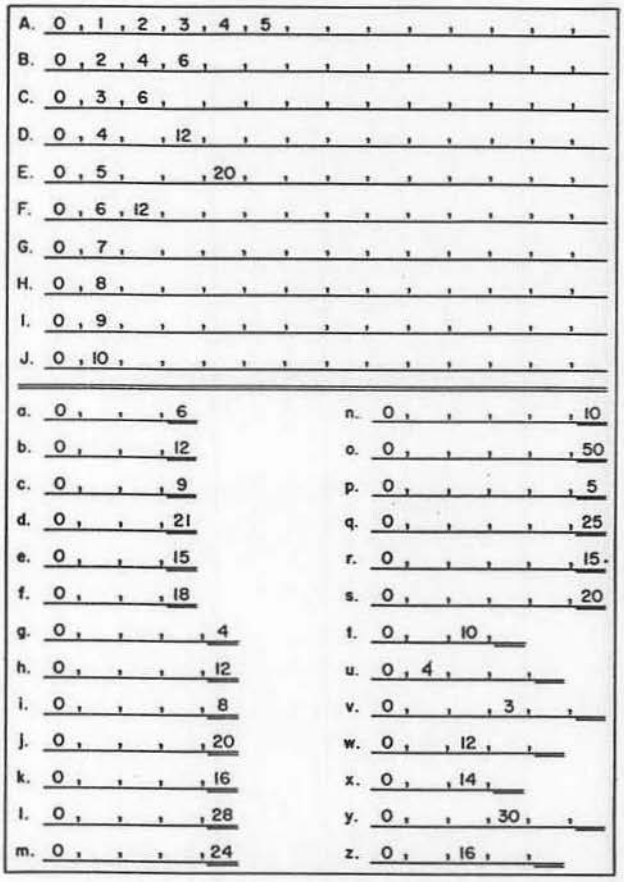
4. $17 \times 10 = \bigcirc$ **5.** $10 \times 4 = \bigcirc$

6. $10 \times 9 = \bigcirc$ **7.** $12 \times 10 = \bigcirc$

8. $25 \times 10 = \bigcirc$ **9.** $36 \times 10 = \bigcirc$

10. $47 \times 10 = \bigcirc$

Here is a suggestion for a drill sheet. Note the concealed division practice in exercises *a-s*.



What Are My Rules?

A

How many letters?

A row of G's: G G G G G G G G G G A row of H's: H H H H		1 row of G's	10	6 rows of G's	60
		1 row of H's	4	6 rows of H's	24
		1 row of each	14	6 rows of each	84
4 rows of G's	40	10 rows of G's	100	14 rows of G's	140
4 rows of H's	16	10 rows of H's	40	14 rows of H's	56
4 rows of each	56	10 rows of each	140	14 rows of each	196

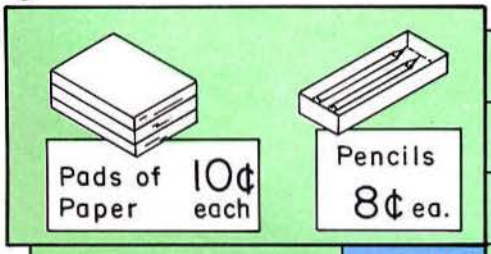
B

What are the coins worth?

		dimes shown	20¢	twice as many dimes	40¢
		pennies shown	3¢	twice as many pennies	6¢
		amount shown	23¢	twice amount shown	46¢
5 × pennies shown	15¢	10 × pennies	30¢	15 × pennies	45¢
5 × dimes shown	$1.00	10 × dimes	$2.00	15 × dimes	$3.00
5 × all coins	$1.15	10 × all coins	$2.30	15 × all coins	$3.45

C

What is the cost?

		2 pencils	16¢	2 of each	36¢
		5 pencils	40¢	5 of each	90¢
		7 pencils	56¢	7 of each	$1.26
6 pads	60¢	6 pencils	48¢	6 of each	$1.08
20 pads	$2.00	20 pencils	$1.60	20 of each	$3.60
26 pads	$2.60	26 pencils	$2.08	26 of each	$4.68

D

7x	5	10	15
	35	70	105

10x	5	10	15
	50	100	150

17x	5	10	15
	85	170	255

Notes on Pupil Page 94

Tasks: *Pupils determine the number or quantity suggested by the picture in each of exercises A-C and complete each exercise by carrying out the computations in accordance with the given instructions.*

Purpose: To further extend understanding of multiplication and to help develop (via an understanding of the distributive principle for multiplication over addition) the necessary methods and procedures for a multiplication algorithm.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Develop this page by discussing examples in which the distributive principle is operating. If necessary, do several examples from pupil page 94 together at the board, eliciting from pupils examples of uses of the principle.

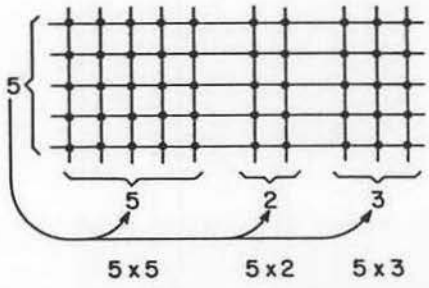
Discuss with pupils how a person who knows that

5×4 is 20 and 7×4 is 28 also knows that
$\_\_ \times 4$ is $20 + 28$;

6×3 is 18 and 8×3 is 24 also knows that
$\_\_ \times 3$ is $\_\_ + \_\_$.

Etc.

Draw pictures like the following and record the corresponding arithmetic with pupil help.



$5 \times (5+2+3) = (5 \times 5) + (5 \times 2) + (5 \times 3) = 5 \times \_\_ = \square$

The exercises on pupil page 94 are designed to pave the way for an algorithm for multiplying a 2-digit number by a 2-digit number. For example, the last problem in exercise A asks for the total number of letters in 14 rows of G's and 14 rows of H's. Here are these letters arranged in a rectangular array with 14 rows:

G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H

Headline for the Day: Schuyler bought a present for his sister Ivanna. The clerk gave him the following change: a half-dollar, a quarter, a dime, a nickel, and a penny. Make a table to record what you can say about this situation.

Since there are 10 G's and 4 H's in each row of the array, our problem amounts to finding the product of 14 by 14.

The previous problems in exercise A give us the clues we need to find the total in an easy way. The first two problems in the bottom row ("4 rows of each" and "10 rows of each") suggest the following partition of the array of letters:

G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H

We have already found that in 4 rows of each there are 56 letters and in 10 rows of each there are 140 letters. So, in 14 rows of each there must be 56 + 140, or 196, letters.

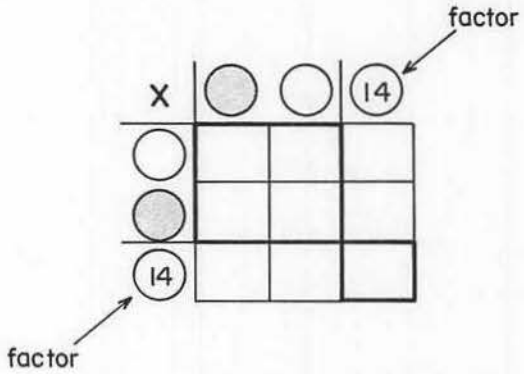
But, how did we find the numbers 56 and 140? We did this by partitioning each of the component arrays shown above by drawing vertical lines:

G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H
G	G	G	G	G	G	G	G	G	G	H	H	H	H

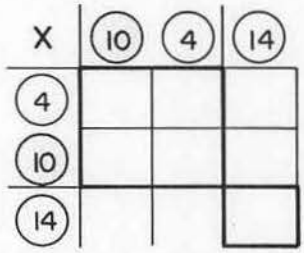
Thus, we found that in 4 rows of each there are 40 + 16, or 56, letters; and that in 10 rows of each there are 100 + 40, or 140, letters.

Cross Number Puzzles Again

The partitioning of a rectangular array into four component arrays by drawing a line parallel to one edge of the array and a second line parallel to an adjacent edge is the key idea behind the standard algorithm for multiplying a pair of 2-digit numbers. (This algorithm is sometimes called an algorithm for long multiplication.) The procedure can be nicely pictured by using a variation of the *Cross Number Puzzle* format:



The numbers to be multiplied are written in the end circles as shown. Next, each factor is partitioned into two components, and the components are written in the other circles. Since we are concerned with the algorithm based on decimal partitioning, we agree that the shaded circular regions are reserved for multiples of 10 and that the remaining circular regions are reserved for numbers less than 10. Thus, in the case at hand:



Now we regard the entire figure as if it were a two-dimensional multiplication table. Each of the nine cells should be filled with the product of the two circled numbers whose row and column cross in that cell. Of course, our main interest is in the cell in the lower right-hand corner, for this cell is to carry

the product of 14 by 14.

Let's find the products for the basic four cells:

X	10	4	14
4	40	16	
10	100	40	
14			

We can now find the numbers for the remaining cells without doing any more multiplying, thanks to the distributive principle for multiplication over addition. Since our main job is to get to the lower right-hand cell, we proceed in *Cross Number Puzzle* fashion but we need to compute just 2 row-sums and then 1 column-sum:

X	10	4	14
4	40	16	56
10	100	40	140
14			196

Let's use our extended *Cross Number Puzzle* device to find the product of 15 by 23, the problem of exercise B on pupil page 94:

X	20	3	23
5			
10			
15			

X	20	3	23
5	100	15	
10	200	30	
15			

X	20	3	23
5	100	15	115
10	200	30	230
15			345

$$\begin{array}{r} 23 \\ \times 15 \\ \hline 115 \\ 23 \\ \hline 345 \end{array}$$

If we had computed the first two column-sums and then a row-sum, we would have been tracing the steps in the following:

$$\begin{array}{r} 15 \\ \times 23 \\ \hline 45 \\ 30 \\ \hline 345 \end{array}$$

Here is a solution to the problem of exercise C in *Cross Number Puzzle* format and a solution in algorithm form:

X	20	6	26
8	160	48	208
10	200	60	260
18			468

$$\begin{array}{r} 26 \\ \times 18 \\ \hline 208 \\ 26 \\ \hline 468 \end{array}$$

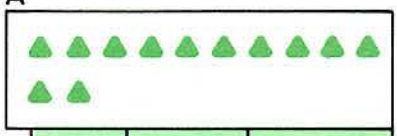
We suggest that you solve the problem of exercise D ($17 \times 15 = ?$) in *Cross Number Puzzle* format. First maintain the decimal restriction on partitioning the factors. Then, remove the restriction and solve the problem using a different partitioning. (See the drill sheet suggestions under "Drill" in Notes on Pupil Page 95 and Notes on Pupil Page 86.)

Drill

Here are some stories for headline practice:

1. Joel's class visited the post office. They watched the postal clerk sort 498 letters and 285 packages.
2. An astronaut plans a ten-orbit flight around the earth. He estimates he will travel 30,000 miles each time he orbits the earth.
3. At Westside Elementary School 54 children played in the orchestra. Twenty-six members of the orchestra were boys.
4. Lucille wanted to watch 3 programs on television. Each of them was 30 minutes long. Her mother only let her watch for 60 minutes.

A

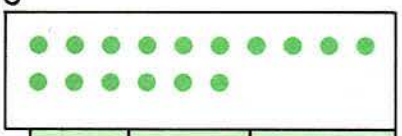


2	10	12
x 3	x 3	x 3
6	30	36

B

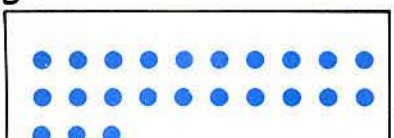
5	10	15
x 2	x 2	x 2
10	20	30

C



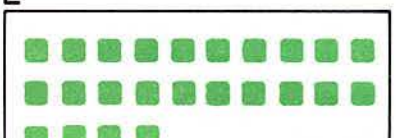
6	10	16
x 3	x 3	x 3
18	30	48

D



3	20	23
x 3	x 3	x 3
9	60	69

E

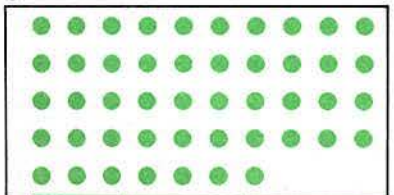


4	20	24
x 3	x 3	x 3
12	60	72

F

6	20	26
x 4	x 4	x 4
24	80	104

G



7	40	47
x 5	x 5	x 5
35	200	235

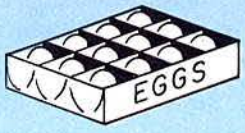
H

8	30	38
x 6	x 6	x 6
48	180	228

I

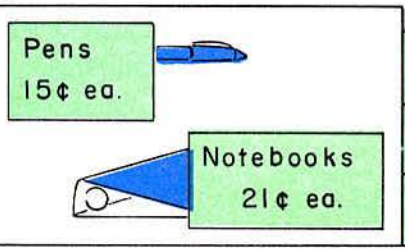
5	30	35
x 7	x 7	x 7
35	210	245

J



3 doz. eggs	36 eggs	5 doz. eggs	60 eggs
6 doz. eggs	72 eggs	10 doz. eggs	120 eggs

K



3 pens	45 ¢	7 pens	$ 1.05
3 notebooks	63 ¢	7 notebooks	$ 1.47
3 of each	$ 1.08	7 of each	$ 2.52

Notes on Pupil Page 95

Tasks: *Pupils determine the number or quantity suggested by the representation in each of exercises A-K and carry out the required computations.*

Purpose: To further extend understanding of multiplication and to develop the necessary skills and techniques used in algorithms for multiplication.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Graph board.

Materials for Pupils: Graph paper or Cuisenaire rods.

The Lesson:

Here is a way to prepare for this page. Let pupils represent various 2-digit numbers by making drawings on graph paper or by using Cuisenaire rods. Request that the drawings or rod representations show a decimal partitioning of the number.

Then, do some multiplication problems in which 2-digit numbers are multiplied by 1-digit numbers. For example:

Represent 23 on your graph paper. Now, multiply 23 by 4. What is the answer?

A pupil might reason as follows:

23 is 2 tens and 3 ones. Four times as much is 8 tens and 12 ones. So, the result is 92.

When pupils are working exercises J and K on pupil page 95, check to see that they are using the distributive principle for multiplication over addition and not repeated addition. A pupil who finds the cost of 7 notebooks by adding:

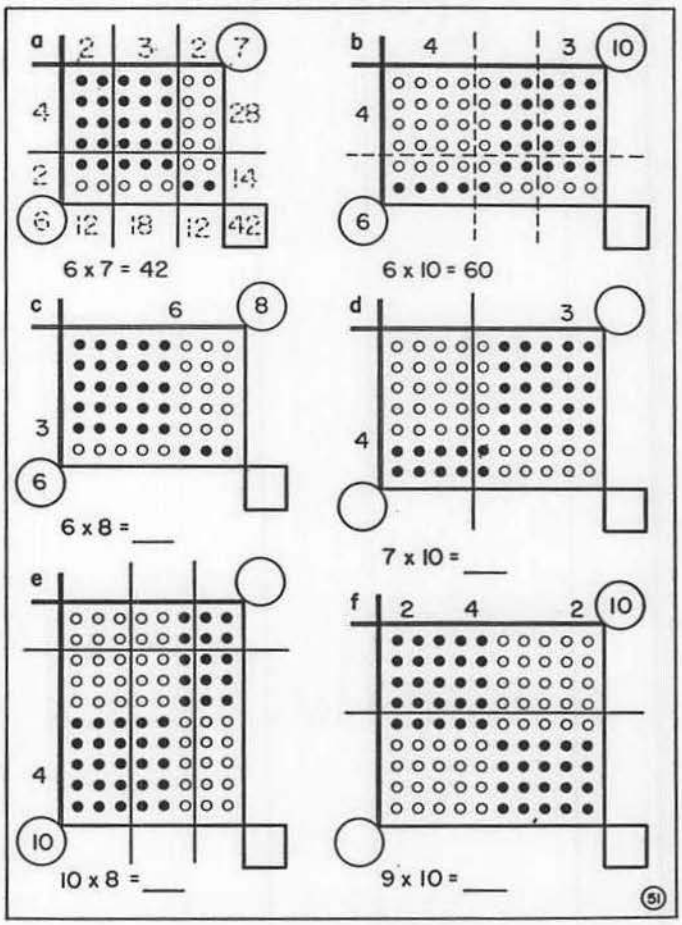
$$21+21+21+21+21+21+21$$

should be told that this is a correct way of solving such a problem but that it takes much too long. You can compel pupils to give up primitive techniques if you place a premium on speed.

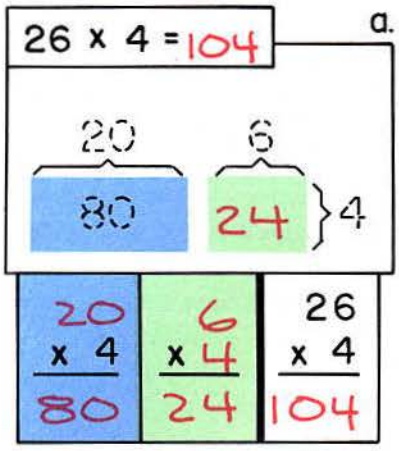
On the other hand, a pupil who solves the problem of the 7 notebooks in K by using available data should be praised. For example, we have already found that 3 notebooks cost 63 cents; so, double 63 to get 126, and then add 21 to get 147. Or, 7 pens cost 105 cents; a notebook cost 6 cents more than a pen; $7 \times 6 = 42$; $105 + 42 = 147$.

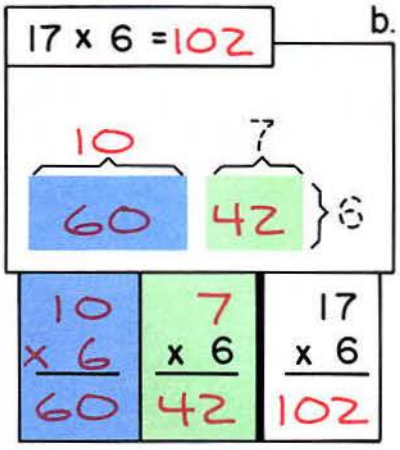
Drill

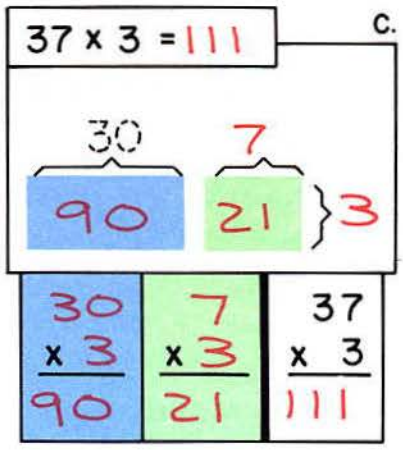
Consider using as a drill sheet page 51 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C. (These exercises provide much insight into the partitioning of arrays in finding products. See "*Cross Number Puzzles* Again" in Notes on Pupil Page 94.)

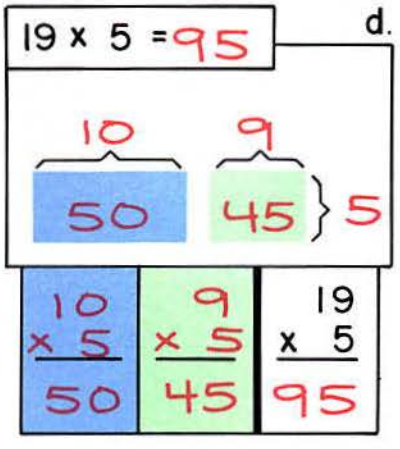


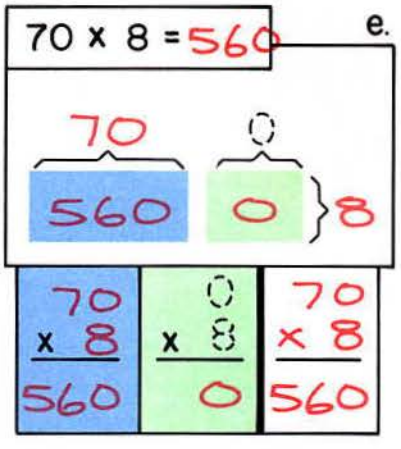
Headline for the Day: Marelda's mother bought some eggs at 43¢ a dozen. Make a table to record what you can say about this situation.

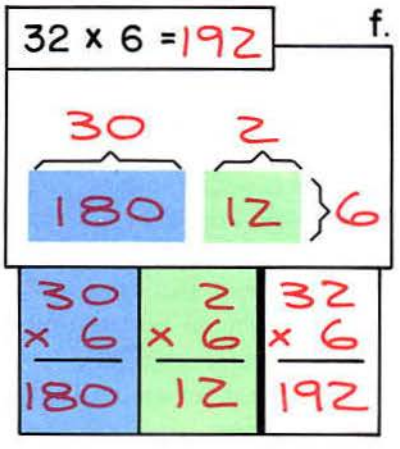




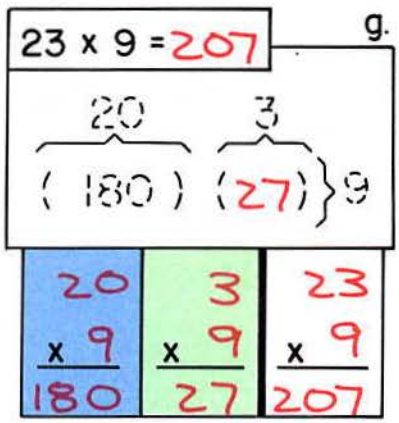


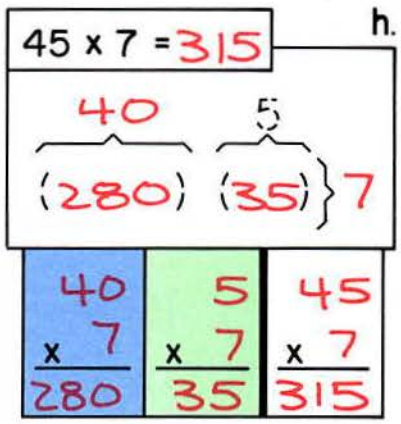


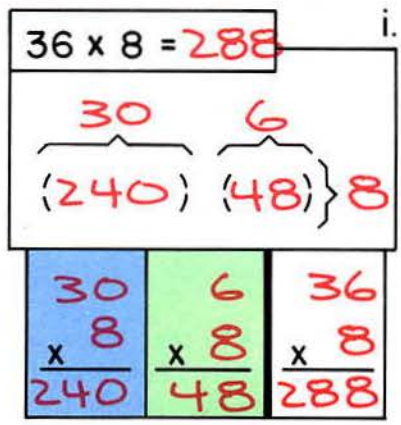




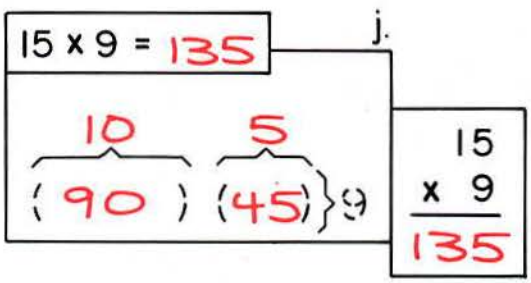
You don't need the colored blocks to show where the crossing points would be.

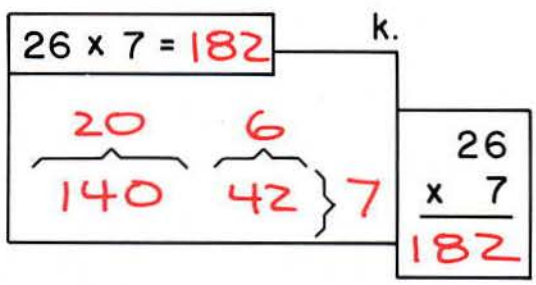






We have left out some of the steps we used above. Work on scratch paper if you wish to write the steps we left out.





Notes on Pupil Page 96

Tasks: *Pupils complete the number sentence at the top of each exercise by using, if necessary, the suggested representation and partitioning.*

Purpose: To perfect skills and techniques used in an algorithm for multiplication.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

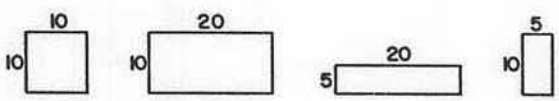
Materials for Teacher: Ditto master.

Materials for Pupils: Scratch paper; Ditto sheets as described below.

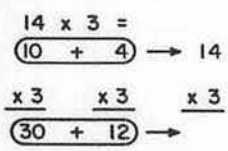
The Lesson:

On pupil page 93 we moved toward greater abstraction by going from crossing points to colored blocks. On the present page we continue the progress toward greater abstraction by going from colored blocks to a simple framework of braces. Encourage pupils to work without reference to pictures if they can. Of course, anyone in trouble can arrange a model from counters or he can draw one.

Again emphasize the partitioning of the 2-digit factors into multiples of 10 and units. Practice recognition of such products as suggested by pictures like these:

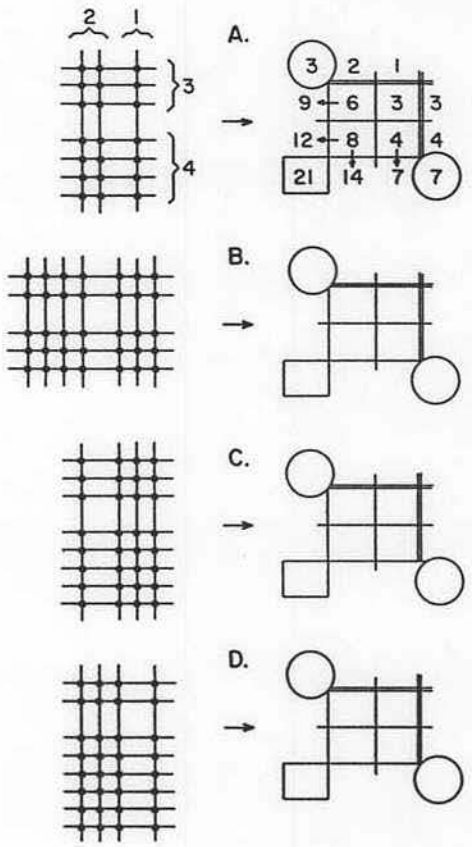


Review the relationship of such partitioning to finding products with examples like these:

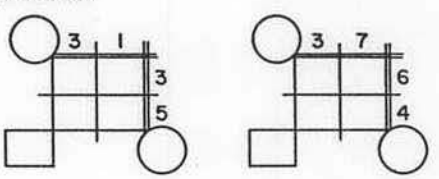


Enrichment Activity

Have pupils explore partitioning exercises like the following in which we create a special kind of *Cross Number Puzzle.* (Compare with the format described in Notes on Pupil Page 94.) Begin by drawing diagram A on the board and gradually filling in the numbers in the *Cross Number Puzzle.* As soon as pupils see what is happening, let them give you the numbers that belong in the cells. Give other examples on worksheets for individuals or groups to work out.



Soon pupils will know how to complete examples with no pictures.



Headline for the Day: Fill in the blanks to complete the pattern.

300, 291, 282, , , , 246, , , , 210

In doing the following examples, you should imagine the sketches. Use a piece of scratch paper to draw a sketch whenever you are in doubt.

A

14 x 6 = 84

10	4	14
x 6	x 6	x 6
60	24	84

B

21 x 7 = 147

20	1	21
x 7	x 7	x 7
140	7	147

C

18 x 5 = 90

10	8	18
x 5	x 5	x 5
50	40	90

D

12 x 3 = 36

10	2	12
x 3	x 3	x 3
30	6	36

E

16 x 6 = 96

10	6	16
x 6	x 6	x 6
60	36	96

F

13 x 7 = 91

10	3	13
x 7	x 7	x 7
70	21	91

G

16 x 5 = 80

10	6	16
x 5	x 5	x 5
50	30	80

H

25 x 6 = 150

20	5	25
x 6	x 6	x 6
120	30	150

I

24 x 6 = 144

20	4	24
x 6	x 6	x 6
120	24	144

J

12 x 4 = 48

10	2	12
x 4	x 4	x 4
40	8	48

K

25 x 5 = 125

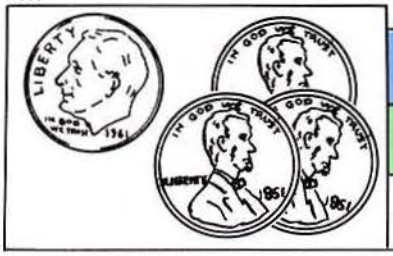
20	5	25
x 5	x 5	x 5
100	25	125

L

27 x 6 = 162

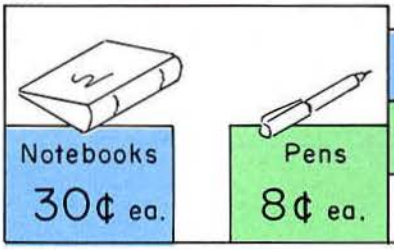
20	7	27
x 6	x 6	x 6
120	42	162

M



twice as many dimes	20¢	9 times as many dimes	$ .90
twice as many pennies	6¢	9 times as many pennies	$ .27
twice as much as 13¢	26¢	9 times as much as 13¢	$ 1.17

N



½ dozen notebooks	$ 1.80	a dozen notebooks	$ 3.60
½ dozen pens	$ .48	a dozen pens	$ .96
½ dozen of each	$ 2.28	a dozen of each	$ 4.56

Notes on Pupil Page 97

Tasks: *Pupils use the clues provided to carry out the computations required to complete each exercise. In A-L only the horizontal sentences need be completed.*

Purpose: To further perfect skills and techniques used in an algorithm for multiplication.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupil teams might be asked to attempt this page. Or you may prefer to discuss the exercises, one by one, eliciting from the pupils the reconstruction of the exercises. Encourage various ways of reasoning about a problem to get to the correct answer.

Permit pupils who can proceed on their own to do so at any time.

Let's look at one such exercise and one way it might be reasoned through. Our job is to fill the blanks in:

×5 =			
	× ___	× ___	× ___ 90

We know that the product of some number and 5 is 90; so we now have:

(a)

×5 =90			
	× 5	× 5	× 5 90

We know that the missing factor in the left-hand cell of the bottom row is a multiple of 10, and since 20 is too big it must be 10. The product of 10 by 5 is 50.

(b)

×5 =90			
	10 × 5 50	× 5	× 5 90

Since we have 50, we need 40 more to make 90, and $8 \times 5 = 40$. So now we have:

(c)

×5 =90			
	10 × 5 50	8 × 5 40	× 5 90

The missing factor in the top row is the sum of the two factors, 10 and 8; so we have:

18×5 =90			
	10 × 5 50	8 × 5 40	18 × 5 90

There are, of course, other ways to reason through this problem.

Drill

Here are some stories for headline practice:

1. To reach the sewing machine from her work table Jane has to walk 2 feet more than 4 yards.
2. Jan's father planned a trip to New York City. He told Jan they first needed to travel 140 miles to Chicago and then 800 miles from Chicago to New York City.

Headline for the Day: Fill in the blanks to complete the pattern.

430, 417, 404, ___, ___, ___, 352, ___, ___, ___, 300

A

31 x 5 = 155

30 x 5 = 150	1 x 5 = 5

31 x 5 = 155

B

15 x 3 = 45

10 x 3 = 30	5 x 3 = 15

15 x 3 = 45

C

23 x 8 = 184

20 x 8 = 160	3 x 8 = 24

23 x 8 = 184

Try to complete the following in the shortest way you can. Use scratch paper if you wish.

D

I	II	III	IV	V	VI	VII	VIII	IX
64 x 2 = 128	13 x 4 = 52	26 x 4 = 104	40 x 6 = 240	58 x 3 = 174	25 x 6 = 150	75 x 5 = 375	15 x 9 = 135	60 x 5 = 300

E

20 x 8 = 160	40 x 5 = 200	24 x 9 = 216	24 x 3 = 72	24 x 5 = 120	24 x 8 = 192	32 x 7 = 224	32 x 2 = 64	32 x 9 = 288

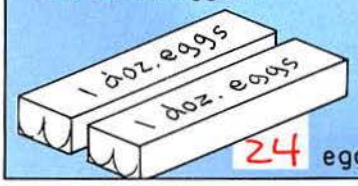
F

36 x 3 = 108	36 x 7 = 252	36 x 10 = 360	25 x 7 = 175	25 x 9 = 225	25 x 16 = 400	16 x 5 = 80	16 x 10 = 160	16 x 15 = 240

If you found shortcuts as you completed the examples above, talk to others about your findings. Share your discoveries.

G

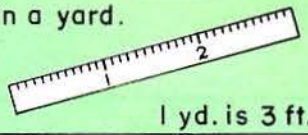
Two dozen eggs is 24 eggs.



3 times as many	72 eggs	4 times as many	84
5 times as many	120	6 times as many	132
8 times as many	192	10 times as many	216

H

There are 36 inches in a yard.



1 yd. is 3 ft.

2 yards is	72 in.	3 yards is	108 in.
4 yards is	144 in.	7 yards is	252 in.
6 yards is	216 in.	10 yards is	360 in.

I

There are 16 ounces in one pound.

3 pounds is	48 oz.	5 pounds is	80 oz.
8 pounds is	128 oz.	10 pounds is	160 oz.
11 pounds is	176 oz.	15 pounds is	240 oz.

Tasks: *Pupils compute missing factors and products in A-F. In G-I pupils determine the number or quantity suggested by each representation and carry out the required computations.*

Purpose: To develop shortcuts, i.e., to develop an efficient algorithm for multiplication.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Examples on this page should be worked at the board until pupils are ready to proceed on their own. At any time pupils who can go on should be encouraged to do so.

Pupils may reason through the following problem:

$$\begin{array}{r} \times\ 7 \\ \hline 224 \end{array}$$

in this way:

$$7 \times 10 = 70$$
$$7 \times 20 = 140$$
$$\boxed{7 \times 30 = 210}$$

Hence, the missing factor is 30 plus something.

$$224 - 210 = 14 \text{ and } 7 \times 2 = 14$$

Therefore, the missing factor is 30 + 2, or 32. Thus, pupils create their own ways of doing division problems.

Note the applications of the distributive principle for multiplication over addition in exercises G-I.

Drill

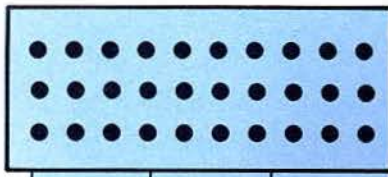
Here is a suggestion for a drill sheet:

Here are some stories for headline practice:

1. Noel is 4 feet from the bulletin board. His right arm is 28 inches long.
2. Jonas noticed that there are 4 months with exactly 30 days and 7 months with exactly 31 days.

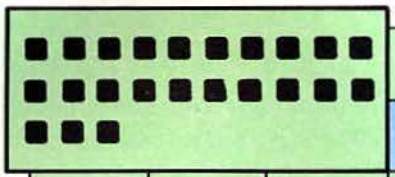
Headline for the Day: $\square - \triangle = 800$. Make a table to record what you can say about this situation.

A



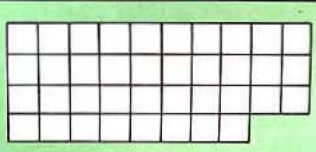
			+25	−8
			55	22
−13	+16	−20	−19	+70
17	46	10	11	100

B



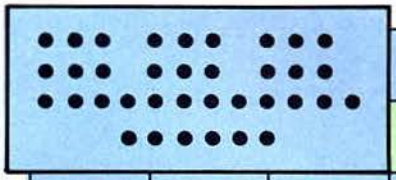
			+27	−15
			50	8
−20	+7	+52	−5	+77
3	30	75	18	100

C



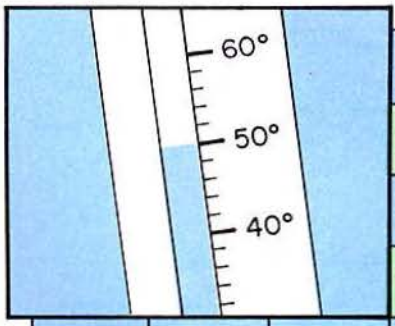
			+2	−20
			40	18
−23	+13	−29	+38	+62
15	51	9	76	100

D



			−6	+8
			30	44
+14	−25	−18	−36	+64
50	11	18	0	100

E



			+10°	−5°
			60°	45°
			+33°	−13°
			83°	37°
+34°	−23°	+25°	−38°	+50°
84°	27°	75°	12°	100°

F

			+9¢	−9¢
			50¢	32¢
			−25¢	+50¢
			16¢	91¢
−14¢	−36¢	−30¢	+59¢	+39¢
27¢	5¢	11¢	$1.00	80¢

G

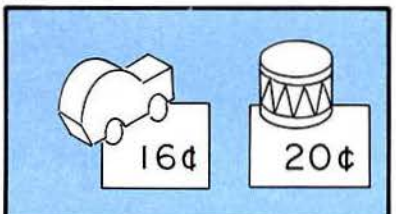
twice as much

60 ¢

half as much

15 ¢

H



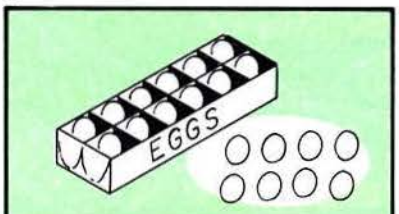
3 drums and 1 car

76 ¢

2 of each

72 ¢

I



twice as many eggs

3 doz., 4 eggs

half as many eggs

0 doz., 10 eggs

99 Test page - a variety of multiple-choice items covering computations in addition, subtraction, multiplication, and division.

ıks: Pupils determine the number or quantity suggested by the representation in each exercise and determine the missing operation or carry out the given operation.

Purpose: To test computational skills via applications.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

Score 7 points for each of A-D, 9 points for each of E and F, and 2 points for each of G-I for a total of 52 points.

Supplementary Activities

All children need to tackle "hard work" problems once in a while. A very significant aspect of these problems is that they often provide dramatic examples of how costly a careless error can be. One little slip can lead to a wrong result.

For example, John got the wrong answer because of a mistake in adding the hundred-dollar column:

$$\begin{array}{r} \$\ 235.16 \\ 85.26 \\ 301.00 \\ 625.75 \\ 823.44 \\ \hline \$3,070.61 \end{array}$$

John committed a $1,000 error. His answer should have been $2,070.61. This is a good example of how 22 correct additions were thrown away in one instance of carelessness. Perhaps the worst of it all was that John was robbed of the feeling of confidence that comes from performing flawlessly a long chain of little steps.

It takes practice to be able to complete the problem above quickly and accurately. This practice can be provided in more interesting surroundings—surroundings which are also in the foothills of big mathematical mountains. "Chain sequence" problems are in this category.

more

Headline for the Day: Manuella said it is 14 degrees warmer than it was at the same time yesterday. Make a table to record what you can say about this situation.

Let's define a chain sequence to be a sequence in which, for each three consecutive terms, the third is the sum of the first and second. We could start the sequence with any pair of numbers, but as soon as the pair is selected, the rest of the sequence is determined. For example, here are the first ten terms of a chain sequence:

2, 4, 6, 10, 16, 26, 42, 68,110,178

The eleventh term is 110 + 178, or 288. What is the twelfth term?

In giving pupils chain sequences to complete, it is good practice to provide some checkpoints along the way. Sequences *a*, *b*, and *c*, below, are to be completed into chain sequences:

a. 1, 1, , , , 8, , , , 55
b. 2, 2, , , , 16, , , , 110
c. 3, 3, , , , 24, , , , 165
d. , , , , , , , , , ,

Slow pupils can stop at the first checkpoint in each sequence—at 8, 16, and 24.

The sequence *c* is the sum of *a* and *b*. (Each term in *c* is the sum of the two numbers listed above it: 3 is the sum of 1 and 2, 24 is the sum of 8 and 16, etc.)

Now let each term of *d* be the sum of the three numbers listed above it. Compute the missing terms of *d*. Is the sequence thus generated a chain sequence? Is each term of *d* after the second the sum of its two immediate predecessors?

Here are some more chain sequences:

e. 1, 2, 3, 5, , 13, , , 55
f. 1, 3, , , , 18, , , 76
g. 1, 4, , , , 23, , , 97
h. 1, 5, , , , 28, , , 118
i. , , , , , , , , ,

Let sequence *i* be the sum of the sequences *e*, *f*, *g*, and *h*. Is sequence *i* a chain sequence? Compute the first ten terms of *i* and check your conjecture.

It is instructive to give pupils a pair of consecutive terms (other than the first two) of a chain sequence and ask them to find missing terms.

j. , , , , , , , 54, 87, ,
k. , , , , , , , 31, 50, ,
m. , , , , , , , , , ,

In sequence *j* the term immediately preceding 54 must be 33 because 33 is the only number whose sum with 54 is 87. Find the missing terms of *j* and *k*.

Let *m* be the difference of *k* and *j*. Is *m* a chain sequence?

We have seen that a chain sequence is determined by any pair of consecutive terms. Is a chain sequence also determined by a given pair of terms which are separated by a missing term? Try to complete the following into chain sequences:

n. , , , 9, , 25, , , , , , 453
o. , , , , 32, , 84, , , , , 932

Let's define a new kind of chain sequence. Let this new chain sequence be a sequence in which, for each three consecutive terms, the third is the product of the first two.

p. 1, 2, , , , 32, , , ,
q. 2, 3, 6, 18, ,1944, , , ,
r. 2, 6, , , , , , , ,

Let *r* be the sequence which is the product of *p* and *q*. Is *r* a chain sequence according to the new definition? Yes, it is. It is quickly evident that these chain sequences have a very rapid growth. For example, the 7th term of *q* is 202176.

Try another kind of chain sequence. This time let a chain sequence be one in which, for each pair of consecutive terms, the second is twice the first.

s. 1, 2, 4, , , 32, , , ,
t. 3, , , , , 96, , , ,
u. 5, , , , ,160, , , ,
v. 9, , , , , , , , , ,

Let *v* be the sum of *s*, *t*, and *u*. Is *v* a chain sequence according to the new definition?

Here are some more chain sequences of the type just defined:

w. ¼, ½, 1, 2, 4, 8, 16, , ,
x. , , , 7, , , , , , ,

Of course, the definition can be changed to consider a chain sequence as a sequence in which, for each pair of consecutive terms, the second term is (a) three times, or (b) four times, or (c) two and one half times the first term. (Chain sequences of this

type are sometimes called *geometric sequences* or *geometric progressions.* Those of the type defined at the outset of this discussion are called *Fibonacci sequences.*)

Chain sequences are a demanding exercise. One careless error, and the whole endeavor gets into a mess. The faster pupil will quickly see that he can invent his own definitions and build his own structures.

There is excitement in getting a chain sequence going from one or two terms and a simple definition. Such problems involve almost any dimension of drill desired, and the whole business takes place in a deeply satisfying mathematical environment.

Drill

Here are some stories for headline practice:

1. Marco and Leeta were playing "Pin the Tail on the Donkey." Marco placed the tail 11 inches from the correct spot. Leeta placed hers 5 inches less than 1 foot from the correct spot.
2. Last month 299 children visited the firehouse. This month 400 children also visited the firehouse.
3. Tod's little sister was playing with her building blocks. She built 2 layers of 2 rows with 3 blocks in each row.
4. Natalie saved coins. She had 5 coins on each card. She had seven cards of coins.
5. A train started out with 9 passengers. It made 7 stops. At each stop it took on 6 passengers.

$$9 \times 7 \times 6 = \square$$

Here is a suggestion for a drill sheet:

1. $71 \times 5 = (70+1) \times 5 = (70 \times 5) + (1 \times 5)$
   $= 350 + 5$
   $= \bigcirc$
2. $43 \times 6 = (40+3) \times 6 = (40 \times 6) + (3 \times 6)$
   $= 240 + \square$
   $= \bigcirc$
3. $57 \times 3 = (50+7) \times 3 = (50 \times 3) + (7 \times 3)$
   $= \square + 21$
   $= \bigcirc$
4. $65 \times 2 = \bigcirc$
5. $74 \times 3 = \bigcirc$
6. $28 \times 5 = \bigcirc$
7. $36 \times 7 = \bigcirc$

A

wk.	da.	da.
7	1	50
2	0	14
5	1	36

<7

B

wk.	da.	da.
8	6	62
2	6	20
6	0	42

C

wk.	da.	da.
10	0	70
3	1	22
6	6	48

D

wk.	da.	da.
11	3	80
3	5	26
7	5	54

E

wk.	da.
10	5
3	4
7	1

F

wk.	da.
12	6
4	2
8	4

G

wk.	da.
15	0
5	0
10	0

H

wk.	da.
17	1
5	5
11	3

I

wk.	da.
19	2
6	3
12	6

I

10¢	1¢	cents
8	7	87
3	4	34
5	3	53

<10

II

10¢	1¢	cents
13	8	138
9	7	97
4	1	41

III

10¢	1¢	cents
7	6	76
5	7	57
1	9	19

IV

10¢	1¢	cents
12	9	129
9	8	98
3	1	31

V

10¢	1¢
10	0
8	3
1	7

VI

10¢	1¢
13	8
10	9
2	9

VII

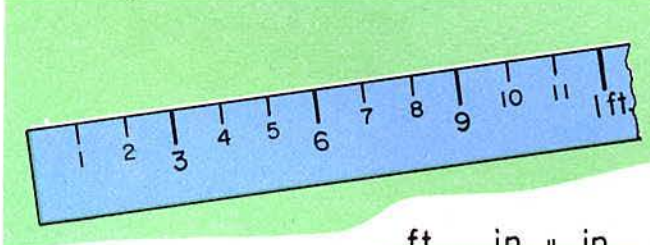
10¢	1¢
10	0
8	9
1	1

VIII

10¢	1¢
32	4
23	5
8	9

IX

10¢	1¢
15	3
6	2
9	1



a

ft.	in.	in.
3	0	36
5	4	64
8	4	100

<12

b

ft.	in.	in.
5	3	63
3	10	46
1	0	12
10	1	121

c

ft.	in.	in.
3	9	45
3	8	44
2	4	28
2	3	27
12	0	144

d

ft.	in.	in.
1	7	19
3	3	39
4	11	59
2	5	29
1	11	23
14	1	169

100 Numeration exercises (bases 7, 10, 12,) in translation and in addition and subtraction using these notations.

Tasks: *Pupils carry out the computations necessary to determine the missing numbers.*

Purpose: To review numeration systems and further perfect computational skills via applications.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

Pupils can begin work on this page without any special preparation from you.

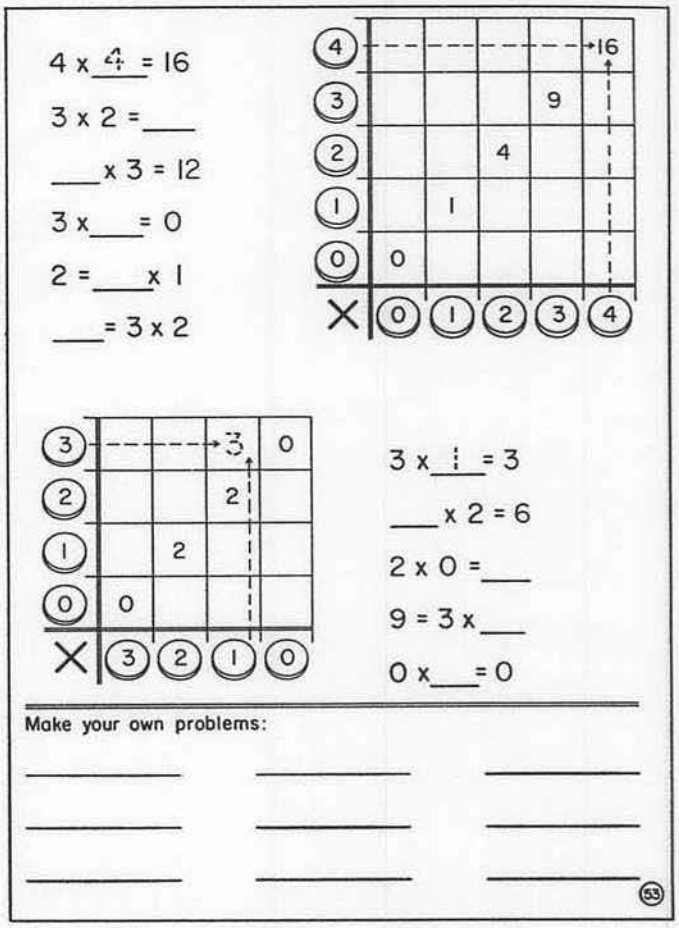
Note that with the exception of E-I and V-IX, each exercise is in *Cross Number Puzzle* format. For example, in exercise *a* at the bottom of the page, we find the sum of 36 and 64 by partitioning 36 and 64 duodecimally (base twelve), and add the parts in columns. Sure enough, 8 twelve-groups and 4 one-groups make 1 hundred-group.

Drill

Here are some stories for headline practice:

1. Jesse and James measured the height of their room. They used a yardstick. They found the height of the room to be 2 feet more than 2 yards.
2. Of the 564 children who went to Parkside Elementary School, 398 rode busses to school.
3. The seat of Jack's chair is 15 inches from the floor. Jack is 43 inches tall.

Consider using as a drill sheet page 53 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Headline for the Day: Fill in the blanks to complete the pattern.

250,238, , , ,190,178, , , ,130

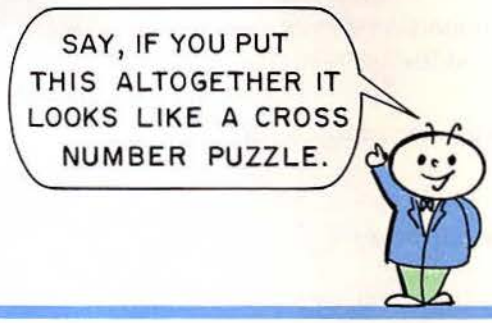
All About Weeks and Days

wk.	da.
0	3
+ 0	6
1	2

wk.	da.
1	0
+ 1	1
2	1

wk.	da.	wk.	da.	wk.	da.
1	2	2	1	3	3

wk.	da.	wk.	da.	wk.	da.
0	3	1	0	1	3
0	6	1	1	2	0



wk.	da.
1	3
+ 2	0
3	3

A

wk.	da.	wk.	da.	wk.	da.
1	3	1	2	2	5
3	1	1	0	4	1
4	4	2	2	6	6

B

wk.	da.	wk.	da.	wk.	da.
2	3	1	2	3	5
2	1	2	1	4	2
4	4	3	3	8	0

C

wk.	da.	wk.	da.	wk.	da.
2	6	1	6	4	5
1	6	2	5	4	4
4	5	4	4	9	2

D

wk.	da.	wk.	da.	wk.	da.
MANY CORRECT ANSWERS				6	3
				5	4
6	3	5	4	12	0

All About Nickels and Pennies

E

5¢	1¢	5¢	1¢	5¢	1¢
0	4	1	1	2	0
0	2	2	3	3	0
1	1	3	4	5	0

F

5¢	1¢	5¢	1¢	5¢	1¢
1	3	1	4	3	2
3	4	2	4	6	3
5	2	4	3	10	0

G

5¢	1¢	5¢	1¢	5¢	1¢
4	3	4	3	9	1
5	2	5	2	10	4
10	0	10	0	20	0

H

5¢	1¢	5¢	1¢	5¢	1¢
9	4	7	4	17	3
7	4	14	3	22	2
17	3	22	2	40	0

101 Addition Cross Number Puzzles using base 7, 10, and 12 notations.

Tasks: *Pupils use the clues provided at the top of the page to carry out the computations necessary to complete* Cross Number Puzzles *A-H*.

Purpose: To review numeration systems and further perfect computational skills in a *Cross Number Puzzle* setting.

Unifying Ideas: Structure; numeration; addition and subtraction.

Materials for Teacher: Ditto master.

Materials for Pupils: Cuisenaire rods; Ditto sheets as described below.

The Lesson:

The *Cross Number Puzzle* approach to the problem of adding "denominate numbers" is particularly useful because of its self-checking feature. In order to make the activities more concrete we suggest that you let pupils work out some of the exercises with Cuisenaire rods or other representational material.

Prepare a large *Cross Number Puzzle* form on a Ditto master and make a copy for each pupil. Let's see how they might work exercise B.

a	b	c
d	e	f
g	h	i

(The letters are used here just to facilitate the discussion that follows. Do not put these letters on the actual form.)

First, they fill cells *a*, *b*, *d*, and *e* with rods: 2 blacks and 3 whites in *a*, 1 black and 2 whites in *b*, 2 blacks

more

Headline for the Day: Fill in the blanks to complete the pattern.

365,358, , ,337,330, , , , ,295

---

and 1 white in *d*, and 2 blacks and 1 white in *e*. Next, they sweep the rods from *a* and *b* into *c*, and from *d* and *e* into *f*.

They now have 3 blacks and 5 whites in *c*; so, they record (3, 5) in the appropriate spaces in exercise B on pupil page 101. Similarly, they record (4, 2).

Next, they sweep the rods from *c* and *f* into *i*. This gives them 7 blacks and 7 whites. They exchange the 7 whites for a black, and record (8, 0).

Now they return to the original distribution of rods in cells *a*, *b*, *d*, and *e* and sweep them into cells *g* and *h*, record, sweep into *i*, exchange, and note that the record has already been made.

Exercise C presents something more of a challenge because it involves more exchanging and because it calls for a reconstruction, an activity which always demands more thinking. One way to start is to fill cells *g* and *h*. Now, the first problem is to partition 4 blacks and 5 whites into two groups such that one of the groups consists of 1 black and 6 whites. A way to accomplish this is to exchange one of the blacks for 7 whites. Now we have 3 blacks and 12 whites in cell *g*. We place 1 black and 6 whites in cell *d* and put the rest in cell *a*. By actual count, there are 2 blacks and 6 whites in *a*; so, we record (2, 6) in the appropriate spaces in exercise C. Etc.

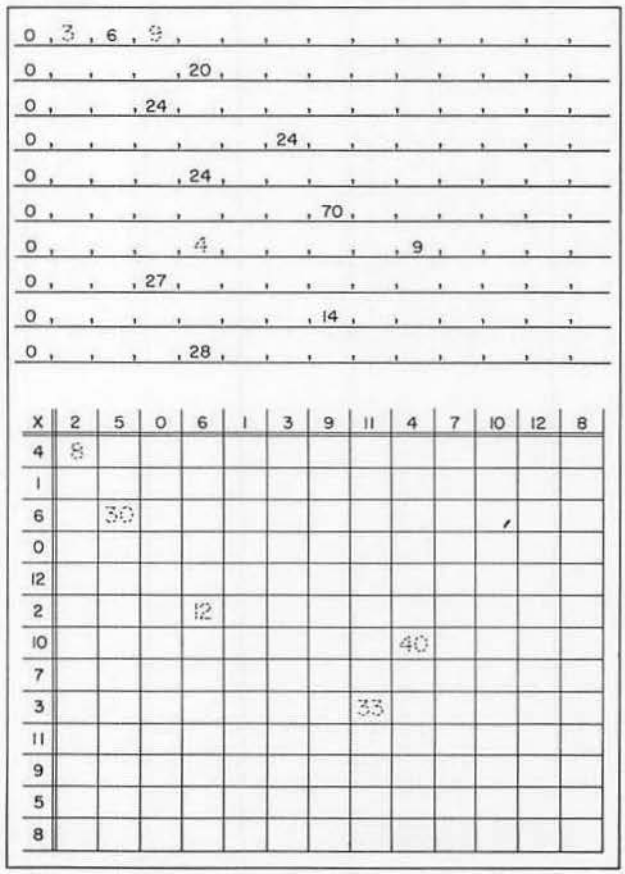
The approach just described for finding the content of cell *a* is sometimes called "the take-away approach" to subtraction. A second approach which reflects what is called "the additive approach" to subtraction starts with putting 1 black and 6 whites in cell *d* and then determining the content of *a*. We notice that the group of 1 black and 6 whites contains the 5 whites and 1 of the 4 blacks required for cell *g*. Move these 5 whites and 1 black from *d* to *g*. We must now put just enough in *a* so that when the content of *a* is combined with the white rod remaining in *d*, the result will be exchangeable for the additional 3 blacks we need in *g*. Clearly, this requires that 6 whites and 2 blacks be placed in *a*.

Exercise D is very interesting for there are many solutions. Let's start by filling cells *g* and *h* and then try to determine the content of cells *a* and *d*. At the moment, it appears that any partition of the 6 blacks and 3 whites in *g* will work. Let's put the 6 blacks in *a* and the 3 whites in *d*. Now, how shall we distribute the 5 blacks and 4 whites of *h* between cells *b* and *e*? Will any distribution do? No, we are constrained by the fact that the combined contents of *a* and *b* must be the equivalent of 6 blacks and 3 whites. We are forced to put the 3 whites from *h* into *b*. This leaves 5 blacks and 1 white for *e*. Everything is as it should be.

The arbitrariness of the choice of partition of the contents of *g* should suggest to pupils that they try another partition. We hope that they will notice that no matter what partition they use, the contents of *d* must be equivalent to that of *b*. Furthermore, the content of *b* cannot exceed the content of *h*; thus, we cannot put more than 5 blacks and 4 whites in *d*. (Notice the similar restraints in exercise H.)

Drill

Here is a suggestion for a drill sheet. Notice that in the exercise in the bottom half the factors are arranged in helter-skelter fashion. This forces the pupil into calling upon individual pieces of information rather than relying upon patterns in the table. However, we hope that some pupils will find ways of subverting our plan. For example, after filling in the 4-row, a pupil might list half of each product in the appropriate cell in the 2-row. Then he might add pairs of these products to find the entries for the 6-row. Etc.



Below is another suggestion for a drill sheet:

1. $20 \times 3 = (10 \times 2) \times 3$
   $= 10 \times (2 \times 3) = 10 \times 6 = \bigcirc$
2. $60 \times 2 = (10 \times 6) \times 2$
   $= 10 \times (6 \times 2) = 10 \times 12 = \bigcirc$
3. $70 \times 3 = \bigcirc$
4. $80 \times 2 = \bigcirc$
5. $60 \times 4 = \bigcirc$
6. $90 \times 5 = \bigcirc$
7. $60 \times 5 = \bigcirc$
8. $70 \times 9 = \bigcirc$
9. $30 \times 4 = \bigcirc$
10. $70 \times 5 = \bigcirc$
11. $6 \times 60 = \bigcirc$

Here are some stories for headline practice:

1. Mr. Jackson measured one of the school play areas. It was 40 feet long and 20 feet wide.
2. Jill leaves her house for school at 8:00 in the morning. She returns home at 4:00 in the afternoon.

   $4 + 4 = \square$ $\qquad$ $8{:}00 + \bigcirc = 4{:}00$
3. The edge of Dolly's desk is 22 inches. She has only a 6-inch ruler.

   $(6 \times 3) + 4 = 22$
4. Ole's father works for a department store. Ole's father told Ole that his store sold 105 radios, 62 TV's, and 40 washing machines during the annual sale.
5. Larry broke open his piggy bank. He counted 1000 pennies. He asked the teller at the savings bank to exchange his pennies for dollar bills.

I

min.	sec.	sec.
0	20	20
0	40	40
1	0	60
1	20	80
1	40	100
2	0	120
2	20	140
2	40	160
3	0	180
3	20	200
	<60	

II

$\frac{1}{4}$hr.	min.	min.
0	8	8
0	13	13
1	3	18
1	8	23
1	13	28
2	3	33
2	8	38
2	13	43
3	3	48
3	8	53
	<15	

III

$\frac{1}{2}$hr.	min.	min.
1	9	39
1	19	49
1	29	59
2	9	69
2	19	79
2	29	89
3	9	99
3	19	109
3	29	119
4	9	129
	<30	

IV

da.	hrs.	hrs.
0	9	9
0	17	17
1	1	25
1	9	33
1	17	41
2	1	49
2	9	57
2	17	65
3	1	73
3	9	81
	<24	

V

wk.	da.	da.
1	0	7
1	3	10
1	6	13
2	2	16
2	5	19
3	1	22
3	4	25
4	0	28
4	3	31
4	6	34
	<7	

VI

yr.	mo.	mo.
0	8	8
1	2	14
1	8	20
2	2	26
2	8	32
3	2	38
3	8	44
4	2	50
4	8	56
5	2	62
	<12	

102 Numeration exercises (bases 60, 15, 30, 24, 7, 12) in translating from base 10 to eachnondecimal base and conversely.

Tasks: *Pupils discover the rule for forming the sequence in each exercise and use this rule to fill the blanks in accordance with the special numeration system used in each exercise.*

Purpose: To review numeration systems and further perfect computational skills.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The exercises on this page amount to nothing more than discovering the patterns in sequences and expressing the terms of each sequence in two numeration systems. For example, in exercise II the list in the blue column contains the terms in decimal notation of an increasing sequence in which consecutive terms differ by 5. The list in the green column contains the same terms but the terms are expressed in a numeration system based on 15. To say "2 ten-groups and 3 one-groups" is to refer to the same number as to say "1 fifteen-group and 8 one-groups." In applying these numeration systems to the measurement of a quantity of time, we would say "23 minutes" or "1 quarter-hour and 8 more minutes."

Drill

Here is a suggestion for a drill sheet:

1. 24×3	2. 37×2	3. 15×6
4. 18×3	5. 26×3	6. 36×2
7. 40×2	8. 22×4	9. 25×3
10. 18×4	11. 36×3	12. 45×3
13. 53×5	14. 73×3	15. 85×6
16. 65×3	17. 63×5	18. 56×3

Here are some stories for headline practice:

1. Lana noticed that it was 9:00 A.M. She knew she would have to wait another 45 minutes until it was mid-morning snack time.

   $9{:}00 + 0{:}45 = \square$

2. Harold's desk is 8 feet from the door while Ray's is 3 yards from the door.

Headline for the Day: Fill in the blanks to complete the pattern.

700,685,670, , , , , , , ,550

I

10¢	5¢	1¢	cents
0	1	0	5
0	1	4	9
1	1	0	15
2	0	3	23
3	0	3	33
4	1	0	45
5	1	4	59
7	1	0	75
9	0	3	93
11	0	3	113
	<2	<5	

II

yd.	ft.	in.	in.
0	0	5	5
0	0	11	11
0	1	8	20
0	2	8	32
1	0	11	47
1	2	5	65
2	1	2	86
3	0	2	110
3	2	5	137
4	1	11	167
	<3	<12	

III

25¢	5¢	1¢	cents
0	1	4	9
0	3	4	19
1	1	4	34
2	0	4	54
3	0	4	79
4	1	4	109
5	3	4	144
7	1	4	184
9	0	4	229
11	0	4	279
	<5	<5	

IV

wk.	da.	da.
0	1	1
0	5	5
1	4	11
2	5	19
4	1	29
5	6	41
7	6	55
10	1	71
12	5	89
15	4	109
	<7	

V

gal.	qt.	$\frac{1}{2}$ pt.	$\frac{1}{2}$ pt.
0	0	1	1
0	0	3	3
0	1	2	6
0	2	2	10
0	3	3	15
1	1	1	21
1	3	0	28
2	1	0	36
2	3	1	45
3	1	3	55
	<4	<4	

VI

$1	10¢	1¢	cents
0	0	4	4
0	1	6	16
0	3	4	34
0	5	8	58
0	8	8	88
1	2	4	124
1	6	6	166
2	1	4	214
2	6	8	268
3	2	8	328
	<10	<10	

103 Numeration exercises (bases 5, 12, 7, 2, 10) in translating from base 10 to each nondecimal base and conversely.

Tasks: *Pupils discover the rule for forming the sequence in each exercise and use this rule to fill the blanks in accordance with the special numeration system used in each exercise.*

Purpose: To review numeration systems and to further perfect computational skills.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The many pages we have studied dealing with measurement systems have three important purposes. The first purpose is to help pupils understand the relationships among units of measurement in each of these systems.

A second purpose is to study patterns that are obviously parallel to those of the decimal system of numeration. Many of the exercises they have worked required the equivalent of carrying and borrowing, or exchanging. For most children these manipulations were deeply rooted in the objects we were talking about. When problems arose, they were solved in terms of what would happen in the world of everyday events. Thus we have seen the inner workings of many systems of numeration from a different viewpoint.

Exercise VI reveals most clearly the parallel we have between the important idea of the base ten numeration system and physical situations that we have been developing. The system is simply:

1 dime is the same as 10 pennies
1 dollar is the same as 10 dimes

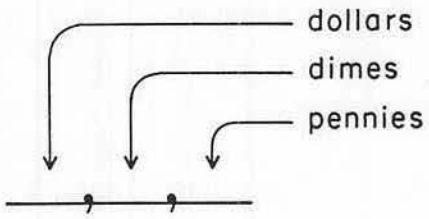
more

Headline for the Day: Fill in the blanks to complete the pattern.

900,884,868, , , , , , , ,740

Each of the larger units is worth 10 times the next smaller unit.

As we work this exercise, we have a strong feeling of familiarity. We've been here before.



We have still a third purpose: the study of systems of numeration with other bases. This topic can be omitted completely from classroom discussion. In fact, you can consider the discussion here as a footnote. Here are four examples of arithmetic each using a numeration system with a different base.

1, 4, 2 +2, 0, 6 4, 0, 3	*Quarters, nickels, and pennies* Each of the larger coins is worth 5 times the next smaller coin.
0, 1, 0, 1, 1 +0, 1, 1, 1, 0 1, 1, 0, 0, 1	*Gallons, half-gallons, quarts, pints, and half-pints* Each of the larger containers holds 2 times as much as the next smaller carton.
2, 3, 1 +1, 3, 4 4, 0, 5	*Cartons, packages, and cookies* Each of the larger containers holds 6 times as many cookies as the next smaller container.
3, 7, 8 +1, 5, 6 5, 3, 4	*Dollars, dimes, and pennies* Each of the larger coins is worth 10 times as much as the next smaller coin.

The shorthand could be simplified by indicating the base in each case and leaving out the commas. Then these four examples would look like this:

142_{FIVE} $+206_{\text{FIVE}}$ 403_{FIVE}	01011_{TWO} $+01110_{\text{TWO}}$ 11001_{TWO}
231_{SIX} $+134_{\text{SIX}}$ 405_{SIX}	378_{TEN} $+156_{\text{TEN}}$ 534_{TEN}

There has been much discussion of *systems of numeration with various bases* as an abstract bit of curriculum content. We have preferred to begin with consideration of the various systems of measurement we use in describing real-world events. In each there is a pattern, a regularity. In some, such as liquid measure, there is a very high degree of regularity, each unit being twice the next smaller unit:

1 gallon is 2 half-gallons.
1 half-gallon is 2 quarts.
1 quart is 2 pints.
1 pint is 2 half-pints.

In the exercises, the children have no rules of a formal kind to guide them. They must continually think of the "numeration systems" as a shorthand for reporting real-world events. They must reason their way to conclusions.

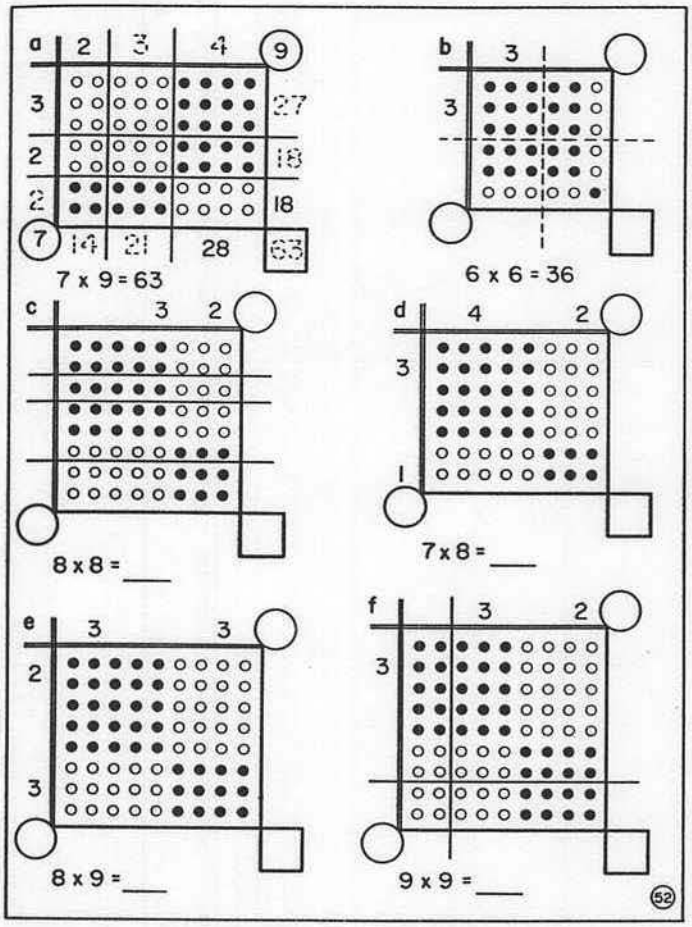
Yet, in the end they will have become aware of ideas that can, if you deem it advisable, be abstracted and extended into a more formal study of systems of numeration to various bases.

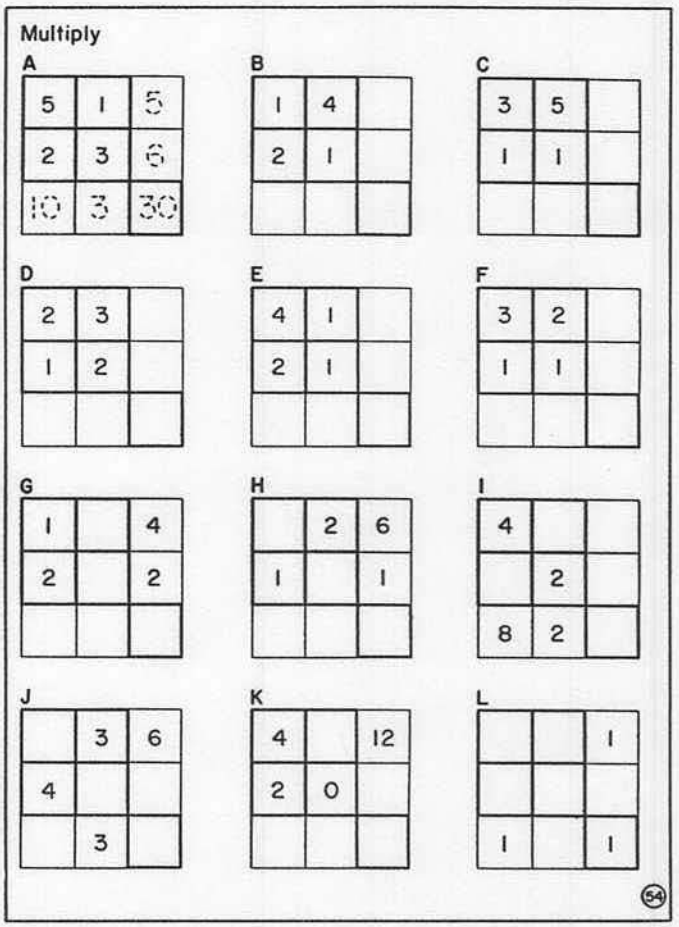
Drill

Here are some stories for headline practice:

1. Samantha's sister is 21 years old. She is 13 years older than Samantha.
2. Cliff said he had five more minutes to finish his homework before class. There are sixty seconds in one minute.
3. At the start of the school year Ronald bought a notebook to do his arithmetic problems in. The notebook contains 109 sheets of paper. By now he has used 86 of the sheets.
4. Tommy can walk to school in twenty-eight minutes. When his father takes him to school in the family car it takes them twelve minutes to make the trip.
5. Mort's father works at the Hometown Bank. He works from 9:00 A.M. to 3:00 P.M. on Monday, Tuesday, Thursday and Friday, and from 9:00 A.M. to noon on Wednesday.

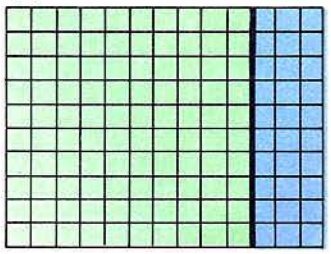
Consider using as drill sheets pages 52 and 54 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, shown on the following page.


a
2
3
4
9
3
27
2
18
2
18
7
14
21
28
63
7 x 9 = 63
b
3
1
3
6 x 6 = 36
c
3
2
8 x 8 = ___
d
4
2
3
1
7 x 8 = ___
e
3
3
2
3
8 x 9 = ___
f
3
2
3
9 x 9 = ___
52


Multiply
A
5 1 5
2 3 6
10 3 30
B
1 4
2 1
C
3 5
1 1
D
2 3
1 2
E
4 1
2 1
F
3 2
1 1
G
1 4
2 2
H
2 6
1 1
I
4
2
8 2
J
3 6
4
3
K
4 12
2 0
L
1
1 1
54

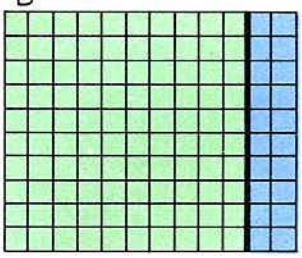
How Many Tiny Squares?

A



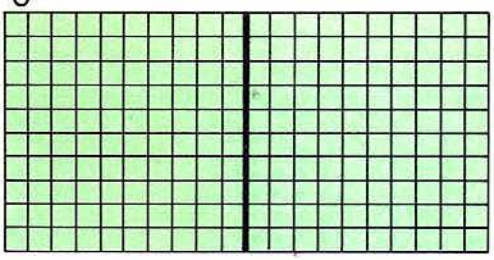
13 x 10 = 130

B



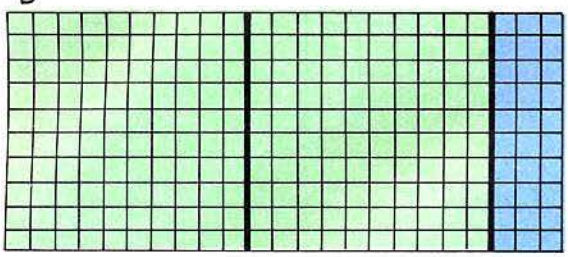
12 x 10 = 120

C



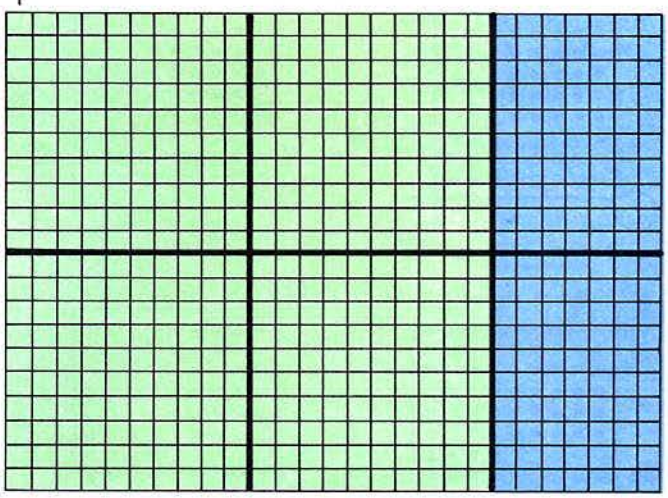
20 x 10 = 200

D



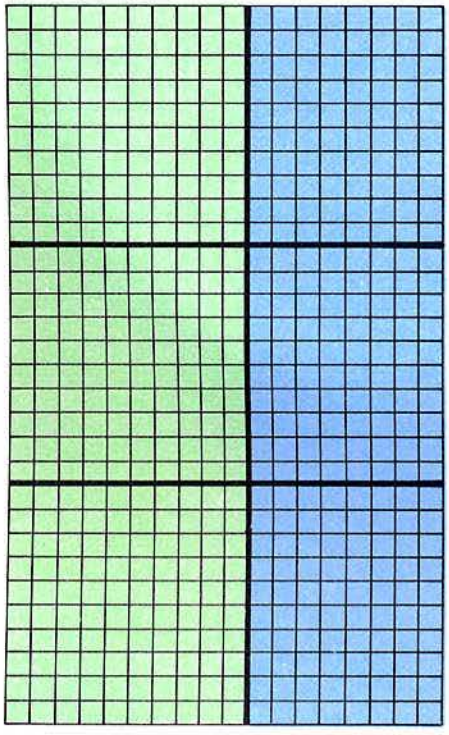
23 x 10 = 230

F



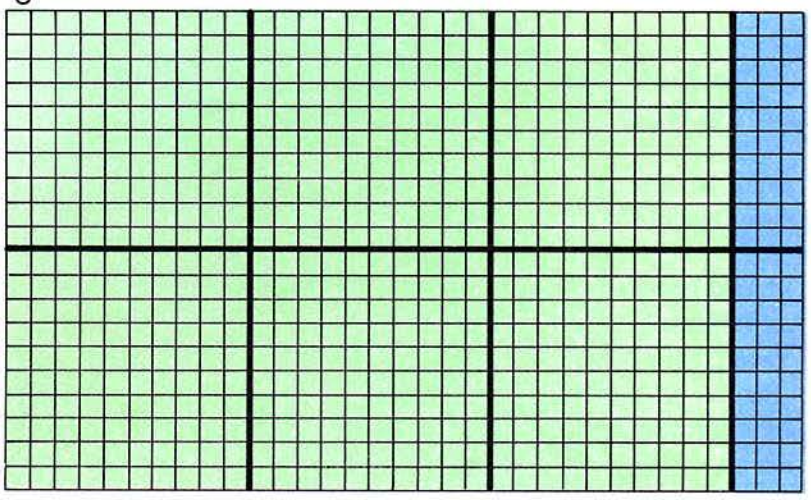
27 x 20 = 540

E



18 x 30 = 540

G



33 x 20 = 660

a.	b.	c.	d.	e.	f.	g.	h.
13	12	20	23	18	27	33	60
x10	x10	x10	x10	x30	x20	x20	x20
130	120	200	230	540	540	660	1200

104-105 Computation exercises dealing with multiplication problems in which one or both factors are multiples of 10.

Tasks: *Pupils use the representations A-G as guides to find the information needed to complete each number sentence (in both horizontal and vertical forms).*

Purpose: To review and extend multiplication concepts by exploring multiplication situations in which one of the factors is a multiple of 10.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

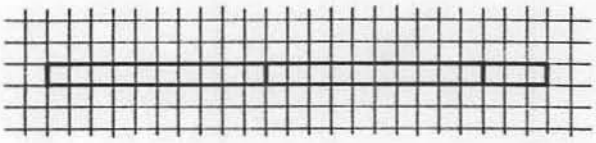
Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

Pupils should work many more exercises on graph paper like those on pupil page 104. The basic procedure reflects the distributive principle for multiplication over addition and the associative principle for multiplication. For example, let's use graph paper to find the product of 23 by 20.

First, we represent 23 as a strip:



The vertical partitioning lines show us that:

$$23 = 10 + 10 + 3$$

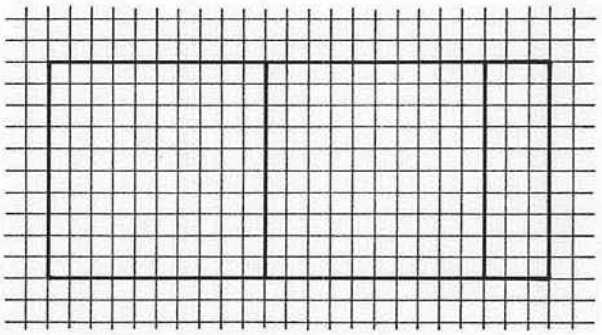
Now we notice that, since $20 = 10 \times 2$,

$$23 \times 20 = 23 \times (10 \times 2) = (23 \times 10) \times 2,$$

by associativity. So we next represent 23×10. We do so by taking 10 of the 23-strips and stacking them one under the other, as shown at the top of the following page.

more

Headline for the Day: Elroy made jumps of 4 on the number line while Paul made jumps of 5. They both started at 0 and landed on 100. What were the other points they both landed on?



The vertical partitioning lines show that:

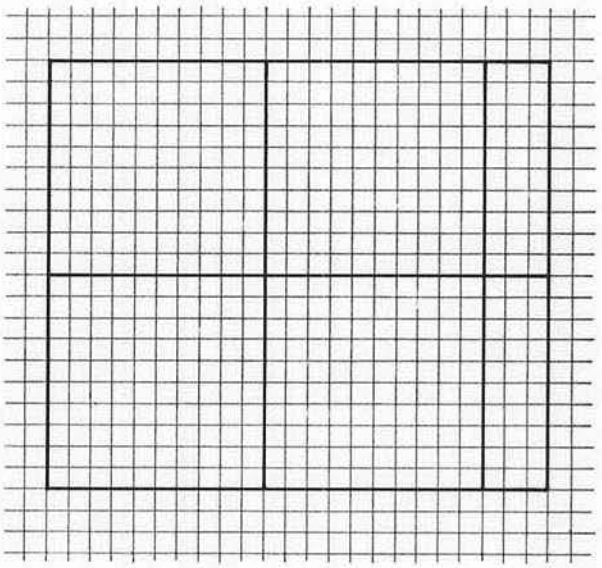
$$23 \times 10 = (10+10+3) \times 10$$
$$= (10 \times 10) + (10 \times 10) + (3 \times 10).$$

Our representation shows us that:

$$23 \times 10 = 200 + 30$$

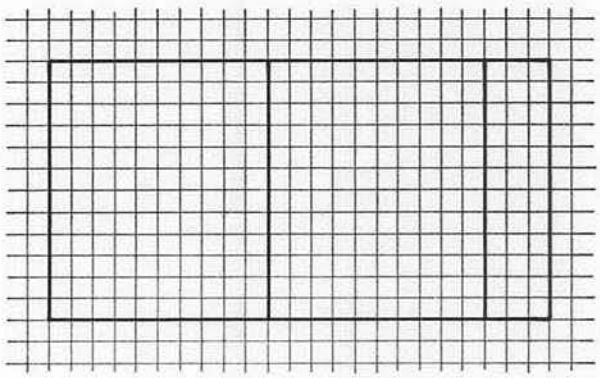
The next step is to represent (23 × 10) × 2. We do so by taking two strips like the wide one above and putting them one under the other:

$$(23 \times 10) \times 2 = (200+30) \times 2 = (200 \times 2) + (30 \times 2)$$

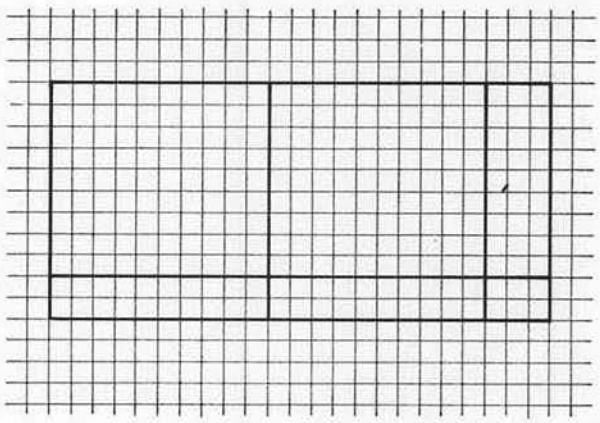


The graph paper approach to multiplying by 10 shows clearly what one means by the imprecise statement about "adding a zero." In the case of 23 × 10, we note that the digit 2 holds the tens place, and the digit 3 holds the ones place. That is, the notation 23 refers to 2 ten-groups and 3 one-groups. When we multiply by 10, we replace each one-group by a ten-group and each ten-group by a hundred-group. So the product of 23 by 10 is 2 hundred-groups and 3 ten-groups. To record this we use the same digits as we did in recording 23 but the digits hold different places than before. We need a 2 to hold the hundreds place and a 3 to hold the tens place. Naturally, we need a digit to hold the ones place. This digit must tell us that when the product has been partitioned into hundred-groups and ten groups, there are no one-groups left over. We use a 0 for this purpose.

It should be perfectly clear to pupils that multiplying by a nonzero multiple of 10 does not differ in substance from multiplying by any nonzero number. The process involves repeated addition in all cases. For example, to find 23 × 12, we could first represent 23 by a strip ahd then stack 12 such strips one above the other:



Now, to find the product we must count the little squares. It is at this point that multiplying by a multiple of 10 differs from multiplying by other numbers. In the former case, all we need do is count the large groups. But, in the case of multiplying by 12 we need to make a further partition:



Now we obtain the answer by counting the larger groups—there are 2 hundred-groups, 7 ten-groups, and 6 one-groups— 276 little squares altogether.

Drill

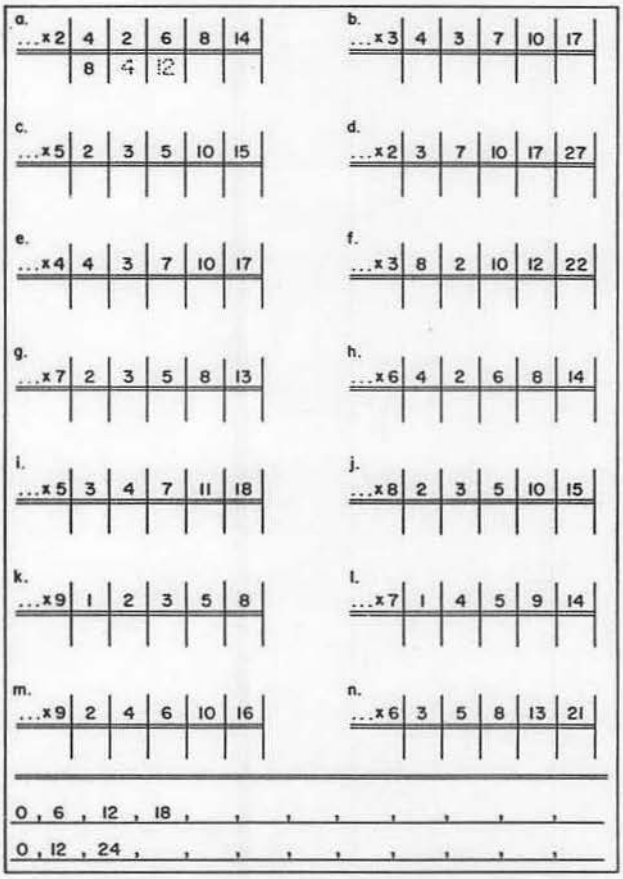
Here are some stories for headline practice:

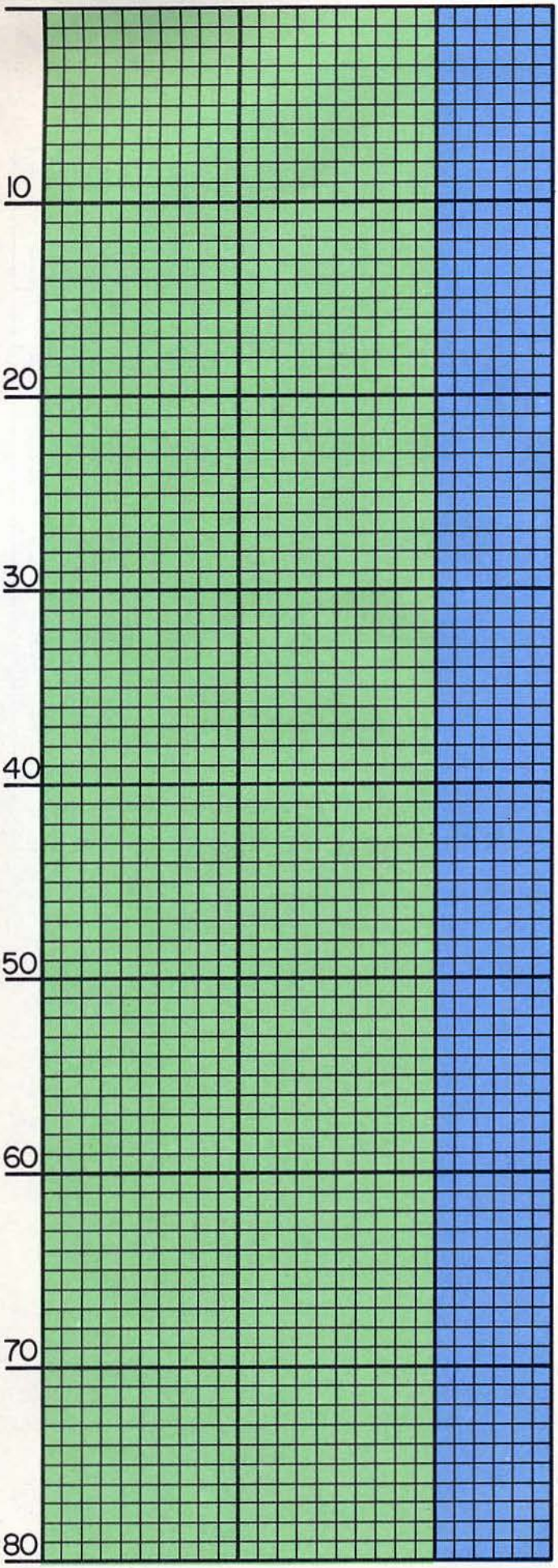
1. Gerald helped his dad clean the garage. They started at 10 o'clock. They finished the job 1½ hours later.

$$10:00 + 1:30 = 11:30$$

2. Petunia measured the window of her room. It was 7 feet high and 3 feet wide.
3. It rained at 10 o'clock. Three and a half hours later it started raining again.
4. Julia's mother bought a roll of 500 stamps for her employer. In one week's time Julia's mother had used 134 of the 500 stamps she purchased.
5. Wally's uncle used three 50-pound kegs of nails to build his house. Twenty nails weigs about 1 pound.
6. In June Southside Elementary School had 528 pupils. During the summer 49 children moved out of the school district.

A suggestion for a drill sheet is shown at the top of the following column. Note the opportunities to use the distributive principle for multiplication over addition.





	I	II	III
a.	10 × 10 = 100	10 × 20 = 200	10 × 30 = 300
b.	10 × 50 = 500	10 × 40 = 400	10 × 90 = 900
c.	10 × 60 = 600	10 × 70 = 700	10 × 80 = 800
d.	6 × 10 = 60	6 × 20 = 120	6 × 30 = 180
e.	6 × 50 = 300	6 × 40 = 240	6 × 90 = 540
f.	6 × 60 = 360	6 × 70 = 420	6 × 80 = 480
g.	20 × 10 = 200	20 × 20 = 400	20 × 30 = 600
h.	20 × 50 = 1000	20 × 40 = 800	20 × 90 = 1800
i.	26 × 10 = 260	26 × 20 = 520	26 × 30 = 780
j.	26 × 50 = 1300	26 × 40 = 1040	26 × 90 = 2340

Tasks: *Pupils use the grid on the left of pupil page 105, if necessary, to carry out the required computations.*

Purpose: To continue perfecting computational skills necessary for developing a multiplication algorithm.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

See Notes on Pupil Page 104 for teaching suggestions.

Note that the answers to the problems in rows *i* and *j* can be obtained from the answers to the problems in rows *d-h*.

Pupils should be aware of the fact that a problem such as finding the product of 32 by 20 can be solved either as on pupil page 104:

$$\begin{aligned} 32\times 20 = 32\times(10\times 2) &= (32\times 10)\times 2 \\ &= [(30+2)\times 10]\times 2 \\ &= (300+20)\times 2 \\ &= 600+40 = 640 \end{aligned}$$

or as follows:

$$\begin{aligned} 32\times 20 = 32\times(2\times 10) &= (32\times 2)\times 10 \\ &= [(30+2)\times 2]\times 10 \\ &= (60+4)\times 10 \\ &= 600+40 = 640 \end{aligned}$$

Drill

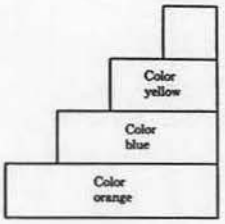
Consider using as a drill sheet page 59 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:

IF the white block is a picture suggesting "one,"

THEN yellow suggests "two,"
blue suggests "three,"
orange suggests "four," and
yellow plus blue suggests "five."

However, if white suggests "five,"
yellow suggests "ten," etc.

Fill in the chart below. In each example you are given a starting point.



	White	Yellow	Blue	Orange
A	1	2	3	4
B			3	
C		4		
D				4
E			6	
F	3			
G		6		
H				12
I		20		

IF:	3 W+Y	4 W+B	6 Y+O	5 B+Y
W suggests 1 W→1	3	4	6	5
B→3	3	4		
B→6				
W→3				
Y→4				
Y→10				
0→8				
Y→20				
B→9				

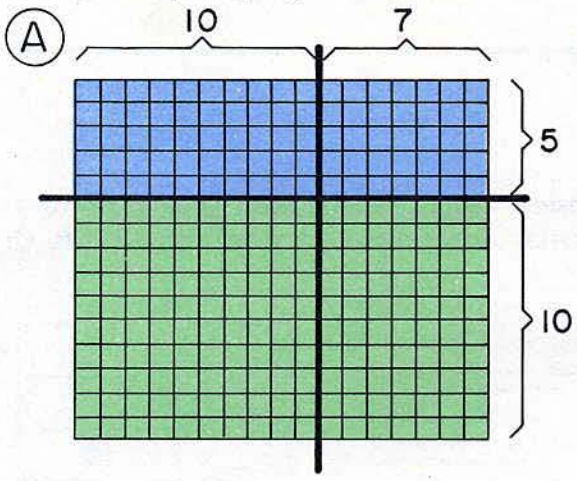
59

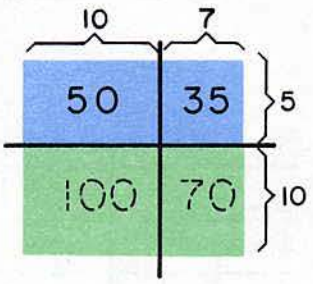
Here are some stories for headline practice:

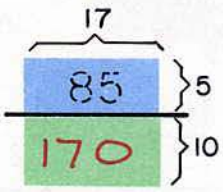
1. At 15 minutes to 7 Marcia's mother called the family in for supper. Supper took 40 minutes.
2. A stop sign is shaped like an octagon. Each of the 8 sides of this sign is 10 inches in length.

Headline for the Day: Joachima made jumps of 3 on the number line while Magda made jumps of 4. They both started at 0 and agreed to stop when either of them reached 100. Name the points they both landed on.

How Many Tiny Squares?

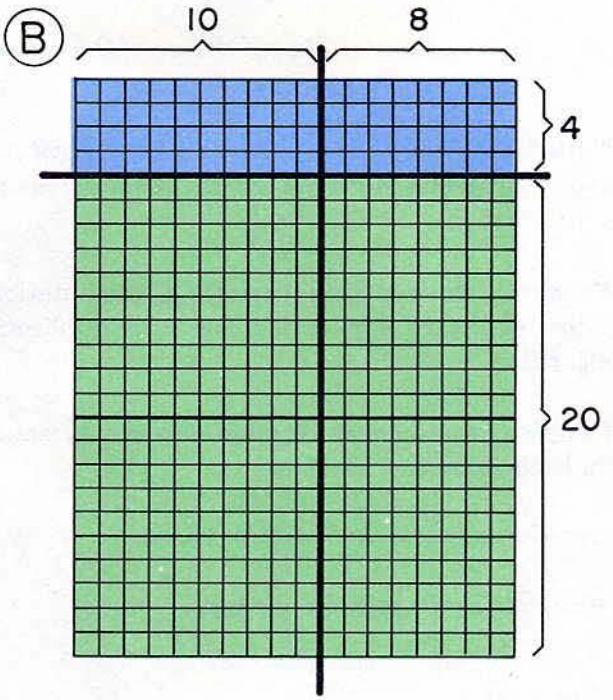


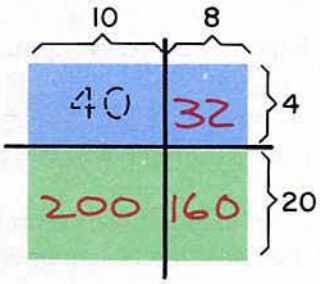


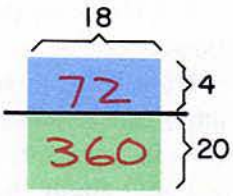


Total 255

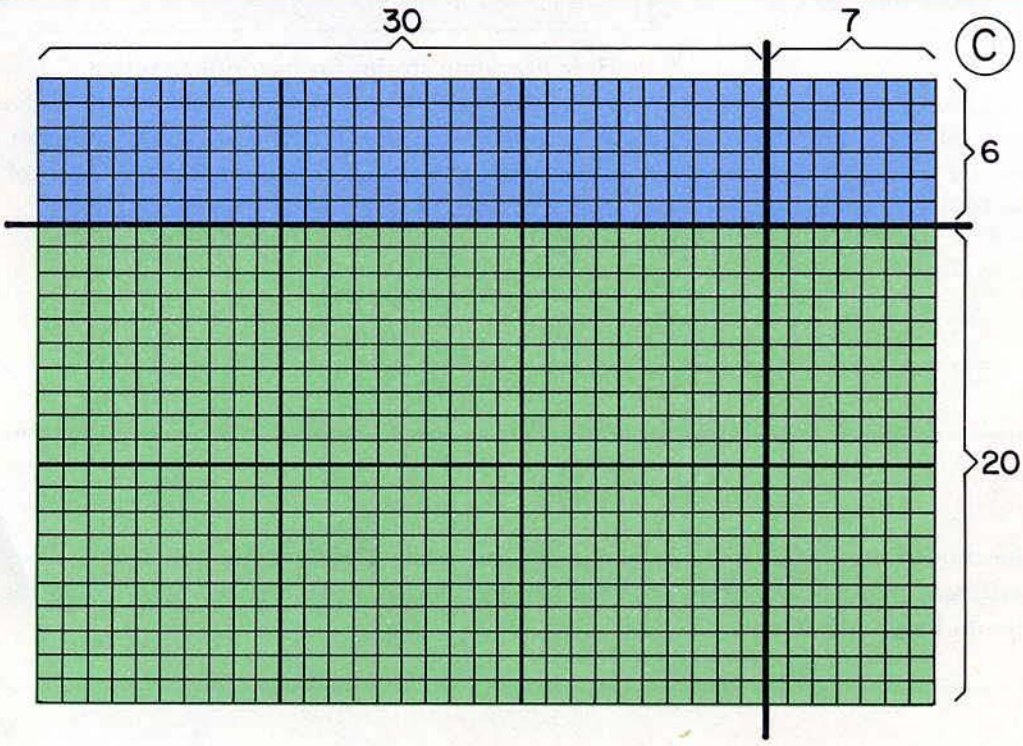
A	B	C
17 x 15 = 85 + 170 = 255	18 x 24 = 72 + 360 = 432	37 x 26 = 222 + 740 = 962

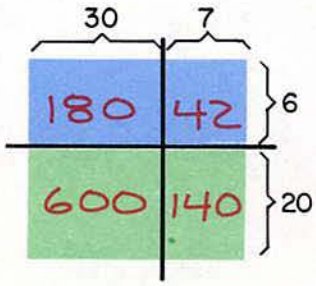


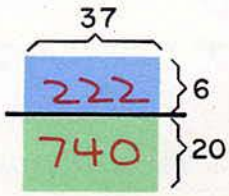




Total 432







Total 962

106-107 Computation exercises in multiplication designed to focus on the use of the distributive principle of multiplication over addition in situations in which neither of 2 factors is a multiple of 10.

Tasks: *For each of exercises A-C pupils record in the tinted regions below the representation the number of small squares in the parts of the representation. They record the totals in the blanks. Finally they complete the summaries at the left center of the page.*

Purpose: To explore in detail an algorithm for multiplication when both factors are 2-digit numbers.

Unifying Ideas: Addition and subtraction; multiplication and division.

Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

Here we have a full-fledged algorithm for long multiplication. The partitioning is illustrated in three ways: (1) literally, (2) in a semi-abstract form, and (3) in a still more abstract form.

A final form of the algorithm can now be interpreted in the light of these diagrams. Now pupils can see where every step comes from and can justify each step.

Cross Number Puzzles

See Notes on Pupil Page 94 for a discussion of the special *Cross Number Puzzle* format to be used with an algorithm for long multiplication:

X	50	2	52
3	150	6	156
40	2000	80	2080
43			2236

```
 52
x43
156
2080
2236
```

The terminal 0 in the 2080 can be omitted if you wish.

Drill

Here is a suggestion for a drill sheet:

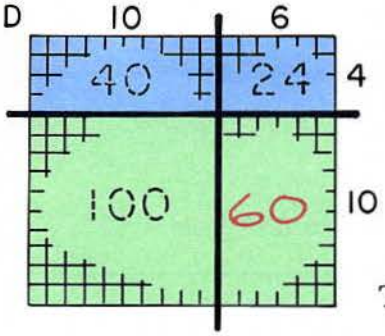
1. 85 × 3	2. 80 × 3	3. 86 × 7
4. 77 × 9	5. 56 × 7	6. 49 × 4
7. 38 × 3	8. 65 × 5	9. 68 × 8
10. 70 × 6	11. 53 × 5	12. 66 × 8
13. 82 × 7	14. 66 × 5	15. 43 × 6
16. 92 × 3	17. 61 × 8	18. 94 × 7
19. 42 × 4	20. 69 × 9	

Here are some stories for headline practice:

1. Barney noticed it was 4:50. His favorite TV show starts at 5:30.
2. Celia's dad built a 4-sided fence around their flower bed. The lengths of the sides are 20 feet 5 inches, 11 feet 7 inches, 20 feet 5 inches, and 11 feet 7 inches.
3. Ted worked for his neighbor Ralph. He began cutting Ralph's lawn at 10 minutes to 9. He stopped at 9:40.
4. It was 1:30 P.M. Joe was hungry. He had eaten lunch 1½ hours earlier.

Headline for the Day: Melita and Rose made some jumps on the number line starting at 0 and stopping at 90. All of Melita's jumps were the same size. So were all of Rose's. Rose made twice as many jumps as Melita. Make a table to record what you can say about this situation.

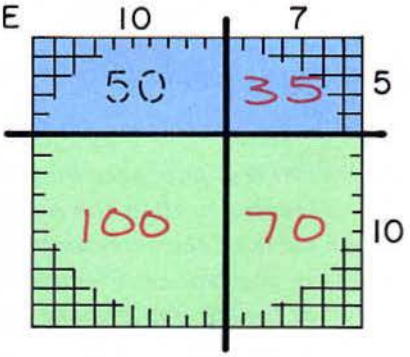
D



64
160

Total 224

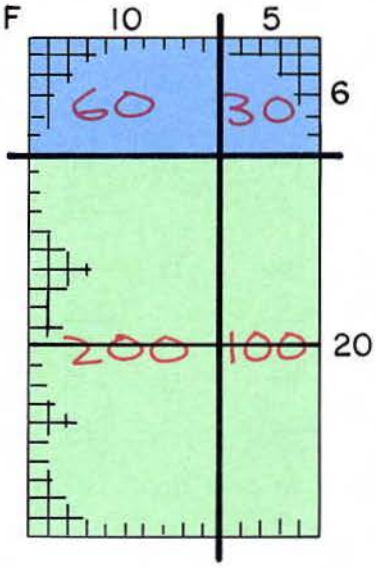
E



85
170

Total 255

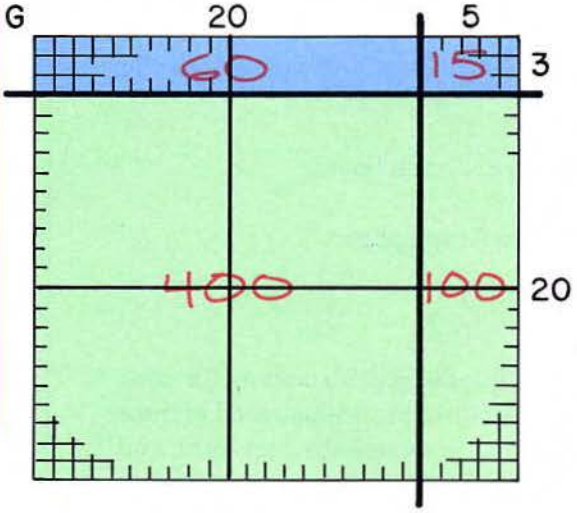
F



90
300

Total 390

G



75
500

Total 575

H

10	8	
40	32	4
100	80	10

72
180

18
x 14
72
180
252

I

20	5	
40	10	2
200	50	10

50
250

25
x 12
50
250
300

J

20	3	
60	9	3
400	60	20

69
460

23
x 23
69
460
529

K

30	2	
210	14	7
300	20	10

224
320

32
x 17
224
320
544

L

20	5	
40	10	2
600	150	30

50
750

25
x 32
50
750
800

M

50	7	
450	63	9
1500	210	30

513
1710

57
x 39
513
1710
2223

Notes on Pupil Page 107

Tasks: *Pupils record the number of blue or green squares in the four parts of the array, the total number of squares of each color, and the total number of squares in the array for each of the representations D-G. In each of exercises H-M pupils use the clues provided in exercise H to carry out the computations suggested by the representation.*

Purpose: To continue to explore an algorithm for multiplication when both factors are 2-digit numbers.

Unifying Ideas: Addition and subtraction; multiplication and division.

Materials for Teacher: Graph board; Ditto master.

Materials for Pupils: Graph paper; Ditto sheets as described below.

The Lesson:

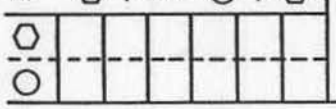
See Notes on Pupil Pages 105 and 106.

Enrichment Activity

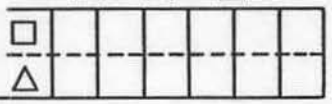
Put the following on a Ditto master:

Rule: When two or more frames of the same shape appear in a mathematical sentence, the same number is used to fill both.

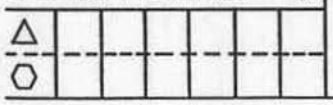
I. $\bigcirc + 2 = \bigcirc + \bigcirc$



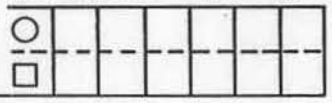
II. $\square \times \triangle = \triangle \times 5$



III. $\triangle + \triangle + \triangle = \bigcirc$

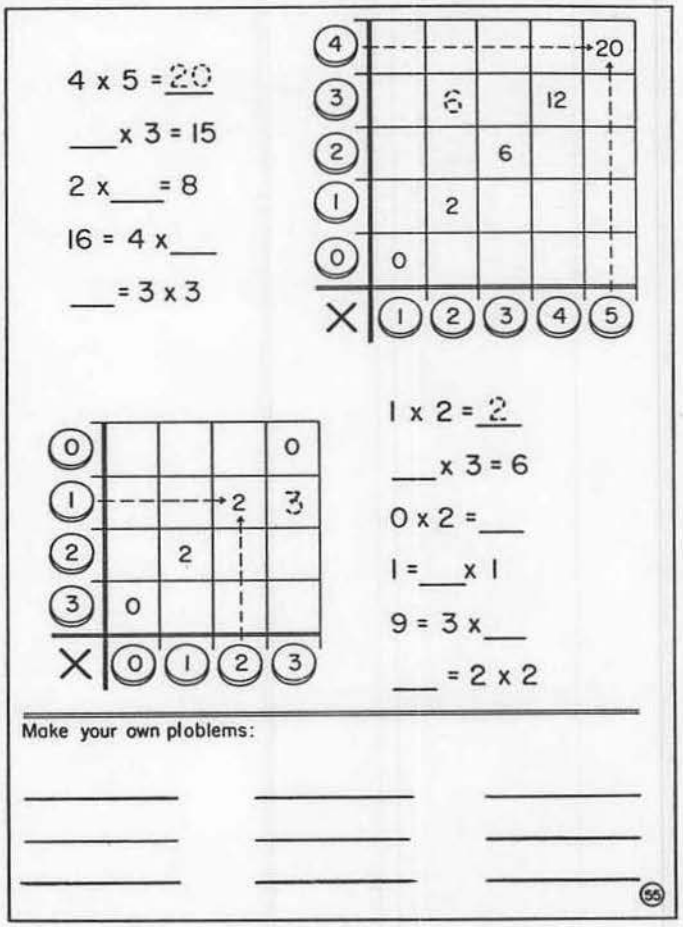


IV. $\bigcirc \times 3 = \square - \bigcirc$



Drill

Consider using as a drill sheet page 55 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Here are two stories for headline practice:

1. Frank's mother drove him to Jean's house. They left at 10:50 and arrived at Jean's house at 12:00.
2. Gertrude noticed it was 11:55 when Tod went home. He walked 20 minutes before he got to his house.

Headline for the Day: Kenne and Latimer made some jumps on the number line starting at 0. Kenne reached 200 in 20 jumps while Latimer reached 200 in 10 jumps. List all the points they both landed on.

Indicate the largest amount by writing an A in the blue block, the next largest by a B, the next largest by a C, etc. Compute the differences asked for in each exercise.

I

seven thousand eighty	B
seventy-eight hundred	A
seven thousand eight	C

difference between

A and B: 720

B and C: 72

A and C: 792

II

four thousand two hundred	A
four thousand two	C
four thousand one hundred two	B

difference between

A and B: 98

B and C: 100

A and C: 198

III

7654	A
4765	D
5476	C
6547	B

difference between

A and B: 1107

B and C: 1071

A and C: 2178

B and D: 1782

C and D: 711

A and D: 2889

IV

3210	A
321	D
1032	C
2103	B

difference between

A and B: 1107

B and C: 1071

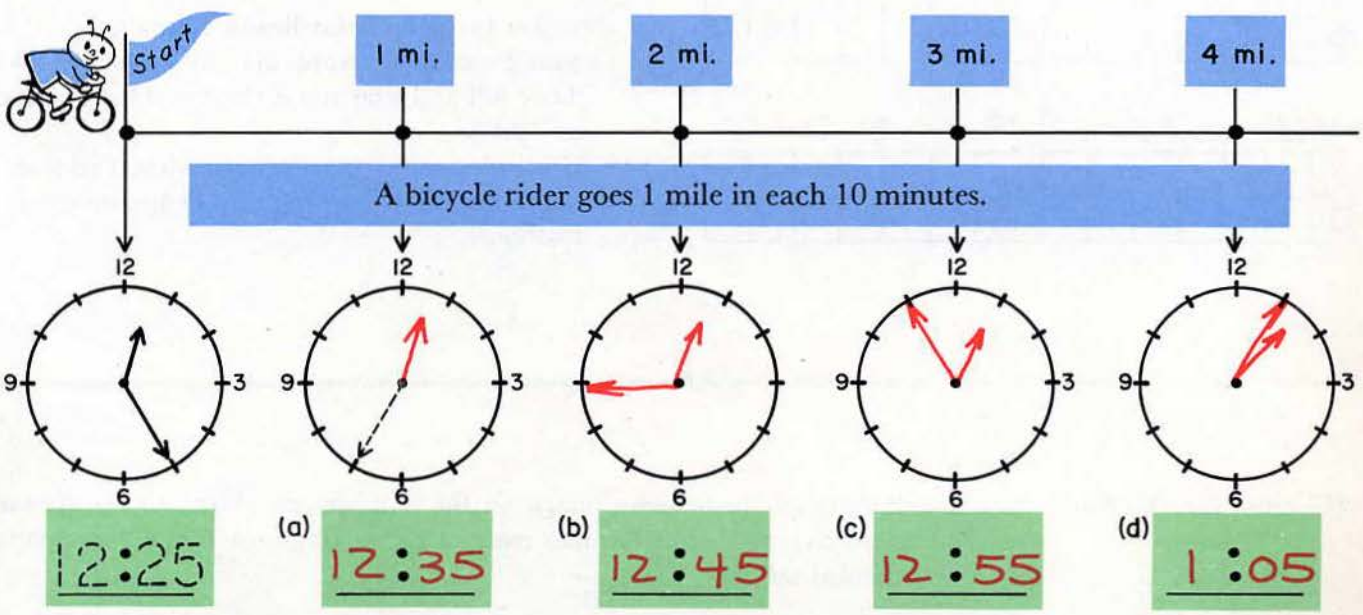
A and C: 2178

B and D: 1782

C and D: 711

A and D: 2889

V



12:25

(a) 12:35

(b) 12:45

(c) 12:55

(d) 1:05

(e) He goes 6 miles in 1 hour.

(f) He goes 15 miles in 2½ hours.

108 A variety of exercises in translating from English word notation to decimal notation, ranking magnitudes, and time-rate-distance problems.

Tasks: *Pupils complete the upper half of the page in accordance with the directions in the green region. For the lower half of the page pupils sketch in the hour and minute hands on each clockface to indicate the specified time at the milepost. They also record in the green boxes the times indicated on the clockfaces and record the required distances in the blanks at the bottom of the page.*

Purpose: To practice reading numbers expressed in word notation and in ordering numbers; to practice doing time-rate-distance problems.

Unifying Ideas: Numeration; addition and subtraction; functions and relations; measurement.

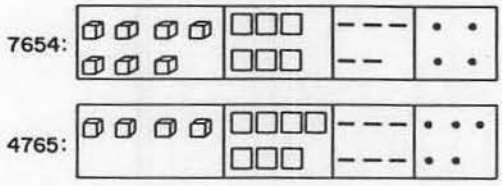
Materials for Teacher: None.

Materials for Pupils: None.

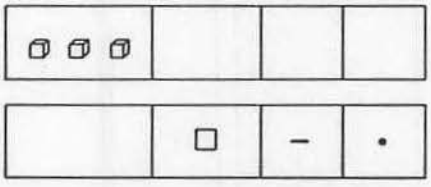
The Lesson:

Notice in exercise III that the numbers in question are represented by rearrangements of the same digits. This is also the case in exercises I and IV. As we illustrated in Notes on Pupil Page 16, differences between pairs of such 2-digit numbers are multiples of 9. This fact that the differences are multiples of 9 also holds for larger numbers. The same kind of physical manipulation we referred to in Notes on Pupil Page 16 will provide an explanation here.

Let's take as an example the numbers 7654 and 4765. The larger number is 7 thousand-groups, 6 hundred-groups, 5 ten-groups, and 4 one-groups—altogether, it is made up of 7 + 6 + 5 + 4 groups. The smaller number is made up of 4 + 7 + 6 + 5 groups, the same number of groups as the larger. At the top of the next column are representations of the numbers.



As a first step in subtraction, let's remove the groups common to both. This leaves us with the following:



Naturally, there are as many groups left in the representation of one of the numbers as there are in the representation of the other. The difference between the original numbers is the sum of the differences (1000 − 100), (1000 − 10), and (1000 − 1). Each of these differences is a multiple of 9. So, by the distributive principle for multiplication over addition, the sum is a multiple of 9. Here are the details:

$$\begin{aligned} &7654-4765 \\ &=(1000-100)+(1000-10)+(1000-1) \\ &=900+990+999 \\ &=(100\times 9)+(110\times 9)+(111\times 9) \\ &=(100+110+111)\times 9 \\ &=321\times 9=2889 \end{aligned}$$

Another point about exercises I-IV that is worthy of interest is that the relation among the differences is the same as that observed in chain reaction games (see Notes on Pupil Page 67). Consider exercise III:

A	−1107	B
	−2178	−1071
		C

A	−1107	B
−2178	−2889	−1782
C	−711	D

B	−1071	C
	−1782	−711
		D

Headline for the Day: Fill in the blanks to complete the pattern.

800,783,766, , , , , , , ,630

Draw a loop around the pair that would make the sentence true.

□ x 3 = 2 x ○

3, 3 | 7, 9
4, 6 | 1, 2

A △ x 5 = △ + □ − 7

20, 3 | 19, 8
9, 9 | 28, 35

B (3 x □) + 2 = (5 x ○) + 3

8, 5 | 9, 5
10, 6 | 7, 4

C 4 x △ = (□ x 7) + 3

6, 3 | 6, 4
5, 2 | 10, 6

D ○ x △ = 4 x 8

12, 12 | 16, 2
4, 12 | 9, 3

E △ + △ = □ x □

5, 4 | 2, 2
3, 2 | 1, 1

F ○ x 8 = (□ x 10) − 10

7, 7 | 4, 12
5, 5 | 8, 54

G (△ x 10) − 1 = 7 x ○

6, 8 | 6, 9
3, 4 | 5, 7

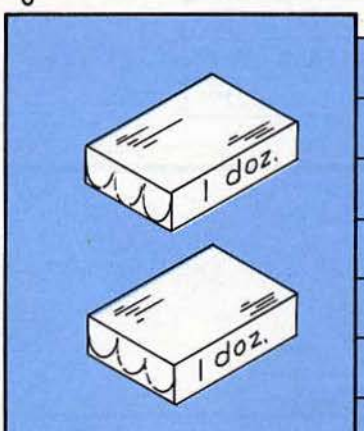
H (○ x 6) − 7 = (□ x 7) − 13

6, 6 | 3, 4
10, 9 | 5, 5

I

twice as much	28 ¢	4 times as much	$ .56
3 times as much	42 ¢	6 times as much	$ .84
5 times as much	70 ¢	10 times as much	$1.40
8 times as much	112 ¢	16 times as much	$2.24
13 times as much	182 ¢	26 times as much	$3.64
21 times as much	294 ¢	42 times as much	$5.88
half as much	7 ¢	68 times as much	$9.52

J



number of eggs	24	3 times as many	72
half as many	12	twice as many	48
one-fourth as many	6	5 times as many	120
one-third as many	8	10 times as many	240
a half-dozen more	30	15 times as many	360
a half-dozen less	18	25 times as many	600
a doz. and a half less	6	40 times as many	960

109 Equation solving - multiplication and division exercises.

Tasks: *In exercises A-H pupils use the directions and clues provided in the green field at the top of the page to complete each number sentence. In each of exercises I and J pupils determine the quantity represented and they record in the short boxes the quantities obtained by carrying out the operations.*

Purpose: To extend understanding of and skill with operations via the completion of number sentences and in applications.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Prepare pupils for the work in exercises A-H by playing some games. Write the open sentence:

$$\bigcirc + 3 = 2 + \square$$

on the chalkboard. Ask a pupil to give you a number for the circle. Write his number in the circle and then ask for a volunteer to give you a number for the square to make the sentence into a true one. Repeat for various choices of numbers for the circle.

Make a table of pairs of numbers and write an open sentence to go with it. For example:

□	○
8	5
10	6
9	5
7	4

$$(3 \times \square) + 2 = (5 \times \bigcirc) + 3$$

Check each pair in the table to see whether or not it satisfies the sentence.

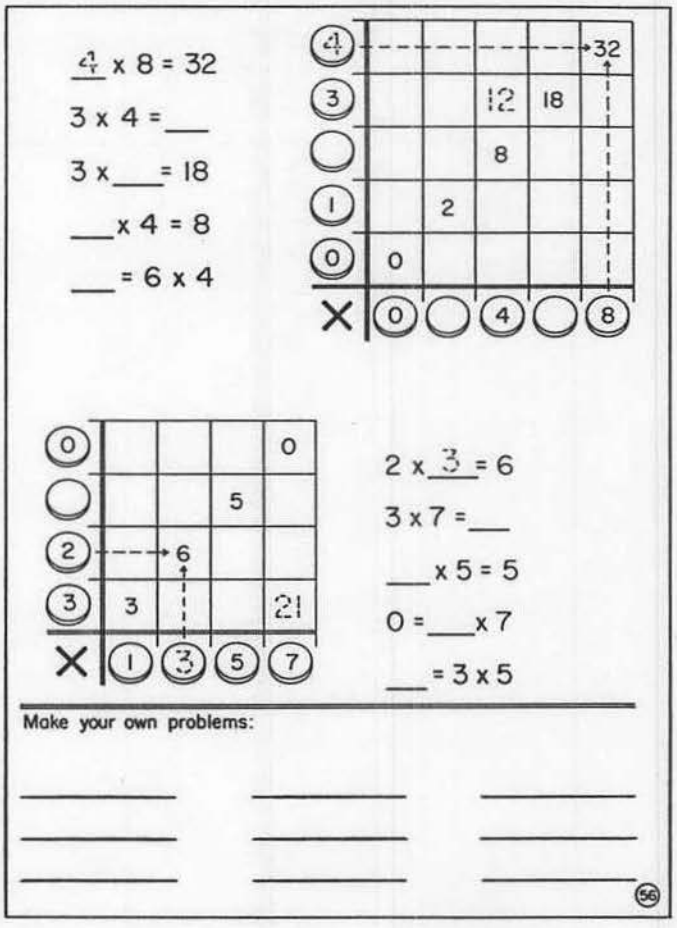
$$\underbrace{(3 \times \boxed{8}) + 2}_{24 + 2 \;\to\; 26} = \underbrace{(5 \times \textcircled{5}) + 3}_{25 + 3 \;\to\; 28}$$

No

Note the applications of the distributive principle for multiplication over addition in exercises I and J.

Drill

Consider using as a drill sheet page 56 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:

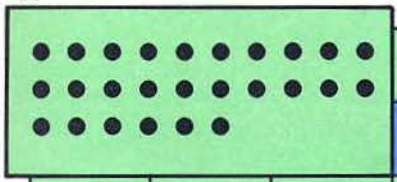


Here are some stories for headline practice:

1. The bus for Center City leaves at 8:32. It arrives at the city hall at 9:07.
2. Carmela had 102 buttons on some cards. Each card had 6 buttons on it.
3. Penelope's third-grade class planted 16 bean plants in 2 equal rows.

Headline for the Day: Kenley and Rima each bought a present. They each had more than $2.00 but less than $4.50 to spend. Kenley spent twice as much as Rima. Make a table to record what you can say about this situation.

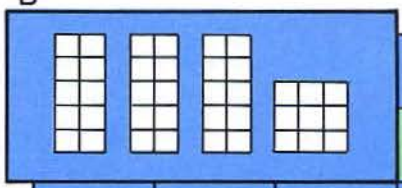
A



How many?	+ 10
25	36

+ 30	− 20	x 2	x 1	x 3
56	6	52	26	78

B



How many?	− 9
39	30

− 20	− 30	− 29	− 39	+ 40
19	9	10	0	79

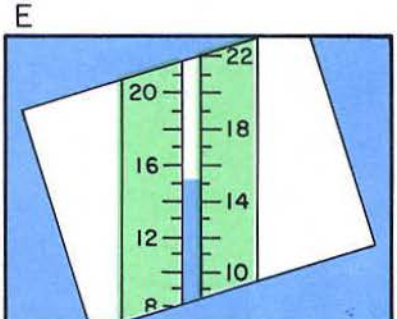
C

A B C D E F G H I J K L M N O P Q R S T U V W X Y				
	twice as many	50	4 times as many	100
	3 times as many	75	5 times as many	125

D

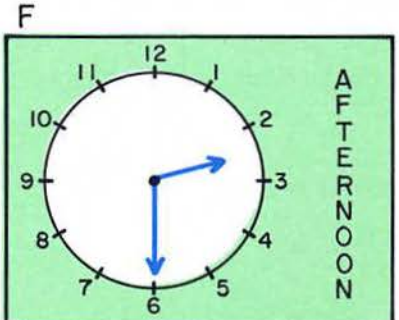
2 times as much	50¢	4 times as much	$1.00
3 times as much	75¢	6 times as much	$1.50
5 times as much	$1.25	10 times as much	$2.50

E



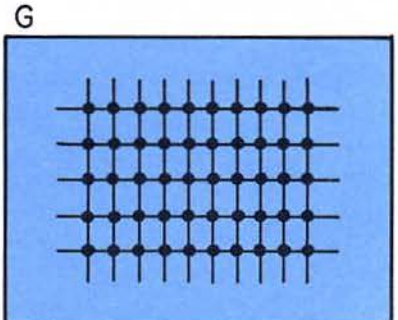
temperature shown
15°
7° colder
8°
7° warmer
22°
15° colder
0°

F



time shown
2:30 PM
20 minutes later
2:50 PM
20 minutes earlier
2:10 PM
2½ hours later
5:00 PM

G



number of crossing points
50
twice the number
100
half the number
25
10 times the number
500

110-112 A variety of exercises providing practice in computational skills.

Tasks: *Pupils determine the number or quantity represented in each exercise and carry out the specified computations.*

Purpose: To extend understanding of and skill with the arithmetic operations in applications.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets as described below.

The Lesson:

Pupils should be ready to work through this page completely on their own.

Enrichment Activity

Here are some *What's My Rule?* games. Exercises 3, 4, and 5 are for the front runners. Let pupils write possible rules at the bottom of the tables. We have given some rules for your convenience.

1.

M → N
3 → 1
10 → 8
7 → 5
2 → 0
9 →
→ 13
→ 4
5 →

N = M − 2

2.

X → Y
2 → 12
10 → 20
0 → 10
13 → 23
724 →
→ 386
299 →
→ 401

Y = X + 10

3.

A, B → C
6, 4 → 5
1, 3 → 2
10, 8 → 9
5, 7 → 6
13, 15 →
15, 13 →
78, 80 →
99, 101 →

(A + B) ÷ 2 = C

4.

P, Q → R
3, 2 → 10
1, 1 → 4
0, 0 → 0
2, 7 → 18
5, 6 →
6, 5 →
10, → 44
, 7 → 107

R = (P + Q) × 2

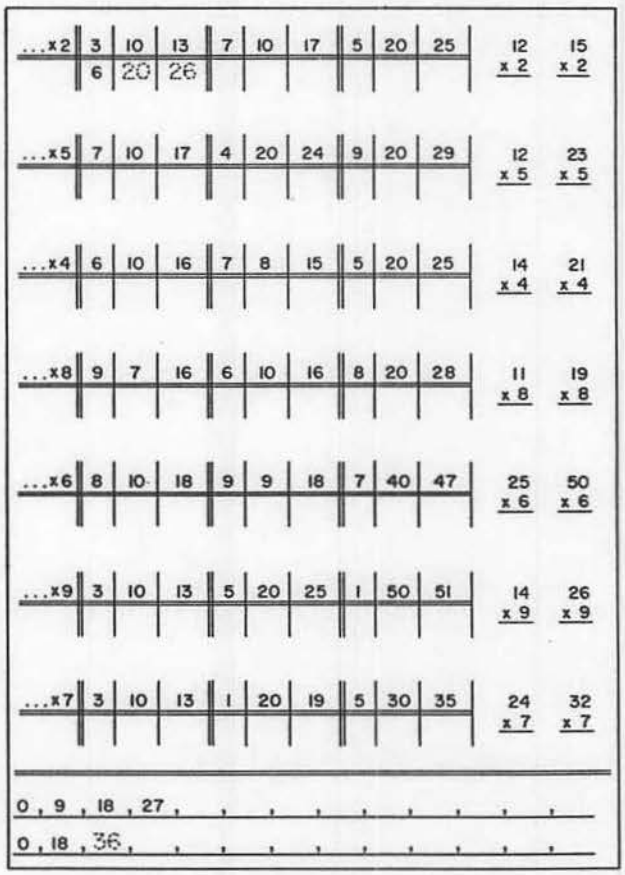
5.

F, L → S
2, 1 → 5
5, 3 → 13
4, 3 → 11
0, 4 → 4
7, 1 →
10, → 23

(F × 2) + L = S

Drill

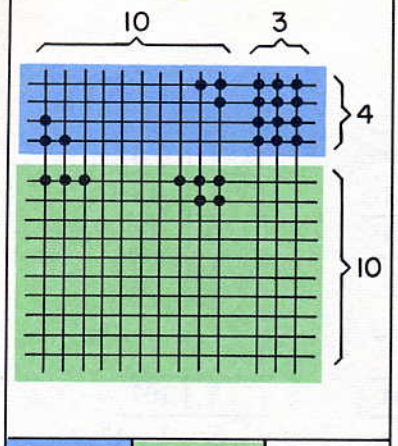
Here is a drill sheet on multiplication of a 2-digit number by a 1-digit number:



Headline for the Day: Tell a story to fit this pair of headlines: $\square \times \triangle = 100$
$\square < \triangle$

A

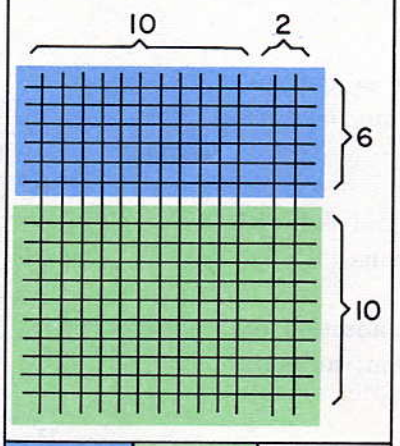
13 x 14 = 182



13 x 4	13 x 10	13 x 14
52	130	182

B

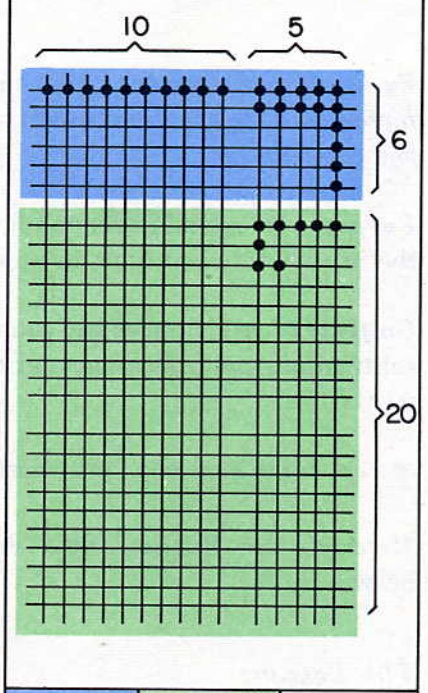
12 x 16 = 192



12 x 6	12 x 10	12 x 16
72	120	192

C

15 x 26 = 390

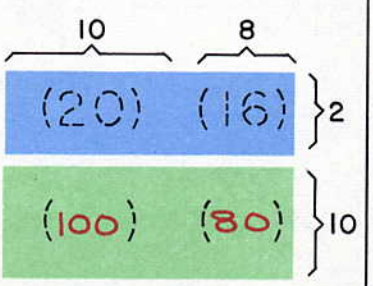


15 x 6	15 x 20	15 x 26
90	300	390

Sketches like these can become very large and hard to draw. Let's use diagrams that suggest large sketches.

D

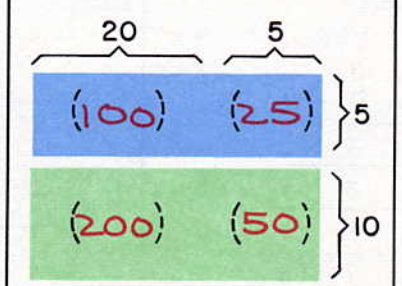
18 x 12 = 216



18 x 2	18 x 10	18 x 12
36	180	216

E

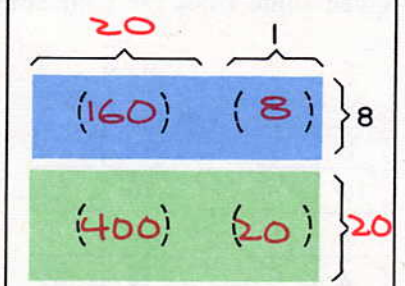
25 x 15 = 375



25 x 5	25 x 10	25 x 15
125	250	375

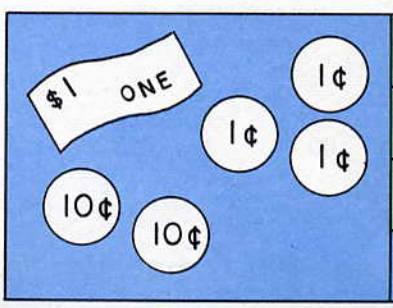
F

21 x 28 = 588



21 x 8	21 x 20	21 x 28
168	420	588

G



6 times as many dollar bills	$6.00	12 times as many dollar bills	$12.00
6 times as many dimes	$1.20	12 times as many dimes	$2.40
6 times as many pennies	$ .18	12 times as many pennies	$ .36
6 times as much as $1.23	$7.38	12 times as much as $1.23	$14.76

Tasks: *In each of exercises A-F pupils complete the number sentence by finding the product represented by computing the products indicated below the representation. In exercise G pupils determine the quantity represented and find the missing data in accordance with the computations specified.*

Purpose: To review and extend understanding of and skill with an algorithm for multiplication.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

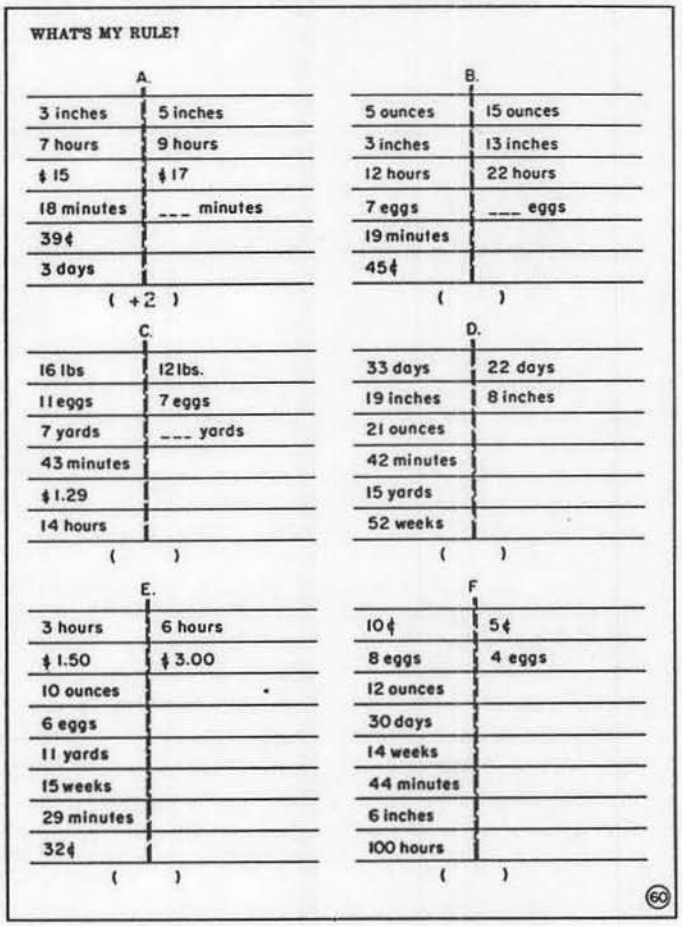
This page and pupil page 113 are designed to lead pupils away from the sketches and into a mechanical use of an algorithm for long multiplication. Note that we show only a few marks for the crossing points in exercises A and C and, of course, no crossing points in exercises D-F.

Drill

Here are some stories for headline practice:

1. Rhonda planted 4 rows of tulip bulbs. Each row had 15 bulbs.
2. There were 19 winners at the county fair. Each winner received 4 free tickets to ride on the ferris wheel.
3. Sam planted 10 pea plants. He was told that, on the average, a pea plant bears 10 pods and that a pod contains 10 peas.
4. Fran decided to figure the area of her 3 foot by 4 foot rug.

Consider using as a drill sheet page 60 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



WHAT'S MY RULE?

A.	
3 inches	5 inches
7 hours	9 hours
$15	$17
18 minutes	___ minutes
39¢	
3 days	
(+2)	

B.	
5 ounces	15 ounces
3 inches	13 inches
12 hours	22 hours
7 eggs	___ eggs
19 minutes	
45¢	
( )	

C.	
16 lbs	12 lbs.
11 eggs	7 eggs
7 yards	___ yards
43 minutes	
$1.29	
14 hours	
( )	

D.	
33 days	22 days
19 inches	8 inches
21 ounces	
42 minutes	
15 yards	
52 weeks	
( )	

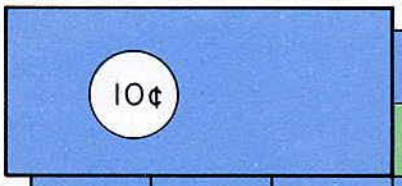
E.	
3 hours	6 hours
$1.50	$3.00
10 ounces	
6 eggs	
11 yards	
15 weeks	
29 minutes	
32¢	
( )	

F	
10¢	5¢
8 eggs	4 eggs
12 ounces	
30 days	
14 weeks	
44 minutes	
6 inches	
100 hours	
( )	

60

Headline for the Day: Tell a story to fit this pair of headlines: $\square \times \triangle = 30$
$\square + \triangle = 11$

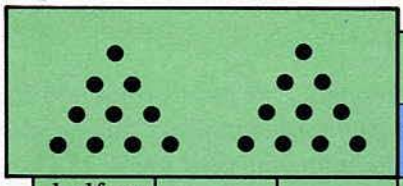
A



How much?	+ 5¢
10¢	15¢

+10¢	+20¢	− 5¢	−10¢	+25¢
20¢	30¢	5¢	0¢	35¢

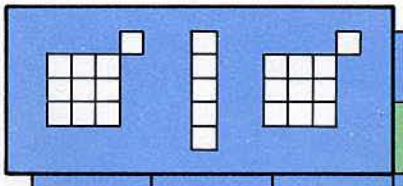
B



How many?	twice as many
20	40

half as many	+ 20	− 20	− 15	+ 18
10	40	0	5	38

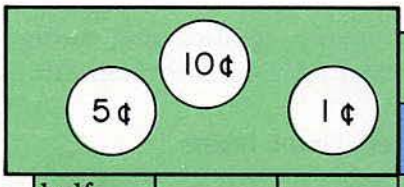
C



double	+ 25
50	50

− 23	+ 50	+65	− 8	twice as many
2	75	90	17	50

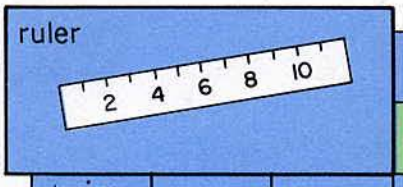
D



How much?	twice as much
16¢	32¢

half as much	+15¢	+50¢	−16¢	− 4¢
8¢	31¢	66¢	0¢	12¢

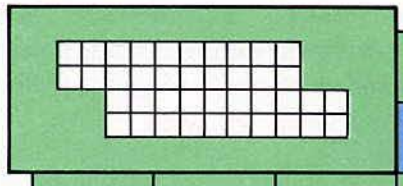
E



total length	half as long
12 in.	6 in.

twice as long	+ 6 in.	−11 in.	+12 in.	+ 1 ft.
24 in. or 2 ft.	18 in.	1 in.	24 in. or 2 ft.	24 in. or 2 ft.

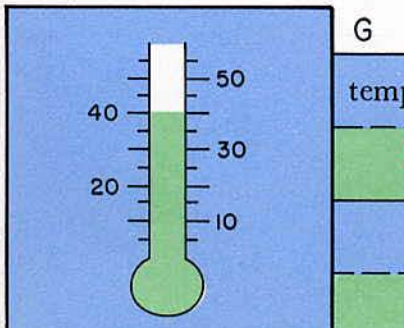
F



double	−20
80	20

+26	−15	−40	+60	half as many
66	25	0	100	20

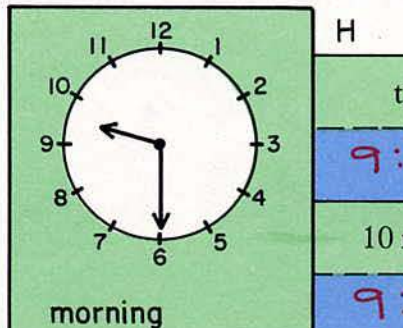
G



temperature shown
40°
up 32°
72°

down 32°	45° colder
8°	−5° or 5° below 0°
45° warmer	60° warmer
85°	100°

H



time shown
9:30 AM
10 minutes later
9:40 AM

10 minutes earlier	2 hours later
9:20 AM	11:30 AM
2 hours ago	2½ hours ago
7:30 AM	7:00 AM

Notes on Pupil Page 112

Tasks: *Pupils determine the number or quantity for each representation and carry out the computations specified.*

Purpose: To review understanding of and computational skill with arithmetic operations via applications.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

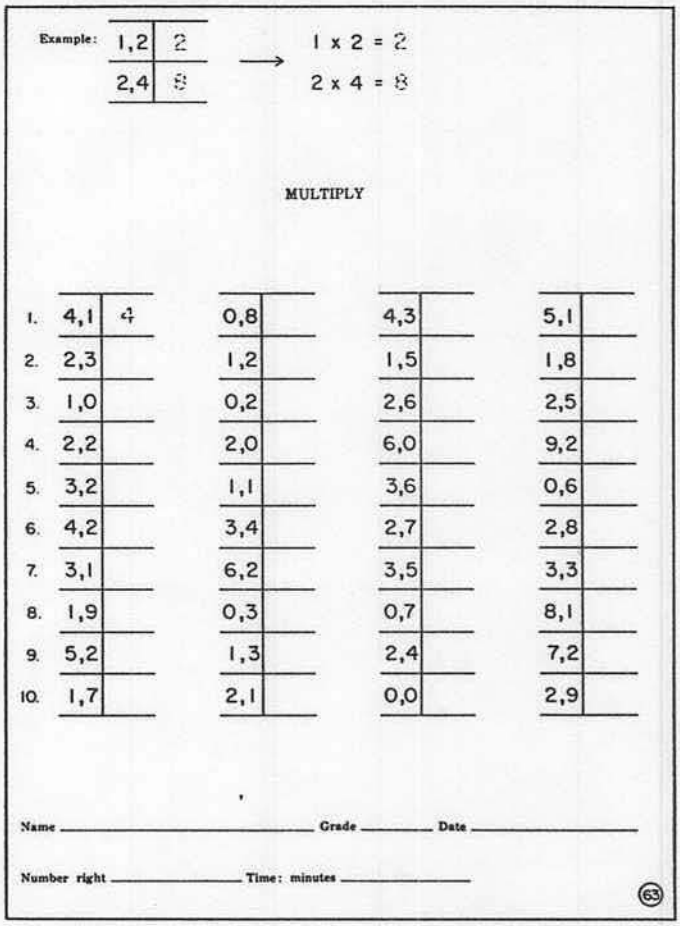
This is an easy review page. Use it as a test if you wish.

Drill

Here are some stories for headline practice:

1. It was 19 blocks from Gail's house to the soda parlor. She made 4 trips (2 trips to and from the soda parlor).
2. Jack and Jill visited the greenhouse. They saw 12 rose plants. Each plant had 8 flowers on it.
3. Mitch had 91 balloons. He gave each of his 13 friends the same number of balloons. When he finished he had none left for himself.
4. At 1:05 Rhonda's family finished eating lunch. At 1:56 she left for the playground.
5. Rodney had 45 pennies. He arranged them into 3 piles.
6. It takes 52 minutes by bus to get to Town Hall. Sid arrived at Town Hall at 10:27 A.M.

Consider using as a drill sheet or as a test page 63 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Example: 1,2 | 2 / 2,4 | 8 → 1 x 2 = 2, 2 x 4 = 8

MULTIPLY

1.	4,1	4	0,8		4,3		5,1	
2.	2,3		1,2		1,5		1,8	
3.	1,0		0,2		2,6		2,5	
4.	2,2		2,0		6,0		9,2	
5.	3,2		1,1		3,6		0,6	
6.	4,2		3,4		2,7		2,8	
7.	3,1		6,2		3,5		3,3	
8.	1,9		0,3		0,7		8,1	
9.	5,2		1,3		2,4		7,2	
10.	1,7		2,1		0,0		2,9	

Name _____ Grade _____ Date _____

Number right _____ Time: minutes _____

63

Headline for the Day: Eunice said the thermometer reading was twice as high as it was the same time yesterday. Make a table to record what you can say about this situation.

Sometimes we need ways to suggest very large sketches.
These sketches would include hundreds of crossing points.

A $132 \times 3 = 396$

100 (300) | 30 (90) | 2 (6) } 3

100 x 3	30 x 3	2 x 3	132 x 3
300	90	6	396

B $256 \times 4 = 1024$

200 (800) | 50 (200) | 6 (24) } 4

200 x 4	50 x 4	6 x 4	256 x 4
800	200	24	1024

C $317 \times 5 = 1585$

300 (1500) | 10 (50) | 7 (35) } 5

300 x 5	10 x 5	7 x 5	317 x 5
1500	50	35	1585

D $196 \times 7 = 1372$

100 (700) | 90 (630) | 6 (42) } 7

100 x 7	90 x 7	6 x 7	196 x 7
700	630	42	1372

If you need sketches for the following
examples, use a piece of scratch paper.

E $214 \times 9 = 1926$

200 x 9	10 x 9	4 x 9	214 x 9
1800	90	36	1926

F $305 \times 8 = 2440$

300 x 8	0 x 8	5 x 8	305 x 8
2400	0	40	2440

G $236 \times 6 = 1416$

200 x 6	30 x 6	6 x 6	236 x 6
1200	180	36	1416

H $204 \times 9 = 1836$

200 x 9	0 x 9	4 x 9	204 x 9
1800	0	36	1836

I

5 x	8	7	5	20	8	10	300	318	3	70	100	173
	40	35	25	100	40	50	1500	1590	15	350	500	865

113 Extending a multiplication algorithm to situations in which one factor is a three-digit number.

Tasks: *Pupils complete the number sentence at the top of each of exercises A-H by computing the products indicated in the boxes. In exercise I pupils do the computations required to complete the table.*

Purpose: To extend understanding of an algorithm for multiplication when one of the factors is a 3-digit number.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: Play money—ten-dollar bills, dollar bills, dimes, and pennies.

The Lesson:

Pupils will get insights into the algorithm developed on pupil page 113 if they work some of the problems using money representations (like that in exercise G on pupil page 111).

Let's consider exercise B as an example. Our problem is to find out how many cents there are in an amount that is 4 times as much as 256 cents. Our first step is to make a decimal partition of 256 cents. Such a partition is represented by 2 dollar bills, 5 dimes, and 6 pennies. Next, we multiply by 4. We do this by using the distributive principle for multiplication over addition and multiplying each component of the partition by 4. We then add the results.

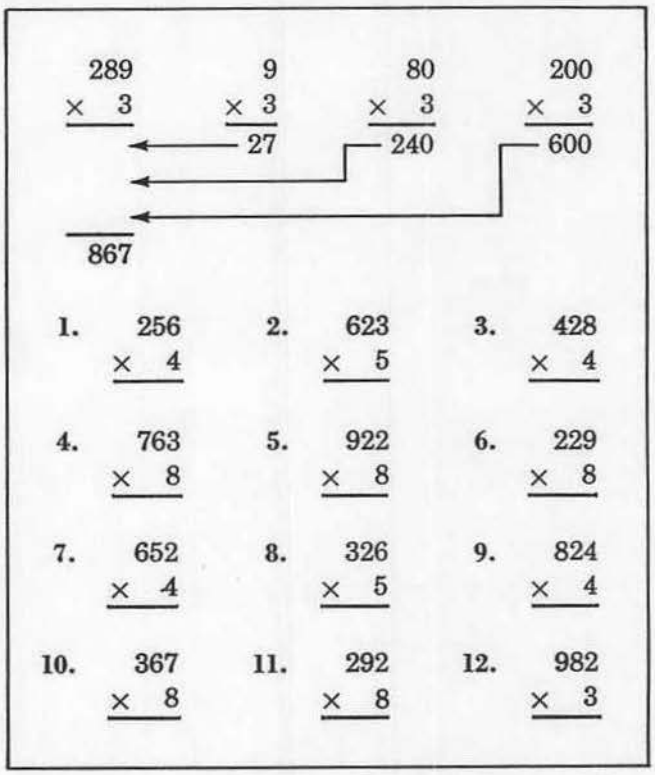
6 pennies	$\times 4 =$ 24 pennies	$=$ 2 dimes, 4 pennies
5 dimes	$\times 4 =$ 20 dimes	$=$ 2 dollar bills
2 dollar bills	$\times 4 =$	8 dollar bills

0 dollar bills, 2 dimes, 4 pennies
2 dollar bills, 0 dimes, 0 pennies
8 dollar bills, 0 dimes, 0 pennies
10 dollar bills, 2 dimes, 4 pennies
$=$ 1 ten-dollar bill, 0 dollar bills, 2 dimes, 4 pennies
$=$ 1024 cents

In exercise I note the opportunities to use the distributive principle for multiplication over addition.

Drill

Here is a suggestion for a drill sheet which deals with the work on page 113:



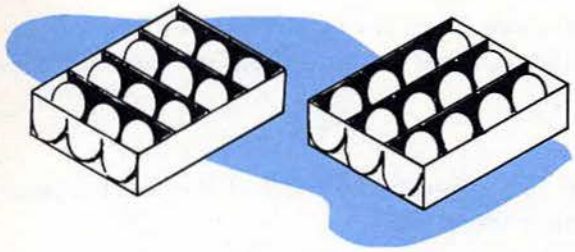
Here are some stories for headline practice:

1. Lena saw 9 people each buy a baker's dozen of donuts at the donut shop. (A baker's dozen is 13.)
2. Wanda and her father and mother planted 78 tulip bulbs. They planted the same number of bulbs in each of 6 window boxes.
3. Lizzy counted the sunflower seeds her mom planted. She counted 126 seeds. Her mom planted the seeds in 14 rows with the same number of seeds in each row.

Headline for the Day: Fill in the blanks to complete the pattern.

900,881,862, , , , , , , ,710

Here are two ways we can think about a dozen eggs:



On the left we see 4 groups of 3 eggs. On the right, we see 3 groups of 4 eggs.

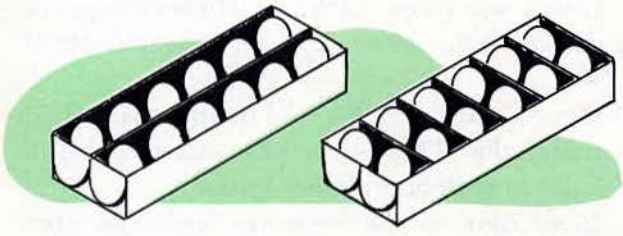
Or, we might say that 12 eggs can be divided into 4 groups of 3 eggs, or into 3 groups of 4 eggs.

This suggests the following mathematical sentences:

$3 \times 4 = 12$ $\quad 4 \times 3 = 12$

$12 \div 3 = 4$ $\quad 12 \div 4 = 3$

Here are two other ways of thinking about a dozen eggs:



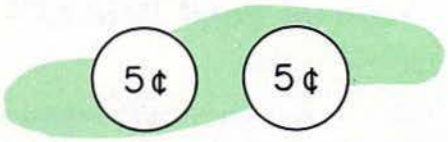
These ways of thinking about a dozen eggs suggest these mathematical sentences:

$2 \times 6 = 12$ $\quad 6 \times 2 = 12$

$12 \div 6 = 2$ $\quad 12 \div 2 = 6$

We might also notice that there are 24 eggs in two dozen eggs.

If we think about 10¢ as two nickels:

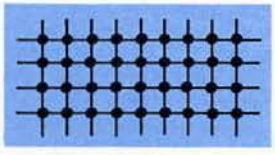


this might suggest the following mathematical sentences:

$2 \times 5 = 10$ $\quad 5 \times 2 = 10$

$10 \div 2 = 5$ $\quad 10 \div 5 = 2$

Thinking about lines and crossing points, this sketch:



suggests the following:

$4 \times 9 = 36$ $\quad 9 \times 4 = 36$

$36 \div 9 = 4$ $\quad 36 \div 4 = 9$

Another way of writing down this idea is:

$4\overline{)36}$ $\quad$ or: $\quad 9\overline{)36}$

These are usually read: "36 divided by 4 equals 9" and "36 divided by 9 equals 4." But you could also think of them as saying:

$4 \times 9 = 36$ and: $9 \times 4 = 36$

Thinking of lines and crossing points, the problem:

$4\overline{)20}$

asks the question: "If there are 20 crossing points on 4 horizontal lines, how many vertical lines must there be?" 5

114 A review of the relationships between multiplication and division and the conventional notation for division.

Tasks: *Pupils complete each number sentence by filling in the blanks with the missing factor or products as required.*

Purpose: To extend the notion of fact teams related to multiplication and division via applications.

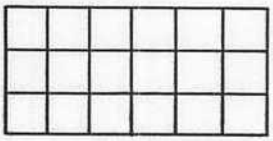
Unifying Ideas: Structure; number and counting; multiplication and division; measurement.

Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets (see below).

The Lesson:

This page contains a brief review which stresses the fact that multiplying by a number and dividing by that number are inverse operations. Each picture or situation that can be headlined by a multiplication statement can also be headlined by a division statement. Thus, as we have noted, such a picture as:



represents each of the following members of a fact team:

$3 \times 6 = 18$ $\quad$ $18 \div 6 = 3$

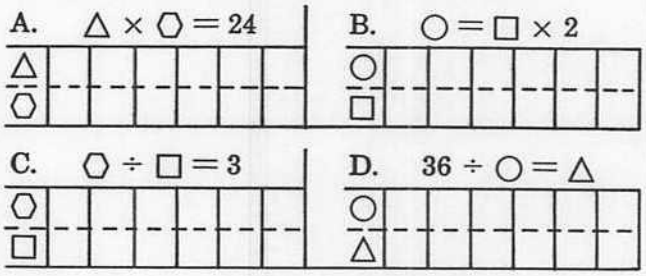
$6 \times 3 = 18$ $\quad$ $18 \div 3 = 6$

Division is not a big idea at all. All its glory is reflected from, and its importance ends with, the fact that it is inversely related to multiplication. If you handle it this way, many of the mysteries, as well as much of the discomfort children commonly feel about division, will never develop.

As one child slyly observed: Division problems are just sneaky ways of finding out how much we know about multiplication.

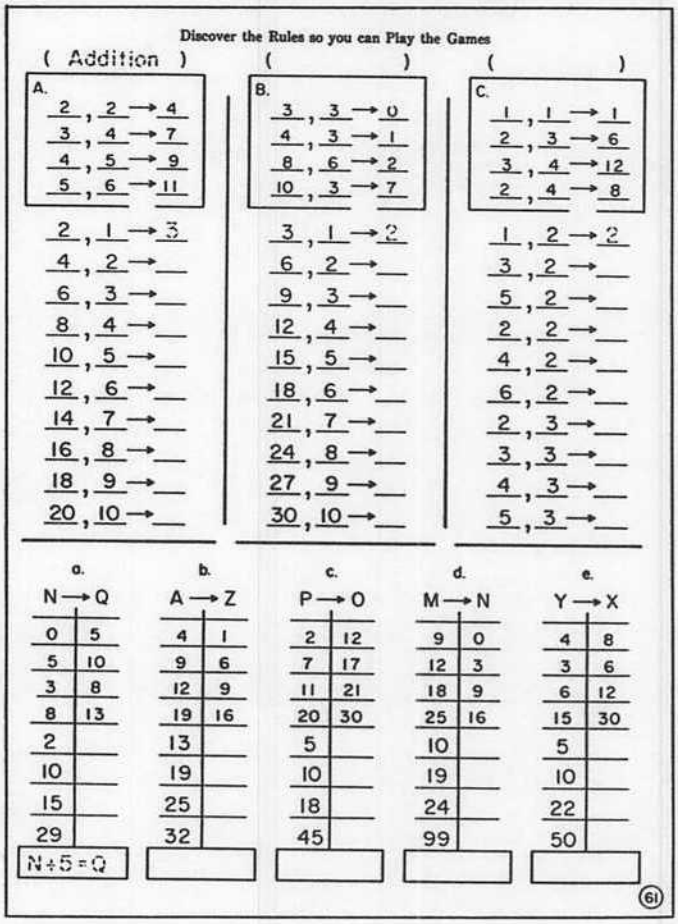
Open Sentences

Put these exercises on a Ditto master:



Drill

Consider using as a drill sheet page 61 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:

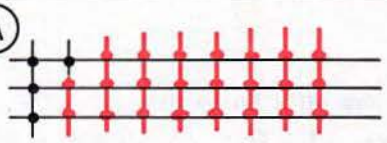


Headline for the Day: Tell a story for this fact team: $4 \times 5 = \square$ $\quad$ $20 \div 5 = \square$

$5 \times 4 = \square$ $\quad$ $4\overline{)20}$ $\square$

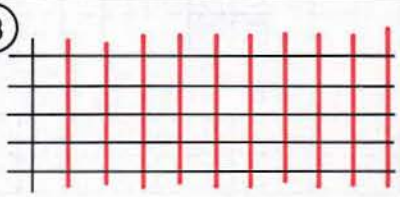
Please finish the following mathematical statements and the sketches they might suggest.

(A)



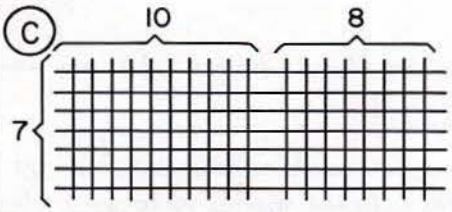
$3\overline{)27}$ $\quad 9$

(B)



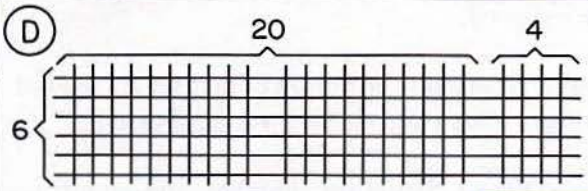
$5\overline{)55}$ $\quad 11$

(C)



$7\overline{)126}$ $\quad 18$

(D)

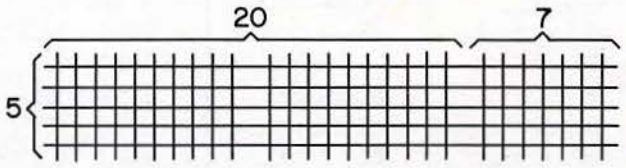


$6\overline{)144}$ $\quad 24$

We might describe sketches C and D more fully in this way:

(C) $7\overline{)70 + 56}$ $\quad 10 + 8$ and (D) $6\overline{)120 + 24}$ $\quad 20 + 4$

Please describe the following sketch in a similar way:



E $\quad 5\overline{)100 + 35} \quad 20 + 7$

Since the sketches take up much space and are hard to draw, let's imagine them. When in doubt, draw the sketch on a piece of scratch paper.

F $\quad 3\overline{)30 + 6} \quad 10 + 2$

G $\quad 5\overline{)100 + 25} \quad 20 + 5$

H $\quad 4\overline{)80 + 12} \quad 20 + 3$

I $\quad 2\overline{)60 + 12} \quad 30 + 6$

J $\quad 8\overline{)160 + 56} \quad 20 + 7$

K $\quad 7\overline{)280 + 42} \quad 40 + 6$

L $\quad 3\overline{)30 + 15} \quad 10 + 5$

M $\quad 9\overline{)90 + 45} \quad 10 + 5$

N $\quad 5\overline{)100 + 30} \quad 20 + 6$

O $\quad 7\overline{)140 + 63} \quad 20 + 9$

P $\quad 8\overline{)80 + 64} \quad 10 + 8$

Q $\quad 9\overline{)270 + 63} \quad 30 + 7$

R $\quad 4\overline{)160 + 32} \quad 40 + 8$

S $\quad 6\overline{)120 + 54} \quad 20 + 9$

T $\quad 12\overline{)120 + 36} \quad 10 + 3$

U $\quad 25\overline{)250 + 100} \quad 10 + 4$

V $\quad 15\overline{)300 + 60} \quad 20 + 4$

W $\quad 11\overline{)330 + 99} \quad 30 + 9$

X $\quad 20\overline{)600 + 140} \quad 30 + 7$

Y $\quad 13\overline{)130 + 0} \quad 10 + 0$

Z $\quad 12\overline{)240 + 36} \quad 20 + 3$

a $\quad 9\overline{)270 + 72} \quad 30 + 8$

115 Introducing a division algorithm based on the distributive principle of division over addition.

Tasks: *Pupils draw sketches where required or necessary to assist them in finding the missing numbers.*

Purpose: To extend understanding of algorithms for multiplication and to develop a division algorithm.

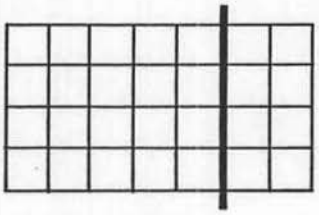
Unifying Ideas: Structure; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board; Ditto master.

Materials for Pupils: Graph paper; Ditto sheets as described below.

The Lesson:

Obviously, such a picture as:



suggests such expressions as:

4×7	$7\overline{)28}$
7×4	$4\overline{)28}$

It also suggests:

$4 \times (5+2)$	$(4 \times 5)+(4 \times 2)$
$(5+2) \times 4$	$(5 \times 4)+(2 \times 4)$
$7\overline{)20+8}$	$4\overline{)20+8}$

Visualized as multiplication:

we are given	*we can determine*
the number of columns and rows	the number of boxes
the number of horizontal and vertical lines	the number of crossing points

Visualized as division:

we are given	*we can determine*
the number of columns *or* rows *and* the number of boxes	the number of rows *or* columns
the number of horizontal *or* vertical lines *and* the number of crossing points	the number of horizontal *or* vertical rows

The reasoning required on this page parallels that on earlier pages when pupils were asked to complete such examples as:

$$\begin{array}{r} \times\ 4 \\ \hline 132 \end{array}$$

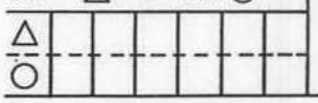
which is just another way of asking:

$$4\overline{)132}$$

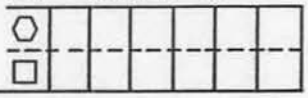
Open Sentences

Put these exercises on a Ditto master:

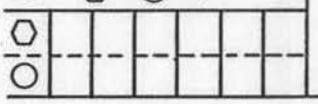
A. $\triangle \div 2 = 2 \times \bigcirc$



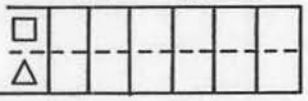
B. $\hexagon \times \square = 48$



C. $\hexagon \div \bigcirc = 7$



D. $\square \div 3 = \triangle \div 2$



Drill

Here are some stories for headline practice:

1. June bought 2 bags of popcorn for 15¢ each. She gave the vendor a one-dollar bill.
2. Naomi mounted 42 butterflies in 6 rows on a piece of cardboard. She put the same number of butterflies in each row.
3. Gina bought a hot dog for 20¢, an orange drink for 15¢, and an ice cream for 15¢.

Headline for the Day: Tell a story to fit this headline: $7\overline{)70+56}$

All About 144

Rule A: The blue-tinted blocks are reserved for whole numbers less than 10.

0, 1, 2, 3, 4, 5, 6, 7, 8 and 9

Rule B: The green-tinted blocks are for multiples of 10.

0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120 .... 190, 200, 210 ..... etc.

(You may wish to think of these examples as referring to sketches with 144 crossing points.)

A

	70 + 2
2	140 + 4

B

	40 + 8
3	120 + 24

C

	20 + 4
6	120 + 24

D

	140 + 4
1	140 + 4

E

	10 + 2
12	120 + 24

F

	0 + 2
72	0 + 144

G

	10 + 8
8	80 + 64

H

	0 + 6
24	0 + 144

I

	0 + 3
48	0 + 144

J

	30 + 6
4	120 + 24

K

	0 + 1
144	0 + 144

L

	0 + 4
36	0 + 144

M

	10 + 6
9	90 + 54

N

	0 + 9
16	0 + 144

All About 216

Follow the same rules — A and B — used in the column on the left.

O

	100 + 8
2	200 + 16

P

	50 + 4
4	200 + 16

Q

	30 + 6
6	180 + 36

R

	70 + 2
3	210 + 6

S

	20 + 4
9	180 + 36

T

	10 + 8
12	120 + 96

U

	10 + 2
18	180 + 36

V

	20 + 7
8	160 + 56

All About 324

Keep following rules A and B.

W

	80 + 1
4	320 + 4

X

	100 + 8
3	300 + 24

Y

	160 + 2
2	320 + 4

Z

	50 + 4
6	300 + 24

a

	20 + 7
12	240 + 84

b

	30 + 6
9	270 + 54

c

	10 + 8
18	180 + 144

d

	10 + 2
27	270 + 54

116 Using a division algorithm to explore ways of thinking and writing.about the numbers 144, 216, and 324.

Tasks: *Pupils follow rules A and B and use the clues provided in exercises A, B, and D to compute missing numbers.*

Purpose: To assist in developing an algorithm for division.

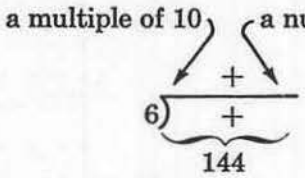
Unifying Ideas: Structure; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board; Ditto master.

Materials for Pupils: Graph paper; Ditto sheets as described below; Cuisenaire rods.

The Lesson:

How do we expect a pupil to attack a problem like those on pupil page 116? Let's try C as an example:



Our job is to partition 144 into a pair of numbers each of which is a multiple of 6 and one of which is the product of 6 by a multiple of 10. The latter must be the largest such number less than 144.

One way to do this problem is to arrange 144 blocks (or white Cuisenaire rods) in a rectangular array with 6 blocks in each column. Then count the columns. This is primitive and tedious.

Another way involves again representing 144 with Cuisenaire rods but using 14 orange rods and 1 purple rod. Partition the rods into 3 groups: 2 groups of 6 orange rods each and 1 group consisting of 2 orange rods and 1 purple rod. Exchange the 2 orange rods for 5 purple rods. This gives us the following rectangular array.

0	0	P
0	0	P
0	0	P
0	0	P
0	0	P
0	0	P

Still a third way to proceed is to try various multiples of 10 for the first addend of the quotient. First try 10. This gives us:

$$6\overline{)\ 60+84}\ \ 10+$$

Since $60 < 84$, the second addend of the quotient will be greater than 10. So try 20. This gives us:

$$6\overline{)\ 120+24}\ \ 20+$$

which is what we are looking for.

Open Sentences

Put the following rule and exercises on a Ditto master:

Rule: When two or more frames of the same shape appear in a mathematical sentence, the same number is used to fill both.

A. ⬡+⬡=○

⬡	3
○	6

B. □−△=□

□	3
△	0

C. △×△=○

△	2
○	4

D. □÷⬡=□

□	5
⬡	1

Drill

Here is a headline story:

1. Tod bought some red mittens. They cost 88¢. He gave the clerk one dollar.

Headline for the Day: Roland and Oliver each bought a present. Roland's present cost $3.56 while Oliver's cost $2.84. They each gave the clerk a five-dollar bill. The clerk gave the correct change to each boy, using the fewest coins and bills possible. Describe the change the clerk gave to each boy.

Work rapidly. KEEP LOOKING FOR TIME-SAVERS.

a.

$3\overline{)30}$ = 10 $\quad 3\overline{)15}$ = 5 $\quad 3\overline{)45}$ = 15

b.

$4\overline{)80}$ = 20 $\quad 4\overline{)16}$ = 4 $\quad 4\overline{)96}$ = 24

c.

$5\overline{)100}$ = 20 $\quad 5\overline{)75}$ = 15 $\quad 5\overline{)175}$ = 35

d.

$8\overline{)160}$ = 20 $\quad 8\overline{)32}$ = 4 $\quad 8\overline{)192}$ = 24

e.

$12\overline{)120}$ = 10 $\quad 12\overline{)60}$ = 5 $\quad 12\overline{)180}$ = 15

f.

$6\overline{)300}$ = 50 $\quad 6\overline{)54}$ = 9 $\quad 6\overline{)354}$ = 59

g.

$9\overline{)180}$ = 20 $\quad 9\overline{)63}$ = 7 $\quad 9\overline{)243}$ = 27

h.

$7\overline{)210}$ = 30 $\quad 7\overline{)42}$ = 6 $\quad 7\overline{)252}$ = 36

i.

$9\overline{)270}$ = 30 $\quad 9\overline{)72}$ = 8 $\quad 9\overline{)342}$ = 38

j.

$15\overline{)300}$ = 20 $\quad 15\overline{)105}$ = 7 $\quad 15\overline{)405}$ = 27

k.

$8\overline{)480}$ = 60 $\quad 8\overline{)56}$ = 7 $\quad 8\overline{)536}$ = 67

l.

$7\overline{)700}$ = 100 $\quad 7\overline{)70}$ = 10 $\quad 7\overline{)21}$ = 3 $\quad 7\overline{)791}$ = 113

m.

$5\overline{)1000}$ = 200 $\quad 5\overline{)150}$ = 30 $\quad 5\overline{)45}$ = 9 $\quad 5\overline{)1195}$ = 239

n.

$8\overline{)800}$ = 100 $\quad 8\overline{)240}$ = 30 $\quad 8\overline{)40}$ = 5 $\quad 8\overline{)1080}$ = 135

o.

$4\overline{)1200}$ = 300 $\quad 4\overline{)200}$ = 50 $\quad 4\overline{)36}$ = 9 $\quad 4\overline{)1436}$ = 359

p.

$9\overline{)1800}$ = 200 $\quad 9\overline{)630}$ = 70 $\quad 9\overline{)54}$ = 6 $\quad 9\overline{)2484}$ = 276

q.

$7\overline{)7000}$ = 1000 $\quad 7\overline{)350}$ = 50 $\quad 7\overline{)49}$ = 7 $\quad 7\overline{)7399}$ = 1057

117 Using the distributive principle of division over addition to extend a division algorithm.

Tasks: *Pupils compute missing numbers.*

Purpose: To perfect computational skills needed in an algorithm for division.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

Materials for Teacher: Graph board; Ditto masters.

Materials for Pupils: Cuisenaire rods; graph paper; Ditto sheets as described below.

The Lesson:

As you will notice, the division problems are clustered in groups of three or four such that the quotients for the first two or three problems lead to the quotient for the last. For example, in exercise *k*, since $536 = 480 + 56$ it follows that $536 \div 8 = (480 \div 8) + (56 \div 8) = 60 + 7 = 67$. The fact that $536 \div 8 = (480 \div 8) + (56 \div 8)$ follows from the distributive principle for multiplication over addition and the fact that multiplying by 8 and dividing by 8 are inverses of each other. Here's how:

$$\begin{aligned}&[(480 \div 8) + (56 \div 8)] \times 8\\ &= [(480 \div 8) \times 8] + [(56 \div 8) \times 8]\\ &= \quad 480 \quad + \quad 56 \quad = 536\end{aligned}$$

Therefore,

$$[(480 \div 8) + (56 \div 8)] \times 8 = 536.$$

So,

$$\begin{aligned}536 \div 8 &= \{[(480 \div 8) + (56 \div 8)] \times 8\} \div 8\\ &= (480 \div 8) + (56 \div 8).\end{aligned}$$

Division Algorithms

It took many fine minds centuries to devise a simple shortcut for finding such quotients as $37250 \div 215$. Slowly, painfully, step by step, a means was evolved that, today, is part of the mathematical heritage we pass on to children in the elementary schools.

Teachers are sometimes so anxious to reveal this important technique with its beautiful simplicity that they hurry into presenting the "standard" division algorithm to pupils as a series of steps to be memorized. The price exacted is a lack of understanding that pupils may never overcome.

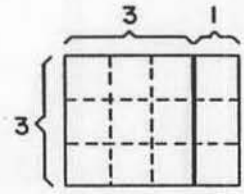
Take the time to play a series of division games which focus the spotlight on meaning. The following games have a touch of the puzzle, and they provide useful drill. These games assume that all pupils would understand the following statements:

$3\overline{)\,12} \;=\; 3\overline{)\,9+3} \;= 3+1=4$

$12 \div 3 = (9+3) \div 3 = (9 \div 3) + (3 \div 3) = 3+1=4$

Is that a big assumption? If so, better stop right there. If the pupils do not understand these statements, they have no basis upon which to build an understanding of the division algorithm.

Discuss problems like the following: I have to divide 12 pieces of fruit among 3 children. If I have 9 apples and 3 pears, what would be a good way to do it? Each child, of course, gets 3 apples and 1 pear; 4 pieces of fruit in all. Now the principle must be brought out. How is it possible to break a group of 12 pieces of fruit into two groups? How does the picture below suggest what was done?



If there is any uneasiness about the example above, don't hurry. Try dozens like it. Make up stories. Draw pictures. Go back to manipulative materials.

Pupils must have a good understanding of addition

Headline for the Day: Tell a story to fit this fact team: $20 \times \square = 200$ $200 \div \square = 20$ $\square \times 20 = 200$ $\square\overline{)\,200}$ 20

---

and multiplication along with subtraction and division, because in performing division they will frequently have to use all four operations. They will multiply and subtract, sometimes over and over again, and finally do a bit of addition to get a quotient!

Each of the division games that follows contains but a few examples of problems that should be presented. This sequence is based on steps that most children can understand completely, and is put together for the purpose of illustrating direction and movement toward a convenient algorithm for division.

A game is invented or selected as soon as there is agreement on a rule or rules. In the games to follow, the special rule for each game is stated.

Game I

The rule is that the player may use no term under the division sign that has more than a single digit.

An example which violates the rule is:

$$2\overline{)12} = 2\overline{)10+2}$$

because the 10 has 2 digits.

$$2\overline{)12} = 2\overline{)8+4} = 2\overline{)4+2+6} = 2\overline{)6+6+6-6}$$

$$3\overline{)12} = 3\overline{)\ +\ } = 3\overline{)\ +\ -\ } = 3\overline{)\ +\ -\ +\ }$$

$$4\overline{)12} = 4\overline{)\ +\ } = 4\overline{)\ -\ +\ } = 4\overline{)\ -\ +\ -\ }$$

$$6\overline{)12} = 6\overline{)\ +\ } = 6\overline{)\ +\ +\ } = 6\overline{)\ +\ -\ -\ }$$

(Note that additions and subtractions are to be performed from left to right.)

Game II

The rule is that no term in the quotient may have more than a single digit.

$$2\overline{)24} = 2\overline{)12+12} = 2\overline{)8+8+8} = 2\overline{)6+6+6+6}$$

$$3\overline{)24} = 3\overline{)21+3\ } = 3\overline{)\ +\ +\ } = 3\overline{)\ +\ +\ -\ }$$

$$4\overline{)24} = 4\overline{)\ +\ } = 4\overline{)\ +\ -\ } = 4\overline{)12+9-6+9}$$

$$6\overline{)24} = 6\overline{)\ +\ } = 6\overline{)\ -\ +\ } = 6\overline{)24+24+24-48}$$

Game III

Same rule as II, except that the quotient is to have many odd terms.

$$2\overline{)36} = 2\overline{)18+18} = 2\overline{)14+10+12} = 2\overline{)10+10+10+6}$$

$$3\overline{)36} = 3\overline{)\ +\ } = 3\overline{)\ +\ +\ } = 3\overline{)\ +\ -\ +\ }$$

$$4\overline{)36} = 4\overline{)\ +\ } = 4\overline{)\ +\ -\ } = 4\overline{)\ +\ +\ -\ }$$

$$6\overline{)36} = 6\overline{)\ +\ } = 6\overline{)\ -\ -\ } = 6\overline{)\ -\ -\ -\ }$$

Game IV

No restrictions.

$$2\overline{)48} = 2\overline{)24+24} = 2\overline{)40+10-2} = 2\overline{)20+20-10+18}$$

$$3\overline{)48} = 3\overline{)\ +\ } = 3\overline{)\ -\ +\ } = 3\overline{)\ -\ +\ -\ }$$

$$4\overline{)48} = 4\overline{)\ +\ } = 4\overline{)\ -\ -\ } = 4\overline{)\ -\ -\ -\ }$$

$$6\overline{)48} = 6\overline{)\ +\ } = 6\overline{)\ +\ +\ } = 6\overline{)\ +\ +\ +\ }$$

Game V

All but one of the terms in the quotient must be multiples of 10.

$$6\overline{)72} = 6\overline{)60+12} = 6\overline{)120+0-48} = 6\overline{)60+0+60-48}$$

$$4\overline{)72} = 4\overline{)\ +\ } = 4\overline{)\ -\ +\ } = 4\overline{)\ -\ -\ +\ }$$

$$3\overline{)72} = 3\overline{)\ +\ } = 3\overline{)\ +\ +\ } = 3\overline{)\ -\ +\ +\ }$$

$$2\overline{)72} = 2\overline{)\ +\ } = 2\overline{)\ -\ -\ } = 2\overline{)\ -\ -\ -\ }$$

Game VI

No restrictions.

$$11\overline{)352} \qquad 12\overline{)294} \qquad 22\overline{)242} \qquad 50\overline{)550}$$

Each method used must be carefully explained by the player and thoroughly discussed by the class. Every technique deserves its day in class. Any method that always leads to a correct result is a correct method.

The basis has been laid for some to go on alone and to make important discoveries as they try their techniques on such tough examples as those to follow.

Meanwhile time is needed to lead others over the hurdles they couldn't clear in Game IV or earlier.

Game VII

No restrictions at all.

$25\overline{)1225}$	$81\overline{)5427}$
$37\overline{)296}$	$13\overline{)104}$
$25\overline{)2575}$	$7\overline{)1001}$
$12\overline{)1164}$	$317\overline{)6357}$

By this time, individual differences will have the class scattered at rough spots along the road. Some will never get to Game VII without help. However, the distance each has come along the way and the further advance he will make with help will contribute understanding when a standard algorithm is finally presented.

Each game should, of course, be played with many, many examples. When children are encouraged to make their own examples, they usually vary the size of the numbers according to their interests and abilities. Some catapult themselves right into a complete mastery of a satisfactory algorithm.

The games cannot fail to develop and strengthen the underlying connections among the basic operations that make any division algorithm possible.

Drill

Here are some stories for headline practice:

1. A window washer can wash 9 classroom windows in 1 hour. He has 36 windows to wash.
2. Patricia bought 3 sacks of lemon drops. In each sack she had 18 pieces of candy.
3. Esther bought a tie for her father, an apron for her mother, and a doll for her little sister. She paid \$1.84 for the tie, \$2.25 for the apron, and \$1.10 for the doll.

Here is a suggestion for a drill sheet:

...x 2	3	20	400	423	7	60	100	167	132 x 2	159 x 2
...x 5	7	20	100	127	8	10	200	218	125 x 5	343 x 5
...x 8	3	20	100	123	7	10	200	217	140 x 8	149 x 8
...x 4	5	30	200	235	6	50	300	356	132 x 4	225 x 4
...x 9	3	10	100	113	9	0	400	409	208 x 9	912 x 9
...x 6	8	40	200	248	6	50	300	356	195 x 6	871 x 6
...x	7	20	400	427	9	10	700	719	136 x 7	218 x 7

0 , 400 , 800 , , , , , ,

0 , 12 , 24 , , , , , ,

0 , 412 , , , , , , ,

Rules of the MONEY GAME:

A. First digit to the left of the period indicates dollars.

B. First digit to the right of the period indicates dimes.

C. Second digit to the right of the period indicates pennies.

D. Digits allowed are: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Sign of the game: $

a. $ 1.00 + $ 2.00 = $ 3.00

b. $ 1.10 + $ 1.20 = $ 2.30

c. $ 0.80 + $ 0.20 = $ 1.00

d. $ 0.50 + $ 0.75 = $ 1.25

e. $ 1.47 + $ 4.37 = $ 5.84

f. $ 4.00 − $ 1.00 = $ 3.00

g. $ 2.00 − $ 0.70 = $ 1.30

h. $ 2.50 − $ 0.75 = $ 1.75

i. $ 3.20 − $ 1.25 = $ 1.95

j. $ 7.36 − $ 3.59 = $ 3.77

We can use the same rules when we multiply and divide in the Money Game.

k. 2 x $ 2.15 = $ 4.30

l. $ 1.50 x 3 = $ 4.50

m. 5 x $ 0.20 = $ 1.00

n. $ 1.25 x 4 = $ 5.00

o. 3 x $ 2.41 = $ 7.23

p. $ 4.30 ÷ 2 = $ 2.15

q. $ 3.00 ÷ 2 = $ 1.50

r. $ 3.39 ÷ 3 = $ 1.13

s. $ 1.20 ÷ 3 = $ 0.40

t. $ 2.19 ÷ 3 = $ 0.73

u.
$ 1.05
x 3
$3.15

v.
$ 2.25
x 4
$ 9.00

w.
$ 0.55
x 2
$ 1.10

x.
$ 1.38
x 3
$ 4.14

y. 6 x $ 1.29 = $ 7.74

z. $ 1.83 x 5 = $ 9.15

A. $1.08 / 3) $3.24

B. $2.50 / 2) $5.00

C. $0.09 / 2) $0.18

D. $0.50 / 5) $2.50

E. $0.39 / 3) $1.17

F. $1.67 / 5) $8.35

G. $2.43 / 4) $9.72

H. $1.21 / 6) $7.26

118 Using the money game to provide practice in computing (all four opera- tions) using dollars and cents notation.

Tasks: Pupils use rules A-D of the Money Game *and the clues provided in exercises* a *and* k *to compute missing sums, differences, products, and quotients.*

Purpose: To review and extend computational skills in decimal notation.

Unifying Ideas: Addition and subtraction; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: Cuisenaire rods; play money.

The Lesson:

First ask: In this shorthand, how can we indicate 7 dimes? How can we indicate 12 dimes? How can we indicate 25 cents?

7 dimes = $0.70
12 dimes = $.
25 dimes = $.

This game is played as a *Fewest Coins* game in which the only "coins" are dollars, dimes, and pennies. What would be the fewest coins we could have with the total value of twelve dimes? Three coins—one dollar, two dimes, and no pennies: $1.20

With only pennies, dimes, and dollars, what would be the fewest coins we could have with the total value of 25 pennies? Seven coins—no dollars, two dimes, and five pennies: $0.25

Read "zero dollars, two dimes, and five pennies."

Use this language as you work through several of the examples. Be sure to include some examples in multiplication and division. No doubt some pupils will have had considerable experience in reading amounts of money in the usual way, reading $1.20 as "one dollar and twenty cents." If all pupils are adept at this, there is little reason to insist on the game language beyond the first few examples. As soon as the pattern of writing dollars, dimes, and pennies is apparent, the game language can be dropped. In any event, several examples should be completely analyzed in terms of the language of the game.

Some of the examples may be worked with Cuisenaire rods or play money.

Drill

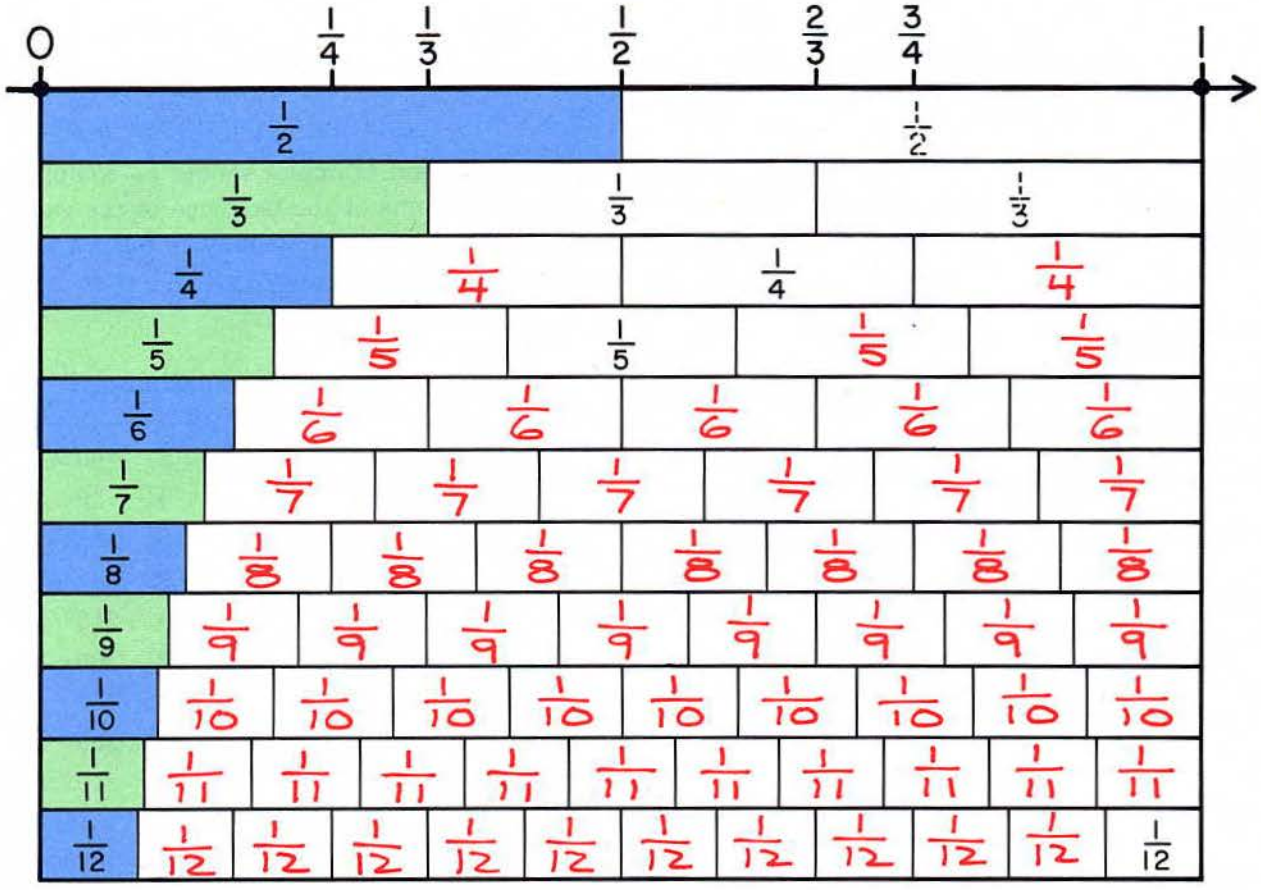
Here is a suggestion for a drill sheet:

1.	$6 \times 90 = \square$	2.	$80 \times 7 = \square$
3.	$3 \times 40 = \square$	4.	$50 \times 7 = \square$
5.	$4 \times 90 = \square$	6.	$5 \times 30 = \square$
7.	$70 \times 7 = \square$	8.	$8 \times 60 = \square$
9.	$80 \times 8 = \square$	10.	$210 \div 3 = \square$
11.	$640 \div 8 = \square$	12.	$300 \div 5 = \square$
13.	$360 \div 9 = \square$	14.	$400 \div 5 = \square$
15.	$180 \div 6 = \square$	16.	$200 \div 4 = \square$
17.	$900 \div 3 = \square$	18.	$240 \div 3 = \square$
19.	$8 \times 400 = \square$	20.	$7 \times 500 = \square$
21.	$500 \times 8 = \square$	22.	$7 \times 800 = \square$
23.	$800 \times 8 = \square$	24.	$5 \times 300 = \square$
25.	$7 \times \square = 350$	26.	$5 \times \square = 200$
27.	$7 \times \square = 630$	28.	$\square \times 3 = 2400$
29.	$\square \times 4 = 3200$	30.	$9 \times \square = 4500$
31.	$400 \div 5 = \square$	32.	$450 \div 9 = \square$
33.	$600 \div 2 = \square$	34.	$3500 \div 7 = \square$
35.	$6300 \div 7 = \square$	36.	$4800 \div 6 = \square$

Here are some stories for headline practice:

1. Mr. Sloan, the third-grade teacher at Central Elementary School, planted 138 bulbs in each of 7 rows.
2. Hector went to the supermarket. He bought 4 cupcakes and 7 cookies. The cupcakes sold for 12¢ each. The cookies sold for 6¢ each.
3. Fanny went to the drugstore. The druggist told her that he had 3 bottles of cough drops. Each bottle held 299 cough drops.

Headline for the Day: Tell a story to fit this fact team: $743 + 257 = \square$ $\square - 257 = 743$ $257 + 743 = \square$ $\square - 743 = 257$



Make TRUE sentences.

= is equal to < is less than > is greater than

	A	B	C	D
I	$\frac{1}{3} < \frac{1}{2}$	$\frac{1}{2} < \frac{2}{3}$	$\frac{3}{8} > \frac{3}{10}$	$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{5}{10}$
II	$\frac{1}{8} > \frac{1}{10}$	$\frac{2}{5} > \frac{1}{4}$	$\frac{6}{12} < \frac{6}{11}$	$\frac{3}{7} > \frac{3}{8} > \frac{3}{9} > \frac{3}{10}$
III	$\frac{1}{4} > \frac{1}{6}$	$\frac{2}{4} = \frac{1}{2}$	$\frac{3}{4} < \frac{4}{5}$	$\frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6}$
IV	$\frac{1}{11} < \frac{1}{10}$	$\frac{2}{12} < \frac{1}{3}$	$\frac{5}{6} > \frac{4}{7}$	$\frac{2}{6} = \frac{1}{3} = \frac{3}{9} < \frac{5}{12}$
V	$\frac{1}{6} > \frac{1}{7}$	$\frac{3}{9} < \frac{2}{3}$	$\frac{9}{11} > \frac{2}{3}$	$\frac{2}{7} > \frac{1}{4} = \frac{3}{12} = \frac{2}{8}$
VI	$\frac{1}{2} > \frac{1}{4}$	$\frac{3}{5} > \frac{1}{2}$	$\frac{6}{12} = \frac{1}{2}$	$\frac{5}{12} > \frac{4}{10} = \frac{2}{5} < \frac{4}{9}$

119 Exploring intuitively the greater than, less than, and equality relations among fractions.

Tasks: *Pupils mark each rectangle with the appropriate fraction; pupils record in the green boxes the relation sign which correctly completes each number sentence.*

Purpose: To review and extend understanding of fractions and of the quality and inequality relations among fractions.

Unifying Ideas: Numeration; functions and relations; measurement.

Materials for Teacher: Materials described below.

Materials for Pupils: Tracing paper to copy the subdivided rectangle on pupil page 119. Other materials as described below.

The Lesson:

Any of the following activities might be useful in introducing the idea of fractions to your pupils.

Fractions and the Real World

Pupils have by now discovered that whole numbers taken singly may not be adequate to describe all real-life situations. List a few situations which demonstrate this and ask pupils to add to this list.

Half of an object
Half of an orange or cookie
Half of a number
Half of a dollar or dime
Half past two
Half price
First half of the baseball game

$\frac{1}{2}$

Each child got a third.
One third of the way around a circle
One third off
One third of a number of objects
One out of every three

$\frac{1}{3}$

A quarter of a dollar
A quarter of an inch
Quarter after four
A quarter of a gallon (one quart)
A fourth of a pie or an apple
A quarter of three

$\frac{1}{4}$

Three fourths of the way around
Three out of four times at bat
Three quarters of an hour until lunch
Three quarters of a football game
Three quarters as large

$\frac{3}{4}$

Two thirds of a pie
Two thirds of the way
Two out of every three
Two thirds off
Two thirds of the number of objects
Two thirds of the apples are ripe.

$\frac{2}{3}$

Two half-dollars
Two halves of a sandwich
Two out of two

$\frac{2}{2}$

Four quarters in the game
Four quarters in a dollar
Four out of four
Four fourths of a pie or an orange

$\frac{4}{4}$

Half for you and half for me
First half and second half of a game
Half shaded and half unshaded

$\frac{1}{2}+\frac{1}{2}=\frac{2}{2}$

One third off
My sister carried one third of my books.
He has gone one third of the way.
He took one third of the pie.
Jump one third of the way to 9.

$\frac{3}{3}-\frac{1}{3}=\frac{2}{3}$

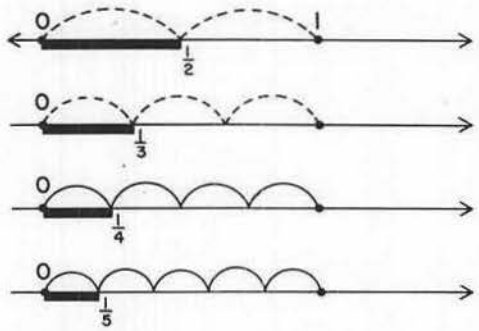
It is not necessary at this point to discuss the mathematical symbols precisely. The purpose of this discussion is to make pupils aware of the ideas referred to by these common expressions.

Headline for the Day: Consider the first completed sentence in each column (A-D) on page 119 as a headline for a story. Tell a story to fit each headline.

Jumps on the Number Line

Indicate jumps on a number line that suggest:

1/2	1/4	1/6	1/8
1/3	1/5	1/7	1/9



When confronted with a problem such as finding the point for 1/4, discuss procedures that might be followed, such as: first find the point halfway between 0 and 1; then find the point halfway between 0 and this new point (or, between 0 and 1/2). When preparing to draw a sketch to suggest 1/6, the first step might be the same as in the previous example, or it might be to indicate the jumps for 1/3, and then to find the midpoints of each jump. In many cases the work can begin exactly as for one of the other examples and then be carried further. In such cases, two examples will show "family resemblances." Since 1/6 is related to both 1/2 and 1/3, either of these can be used to begin the work. One eighth is related in an even closer way to 1/2 and 1/4. One ninth bears a family resemblance to 1/3.

What fractions do not seem to be related to any others in the group under consideration? (Only 1/5 and 1/7)

Do you suppose that 1/5 and 1/7 belong to any families at all? (Yes, 1/5 is related to 1/10, 1/15, 1/20, etc.; 1/7 is related to 1/14, 1/21, 1/28, etc.)

Refer back to the diagrams suggesting 1/2, 1/3 . . . 1/9. If we wish to label all the landing points between 1 and 0, what labels would you suggest in each case? (2/3, 1/3; 3/4, 2/4, 1/4; 4/5, 3/5, 2/5, 1/5; etc.)

Starting from 0, which is larger, a jump to 3/5 or a jump to 1/2?

In the following pairs, circle the fraction that is located farther away from 0 on the number line:

1/3, 2/3	3/4, 1/4
2/4, 1/5	2/3, 5/6
2/8, 1/5	1/2, 3/7

Consider the following. Which is farther from 0?

1/2, 3/6	2/4, 1/2
2/6, 1/3	2/3, 6/9

We have a shorthand that expresses these ideas. I will write some true statements in this shorthand. Can you complete the others?

$1/2 > 1/3$	$1/2 < 3/4$	$1/2 = 3/6$
$3/9 = 1/3$	$3/6 < 4/6$	$7/8 > 1/4$
2/4 3/6	2/3 2/4	1/7 1/4
1/3 2/7	3/5 1/2	4/8 3/6

Continue examples until everyone can use this notation.

Use Sketches

Use sketches to suggest the idea of fractions and to reveal some important relationships that exist among fractions.

Draw a line segment on the board—preferably at a slight angle—and put a dot at each end.

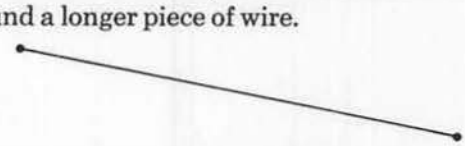
This is my picture of a piece of wire. The ends are sharp and I don't wish to get scratched, so I've covered them. Where should I cut the wire to make 2 pieces the same length? Mark the point with a dot.

Suppose that I wished to cut the wire into 4 pieces that are the same size. Can you mark with a cross the places where I ought to cut it?

Now I'm going to draw another sketch of my piece of wire. This time I will bend it a little bit.

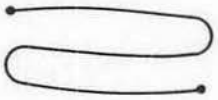
Where should I cut the wire if I wished to have 2 pieces the same size? Can you mark with a dot the place where I should cut it? And where would you put cross marks to indicate where I should cut the wire if I wish to have 4 pieces the same size?

I found a longer piece of wire.

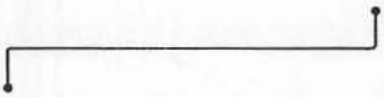


Where should I cut it if I wish to have 2 pieces of the same size? Where should I cut it if I wish to have 4 pieces the same size?

Now I've bent the wire into an S-shape.



Now I put 2 sharp bends in it.



Finally, I've bent the wire into a circle.

In each case, mark with a dot the place where I should cut the wire if I wish to have 2 pieces the same size, and then mark with crosses the places where I should cut if I wish to have 4 pieces the same size.

I finally cut my wire into 2 pieces.

No sooner had I done this than I was sorry. I had really wished to cut it into 4 pieces that were the same size . . . but I only have 2. What can I possibly do? (You will get plenty of advice.)

If I change my mind again and wish 2 pieces after I have cut the wire into 4 pieces, is there anything I can do? (You'll have more advice: solder 2 pieces, twist them together, glue them, nothing, get another piece of wire, etc.)

More Sketches

Today, I have a rope. I've knotted each end so it won't unravel. If I were to lay it out straight, it would look something like this:

If I wished to cut the rope into 3 pieces about the same size, where should I cut it? Mark the places with dots.

Now suppose that I wished to have twice as many pieces of rope, all about the same size. Using a little cross, can you mark each place on the rope where I should cut it?

Why do some of the dots and crosses mark the same place? (Don't expect well-formulated answers. Don't try to settle the matter.)

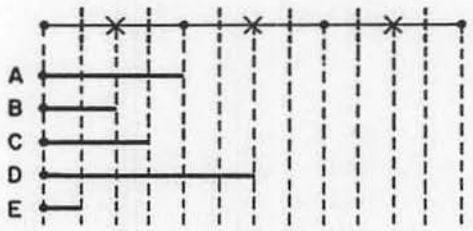
If I wished to have only 2 pieces about the same size, where would I cut? Is the place marked?

Suppose that I wished to have 4 pieces about the same size? (Cut it in half—and then cut each of the 2 pieces in half.) Are all the places to be cut marked? Use a dotted line to mark the new places I would need to cut.

Now, what do you suppose has happened? I've changed my mind again. I need a dozen pieces—all about the same size. Where shall I cut? Discuss this matter for a while. You might cut the rope in half and then cut each of those pieces in half to get 4 pieces. You would then cut each of the 4 pieces into 3 parts. Or you might cut the rope into 6 parts and cut each of the 6 pieces in half. Or, you might cut the rope into 3 parts, then cut each of those in half, and halve each of the 6 pieces, etc.

Are any of the places to be cut already marked? How many more places do we have to mark? (4) Indicate these places with dotted lines.

On the board draw the following:



At one time, I wished to cut my rope into 2 pieces about the same size. Does any line on the chart look something like one of those pieces? (The line marked D.)

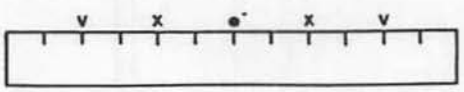
What do the other sketches suggest to you?

How many pieces of rope the size of line B would I be able to cut from a piece of rope the size of line D?

Is the line marked A twice as long as the line marked C? Which line is twice as long as A?

Labeling a Ruler

Here is a picture of a 1-foot ruler:



Where is the middle of the ruler? What number is usually written at that point on a 1-foot ruler? Use small crosses to divide the length of the ruler into 3 parts about the same size. What numbers would be written at these points?

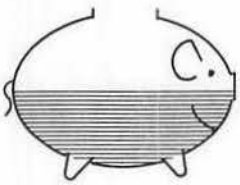
What points would divide the ruler into 6 parts about the same size? Label the points as they usually are labeled on a ruler.

Is it easy to divide the ruler into 4 parts about the same size? Label these points if they are not already labeled.

Are there any more points to be labeled on this ruler?

Piggy Bank Problem

Here is a sketch of a piggy bank. It is half full of pennies:

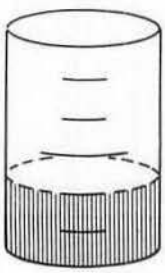


Can you tell me how many pennies there are? What do you need to know before you can tell me the number? If I told you how many pennies were needed to fill the piggy bank, could you tell me how many pennies the bank holds when it is half full?

Value when full	Number of pennies when half full
$1.00	
$ .50	
$3.00	

Water Tank Problem

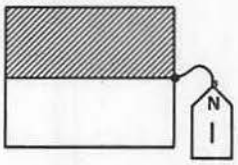
Here is a water tank. The marks on it help to indicate how full it is:



No. of gallons when full	No. of gallons in shaded area
6	
60	
3	
1	

Tag Problem

Here is a sketch of 1 of something. This is indicated by the 1 on the tag:



What does the shaded region suggest? The unshaded region?

Shaded	Unshaded	Tag
1/2	1/2	1
		2
		1/2

Suppose that we change the tag. Let's write 2 in the tag. Now, what do the shaded and unshaded regions suggest? (1 and 1)

Suppose that we change the tag to read 1/2. Now, what does the shaded region suggest? (1/4) Without discussing, make the comment: It seems that a half of one half is one fourth. In money this would mean that half of a half-dollar is the same as a quarter of a dollar.

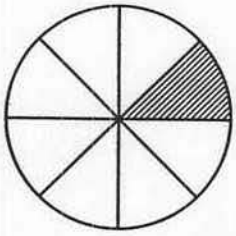
Can you make sketches suggested by each row in the following chart?

Shaded	Unshaded	Tag
3/4	1/4	1
1/3		1
	2/5	1
3/2		2
	5/3	2

Don't hurry. Discuss the adequacy of each sketch, possible improvements, similar sketches that suggest that same idea, similar ideas suggested by the same sketch, etc.

Pie Problem

Here is a pie that is cut into 8 pieces:



There are 16 ounces in a pound. How many ounces would there be in a shaded piece?

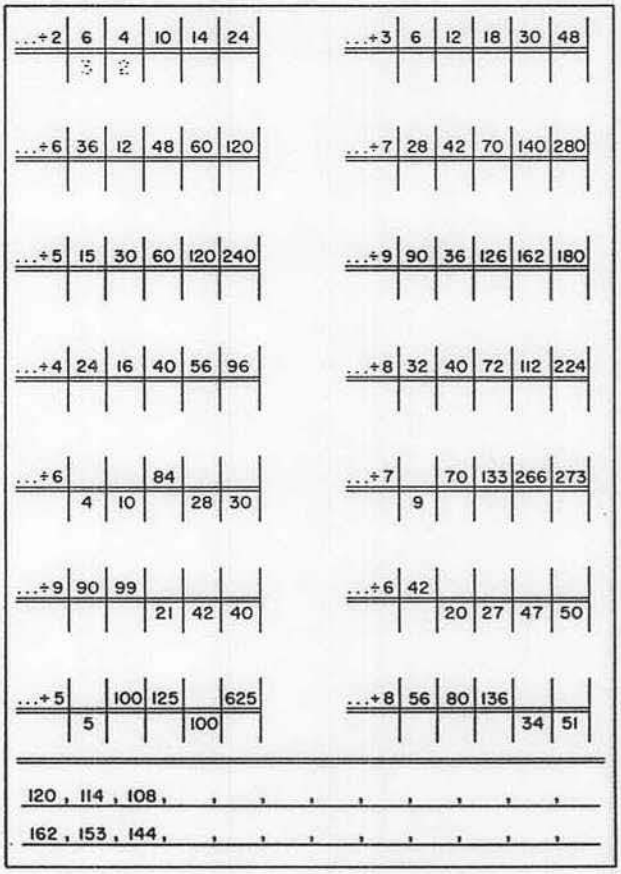
Weight of whole pie in pounds	Weight of 1 piece	
	in pounds	in ounces
1	1/8	2
2		
4		
6		

Drill

Here are some stories for headline practice:

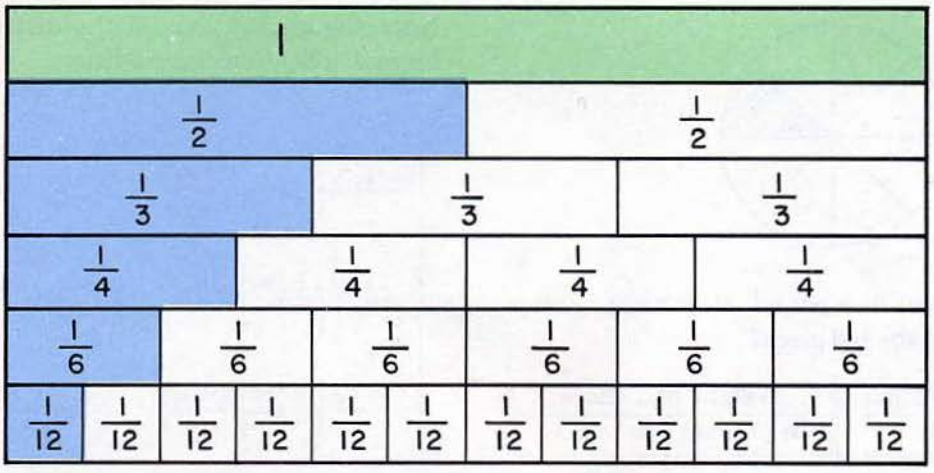
1. There were 34 children in Mrs. Jasper's class. There were the same number of boys as girls.
2. There were 90 people playing baseball at the state park. There were 5 baseball diamonds in use.
3. Perry's third-grade class had a science table. The class had arranged rock collections on the table. There were 9 collections in all. Each collection had 9 rocks in it.

Here is a suggestion for a drill sheet on division. Note the applications of the distributive principle for multiplication over addition:



If we agree that the area inside the heavy black line suggests 1

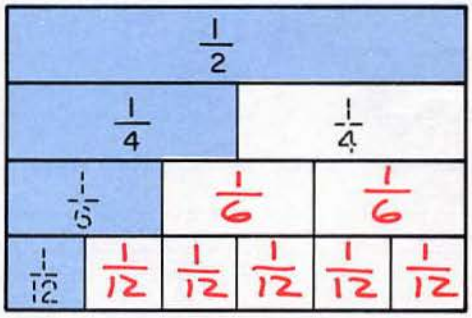
THEN the blocks in the drawing below can suggest the following:



This sketch suggests the idea that $1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{6}{6} = \frac{12}{12}$.

We notice that the area of some groups of blocks is the same as that of some single blocks.

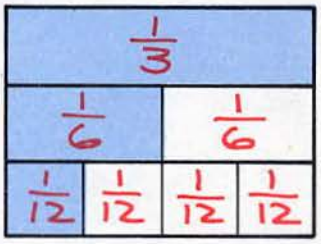
A



This sketch suggests that

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{6}{12}$.

B

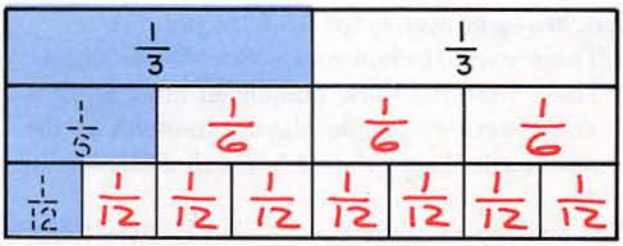


This sketch suggests that

$\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$.

We notice that the area of some groups of blocks is the same as that of other groups.

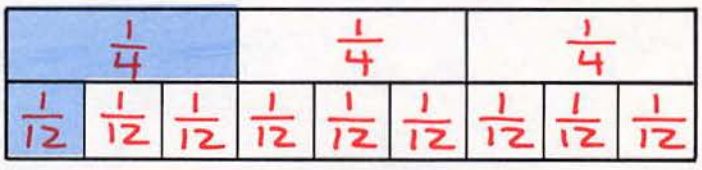
C



This suggests that

$\frac{2}{3} = \frac{4}{6} = \frac{8}{12}$.

D



This suggests that

$\frac{3}{4} = \frac{9}{12}$.

We also notice that the large sketch at the top of the page suggests that $\frac{5}{6} = \frac{10}{12}$.

Tasks: *Pupils complete each number sentence by correctly filling in the blanks provided; pupils mark each rectangle with the appropriate fraction.*

Purpose: To further explore fraction concepts.

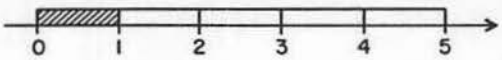
Unifying Ideas: Structure; functions and relations; measurement.

Materials for Teacher: Tagboard strips suggested by pupil page 120; graph board.

Materials for Pupils: Graph paper.

The Lesson:

Without any introduction, put the following diagram on the board and ask: What does this diagram suggest to you?



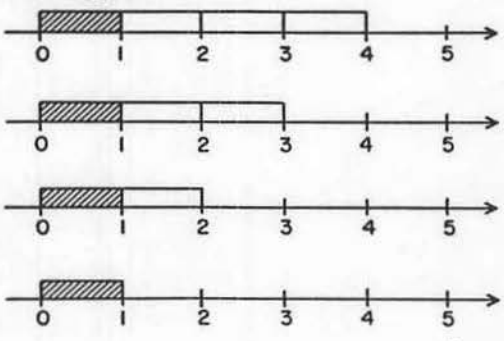
Complete the following mathematical statements and decide which are suggested by the picture:

$5 \times 1 = \_\_\_$ $4 \times 5 = \_\_\_$ $5 \div 5 = \_\_\_$

$\frac{1}{5} + \frac{1}{5} = \_\_\_$ $\frac{5}{5} = \_\_\_$ $1 \times 5 = \_\_\_$ $5\overline{)5}$

Answer: $5 \div 5 = 1$ $5/5 = 1$ $5\overline{)\overset{1}{5}}$

What mathematical sentences do the following diagrams suggest?



$4 \div 4 = 1$ $4/4 = 1$ $4\overline{)\overset{1}{4}}$

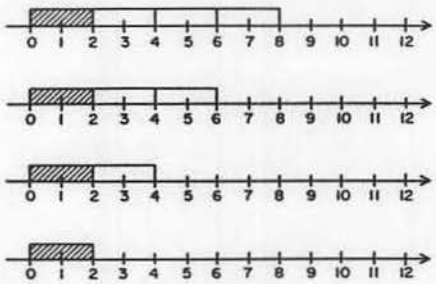
$3 \div 3 = 1$ $3/3 = 1$ $3\overline{)\overset{1}{3}}$

$2 \div 2 = 1$ $2/2 = 1$ $2\overline{)\overset{1}{2}}$

$1 \div 1 = 1$ $1/1 = 1$ $1\overline{)\overset{1}{1}}$

and, further, a statement such as: $1/1 = 2/2 = 3/3 = 4/4 = 5/5$ . . . etc.

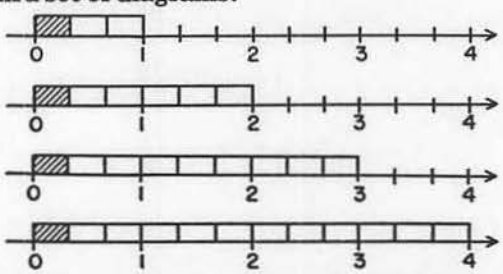
Then consider another group of diagrams:



which suggests:

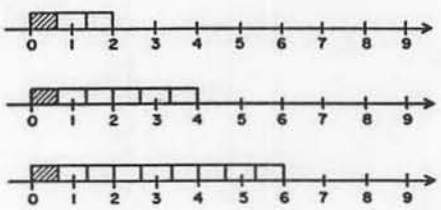
$2/1 = 4/2 = 6/3 = 8/4 = 10/5 = 12/6 = 14/7$ . . . etc.

A third set of diagrams:



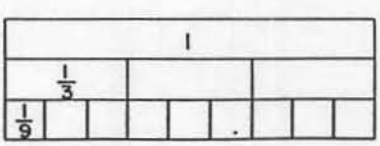
which suggests that $1/3 = 2/6 = 3/9 = 4/12$ . . . etc.

And a final set:



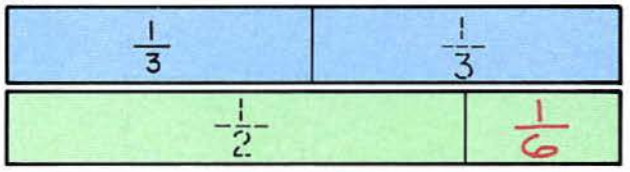
which suggests that $2/3 = 4/6 = 6/9$ . . . etc.

Headline for the Day: Tell some stories about this picture:



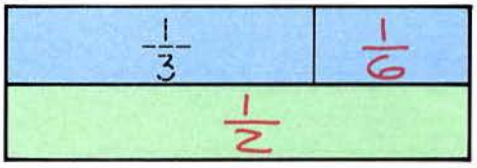
Some combinations of blocks have the same area as that of other combinations or as that of single blocks. Please note the mathematical facts suggested by the sketches.

A



$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ $\frac{2}{3} - \frac{1}{6} = \frac{1}{2}$

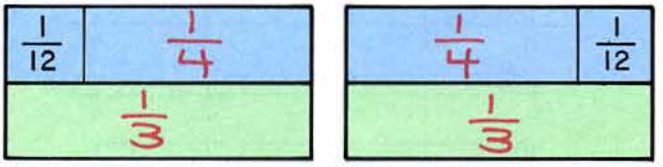
B



$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

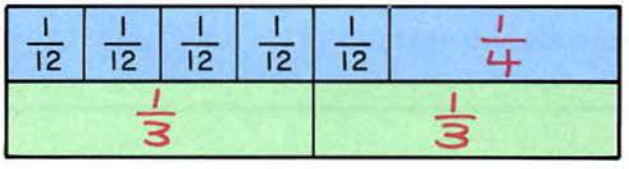
C



$\frac{1}{12} + \frac{1}{4} = \frac{1}{3}$ $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$

$\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$ $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

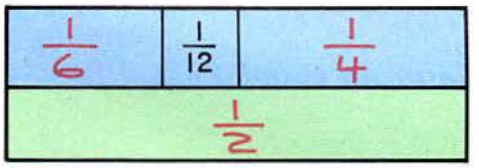
D



$\frac{5}{12} + \frac{1}{4} = \frac{2}{3}$ $\frac{1}{4} + \frac{5}{12} = \frac{2}{3}$

$\frac{2}{3} - \frac{5}{12} = \frac{1}{4}$ $\frac{2}{3} - \frac{1}{4} = \frac{5}{12}$

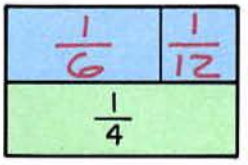
E

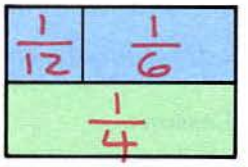


$\frac{1}{6} + \frac{1}{12} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{2} - \frac{1}{12} = \frac{1}{6} + \frac{1}{4}$

F

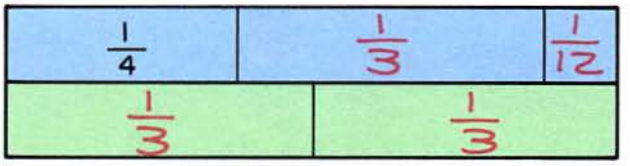




$\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$ $\frac{1}{4} - \frac{1}{12} = \frac{1}{6}$

$\frac{1}{12} + \frac{1}{6} = \frac{1}{4}$ $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

G



$\frac{1}{4} + \frac{1}{3} + \frac{1}{12} = \frac{2}{3}$

$\frac{2}{3} - \frac{1}{4} = \frac{1}{3} + \frac{1}{12}$

Please complete the following as similar sketches would suggest:

H $\frac{1}{2} - \frac{1}{12} = \frac{5}{12}$ I $\frac{7}{12} + \frac{1}{12} = \frac{8}{12}$

J $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ K $\frac{1}{4} + \frac{1}{3} = \frac{7}{12}$

L $\frac{2}{3} + \frac{1}{12} = \frac{3}{4}$ M $\frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

N $\frac{1}{4} + \frac{5}{12} = \frac{2}{3}$ O $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

ACCEPT OTHER CORRECT FRACTIONS.

121 Exploring intuitively sums and differences of fractions and equation solving using fraction notation.

Tasks: *Pupils complete each number sentence by correctly filling in the blanks; pupils mark each rectangle with the appropriate fraction.*

Purpose: To further explore fraction concepts.

Unifying Ideas: Structure; addition and subtraction; measurement.

Materials for Teacher: Demonstration-size tagboard strips suggested by pupil page 121; graph board; colored tagboard squares as described below.

Materials for Pupils: Tracing paper.

The Lesson:

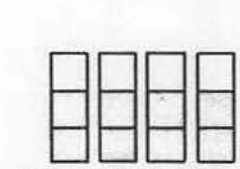
Pupils should copy on tracing paper the large, subdivided rectangle located near the top of pupil page 120 and use their copy to measure the unmarked rectangles in A-G on pupil page 121. Once this is done, it is easy to complete the sentences.

Enrichment Activities

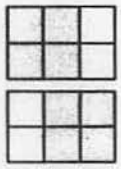
Ratio

Cut squares about 2″ × 2″ out of cardboard that is white on one side and gray on the other, or have the squares colored one color on one side, another color on the other. Put 12 squares in a box. Mix them up and dump them out on a table.

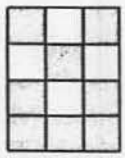
How many squares are white? How many are gray? Without turning any over, arrange them in a 3 × 4 array. Arrange them in different patterns. (Unless the result is 5 of one color and 7 of another, they can usually be arranged and rearranged to indicate many ways of saying the same thing.) Suppose that there are 4 white and 8 gray:



gray – 2 out of 3
white – 1 out of 3



4 out of 6
2 out of 6



8 out of 12
4 out of 12

After several examples, include 4 more squares so that there are 16 in all. Now all even numbers of squares, either white or gray (if one is even, so is the other), can be arranged in more than one way.

The shorthand 1/3, 2/5, 4/8, etc., can be used to keep track of our results. In the example when 8 out of 12 were gray, we could have expressed the idea in the sketch this way:

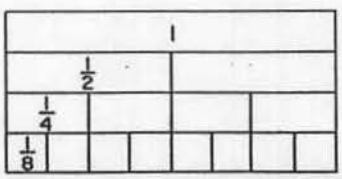
Gray	White	All together
8/12 or 4/6 or 2/3	4/12 or 2/6 or 1/3	12/12, 6/6, 3/3, 11/11, 10/10, 9/9, etc.

We can read this new shorthand: "8 out of 12 are gray" or "4 out of 6 are gray," etc. Write the following on the board, and have pupils complete them:

If 8/12 are gray then are white.
If 3/12 are white then are gray.
If are white then 7/12 are gray.

more

Headline for the Day: Tell some stories about this picture:



Are the following statements true?

If 8/12 are gray then 1/3 is white.
If 1/4 is white then 9/12 are gray.
If 2/12 are white then 5/6 are gray.

Include 8 more squares. Now there are 24 in all. Proceed in same general way.

Talk about each example. Draw sketches. Encourage the pupils to explain their arguments. Keep asking: Does that sound reasonable? Does everyone agree?

Do not expect to reach any general rules about this idea that, later on, will be referred to as *ratio*. We are introducing the idea at this point mainly for discussion purposes.

Probability

If time is available for some or all of the pupils, use the 2″ × 2″ squares of white-gray cardboard (or the equivalent) for an early experience with the idea of chance or probability.

Put 2 squares in a box. Mix them, and drop them on a table. How many are gray? In what different ways can the squares show up?

0 out of 2 may be gray
1 out of 2 may be gray
2 out of 2 may be gray

Are there any other possible outcomes? Which result do you think will happen most often?

Let several groups experiment, and have them each keep a record. After 24 tries their record might be:

How many are gray?

	0/2	1/2	2/2
Totals	6	14	4

The more trials each group makes, the more closely the results will approximate a 1-2-1 ratio. That's a secret.

Then make a compilation of totals:

Group	0/2	1/2	2/2
A	6	14	4
B	4	10	8
C	7	15	4
D	8	13	8
Totals	25	52	24

Expand the activity to an investigation of results when 3 squares are used. What are the possible combinations? How many can be gray each time?

0 out of 3 2 out of 3
1 out of 3 3 out of 3

Again, keep records:

How many are gray?

	0/3	1/3	2/3	3/3
Totals	5	14	16	6

Again, have several groups keep track of as many trials as possible—and then record the totals as shown above. (This time, the outcomes will approximate 1-3-3-1; that's a secret.)

The only goals we have at this point are to plant the idea that certain combinations seem to occur more often than others, and to use familiar notations to express ratios (for example, 2/3, meaning 2 out of 3). However, we do not yet use the term *ratio*.

Another Game of Chance

Prepare 24 pieces of cardboard. Leave one side of each piece white. Color the other side of 12 pieces blue and the other side of the remaining 12 pieces, green.

Start with 6 pieces: 3 with one side blue, and 3 with one side green. Put the 6 pieces in a box—shake them up—and spill them out on a desk. Suppose that 3 show white, 2 blue, and 1 green. We can record this as:

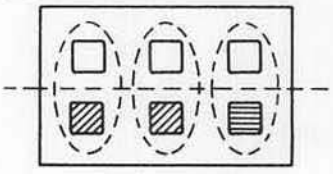
3/6 white; 2/6 blue; 1/6 green

We can also write:

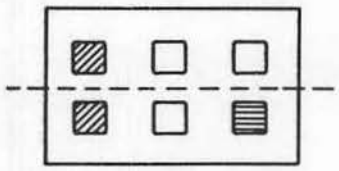
1/2 white; 1/3 blue; 1/6 green

	White	Blue	Green	or	White	Blue	Green
1st trial	3/6	2/6	1/6		1/2	1/3	1/6
2nd trial	4/6	0/6	2/6		2/3	0/3	1/3
3rd trial	2/6	2/6	2/6		1/3	1/3	1/3
4th trial	4/6	1/6	1/6		2/3	1/6	1/6

As each combination appears, arrange the pieces to illustrate each idea. In the first trial, arrange the pieces as follows:

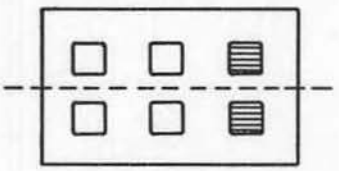


We can see that 3 out of 6 are white. And we can also see that 1 out of every group of 2 is white. This arrangement also shows us that 1 out of 6 is green. Change the arrangement:



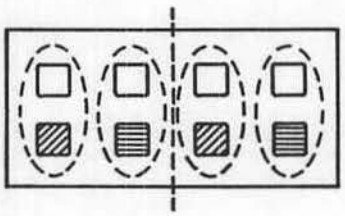
We can see now not only that 2 out of 6 are blue but, also, that 1 out of every group of 3 is blue.

The second trial may result in 4 white and 2 green. Now a single arrangement is enough to bring out the relationship in which we are interested:



Clearly, 4 out of 6 are white; and 2 out of every group of 3 are white. Also, 2 out of 6 are green; and 1 out of every group of 3 is green.

Include two more pieces—one white-blue and one white-green. Suppose that the first trial shows 4 white, 2 blue, and 2 green. A single arrangement will suffice:



Four out of 8 are white. We can also see that 1 out of every group of 2 is white.

Draw a vertical line through the middle. Now, we can see that 2 out of every group of 4 are white. Also, we see that 1 out of every group of 4 is blue and 1 out of every group of 4 is green. We can see that 2 out of 8 are blue and 2 out of 8 are green. We can summarize these results as follows:

	White	Blue	Green
1st trial	4/8 or 2/4 or 1/2	2/8 or 1/4	2/8 or 1/4
2nd trial	3/8	4/8 or 2/4 or 1/2	1/8

Drill

Here is a suggestion for a drill sheet:

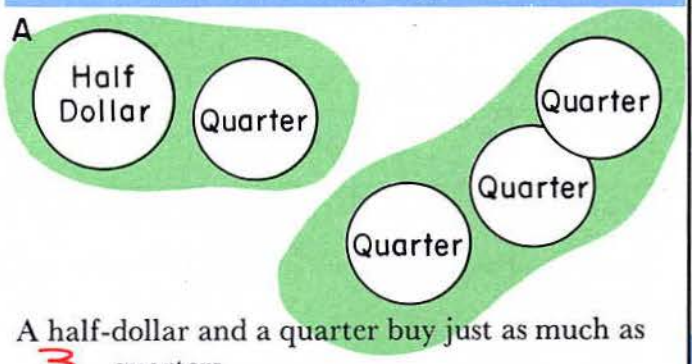
9 x 4	8 x 7	3 x 9	7 x 6	8 x 5	9 x 8
30 x 4	50 x 7	40 x 9	50 x 6	90 x 5	70 x 8
39 x 4	58 x 7	43 x 9	57 x 6	98 x 5	79 x 8
9 x 7	7 x 5	4 x 8	6 x 9	84 x 2	37 x 6
40 x 7	60 x 5	70 x 8	40 x 9	109 x 9	98 x 5
300 x 7	400 x 5	500 x 8	300 x 9	114 x 7	241 x 8
349 x 7	467 x 5	574 x 8	346 x 9	659 x 4	457 x 7

Here are some stories for headline practice:

1. Jay took his older brother to the movies. Jay's ticket cost 50¢ and his brother's cost $1.25. Jay gave the cashier $2.
2. Carter bought 7 packages of seeds. Each package contained 16 seeds.
3. Duncan and Sherry went to the carnival with Mr. Carns. Mr. Carns' ticket cost 35¢ and the children's tickets cost 20¢ each.

Familiar ideas — and the arithmetic they suggest.

A

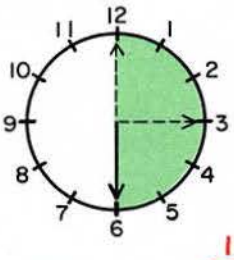


A half-dollar and a quarter buy just as much as 3 quarters.

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ or $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

B

Two quarters of an hour:



Two quarters of an hour is $\frac{1}{2}$ hour.

$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

C

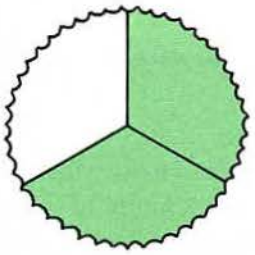
A day is one seventh of a week.

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11

Two sevenths of a week and three sevenths of a week are the same as 5 sevenths of a week.

$\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$ or $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$

D

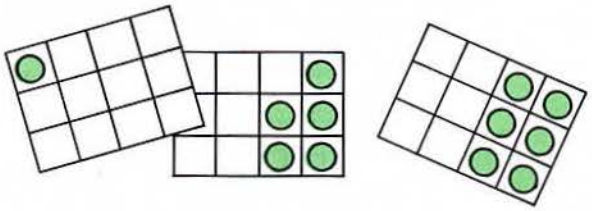


One third of a pie and another third of a pie together are 2 thirds of a pie.

$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ and $\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

E

One egg is one twelfth of a dozen eggs.

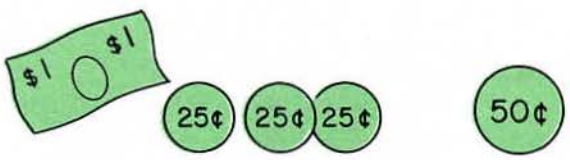


One twelfth of a dozen and five twelfths of a dozen is the same as 6 twelfths of a dozen or 1 half-dozen.

$\frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$

F

A half-dollar added to a dollar and three quarters . . .



is as much as two dollars and 1 quarter.

$1\frac{3}{4} + \frac{1}{2} = 2\frac{1}{4}$ or $\frac{1}{2} + 1\frac{3}{4} = 2\frac{1}{4}$

G $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

H $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

I $\frac{1}{2} + \frac{1}{12} = \frac{7}{12}$

J $\frac{1}{2} - \frac{1}{12} = \frac{5}{12}$

K $1 - \frac{1}{2} = \frac{1}{2}$

L $2 - \frac{1}{2} = 1\frac{1}{2}$

M $1 - \frac{1}{4} = \frac{3}{4}$

N $97 - \frac{1}{4} = 96\frac{3}{4}$

122 Using applications to gain insights into computing sums and differences of fractions-equation solving using fraction notation.

Tasks: *Pupils use the clues provided in each representation to complete the number sentence.*

Purpose: To continue exploring fraction concepts, particularly computations with fractions, "mixed numbers," and fractions and whole numbers.

Unifying Ideas: Structure; addition and subtraction; measurement.

Materials for Teacher: Demonstration-size tagboard representations suggested by pupil page 122.

Materials for Pupils: Play money or Cuisenaire rods to represent half-dollars and quarters.

The Lesson:

Each of the examples on the page can be reasoned through without knowing anything about common denominators and other formal manipulations. Measurement units in the real world are used to give pupils a basis for understanding fractions.

$$\frac{1}{2} + \frac{1}{12} = ?$$

Given such a problem, a pupil can think of a clock, and say that 1/2 hour = 30 minutes; that 1/12 hour = 5 minutes; and 35 minutes is 7/12 of an hour.

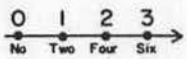
$$\frac{3}{4} + \frac{1}{2} = ?$$

The pupil can think of quarters and half-dollars. 1/2 dollar is the same as 2 quarters. 3 quarters plus 2 quarters is 5 quarters. So,

$$\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

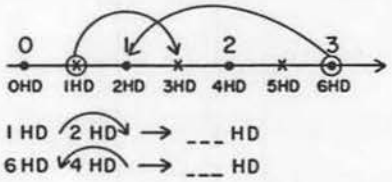
Lands of Half-dollars and Quarters

Let's label a number line for a land in which the only coins are half-dollars.



No half-dollars	Two half-dollars	Four half-dollars	Six half-dollars
0HD	2HD	4HD	6HD

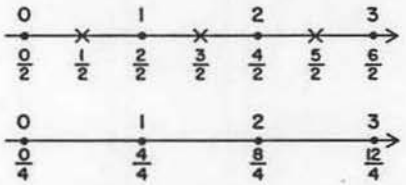
We ought to have points we can label 1HD, 3HD, 5HD, etc. What can you suggest? What meaning could you attach to jumping on such a line?



(1HD and a jump to the right of 2HD goes to 3HD.)
(6HD and a jump to the left of 4HD goes to 2HD.)

	0HD	1HD	2HD	3HD
0HD	0HD	1HD	2HD	3HD
1HD	1HD	2HD	3HD	4HD
2HD	2HD	3HD	4HD	5HD
3HD	3HD	4HD	5HD	6HD

Then develop a parallel number line that might be useful in the Land of Quarters (quarter dollars).

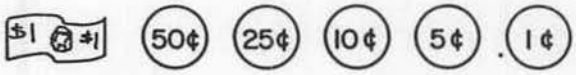


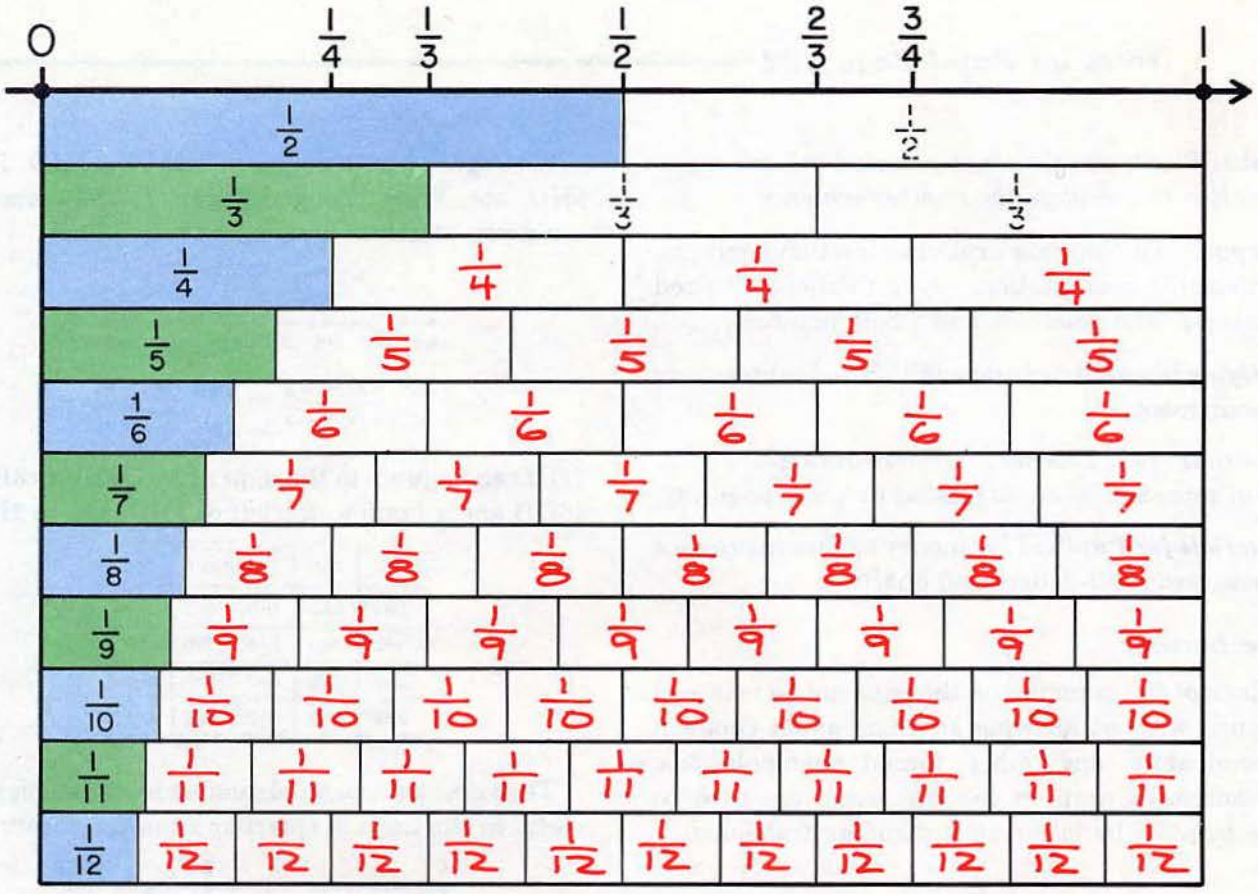
In the Land of Quarters we certainly hope that 4 quarters would buy the same amount of groceries as one dollar might. Also, that we could exchange 8 quarters for 2 dollars, if they had any dollars. How do you think they would label other points on their number line of quarters?

If a pupil wishes to label the midpoint between 0/4 and 4/4 as 1/2, he should not be told he is wrong. Rather, remind him to imagine that we are in the Land of Quarters, and that our labels should suggest quarters. For that reason the label 2/4 is more appropriate for that point than 1/2.

Jump both ways along this number line of quarters and, as for the previous number line, record the jumps to the right on a chart.

Headline for the Day: Compare each coin with a dollar bill.





Make TRUE sentences.

$=$ is equal to $<$ is less than $>$ is greater than

	A	B	C	D
I	$\frac{5}{6} < \frac{6}{7}$	$\frac{2}{5} < \frac{3}{7}$	$\frac{5}{11} < \frac{1}{2}$	$\frac{2}{3} < \frac{5}{7}$
II	$\frac{1}{2} + \frac{1}{2} > \frac{1}{3} + \frac{1}{3}$	$\frac{1}{2} - \frac{1}{2} = \frac{1}{3} - \frac{1}{3}$	$\frac{1}{3} + \frac{2}{3} < \frac{1}{2} + \frac{3}{4}$	$\frac{2}{5} + \frac{3}{5} < \frac{1}{2} + \frac{2}{3}$
III	$\frac{1}{3} + \frac{1}{2} = \frac{1}{2} + \frac{1}{3}$	$\frac{1}{3} + \frac{1}{3} > \frac{1}{2} - \frac{1}{3}$	$\frac{3}{4} - \frac{1}{2} < \frac{1}{4} + \frac{1}{4}$	$\frac{3}{4} + \frac{3}{4} > \frac{2}{5} + \frac{3}{5}$
IV	$\frac{5}{6} - \frac{1}{3} = \frac{1}{8} + \frac{3}{8}$	$\frac{1}{9} + \frac{1}{9} < \frac{2}{12} + \frac{1}{12}$	$\frac{7}{12} + \frac{5}{12} = \frac{3}{8} + \frac{5}{8}$	$\frac{2}{7} + \frac{1}{6} < \frac{1}{5} + \frac{2}{5}$
V	$2 + \frac{1}{3} > 2 + \frac{1}{4}$	$2 + \frac{3}{4} > 3 - \frac{1}{2}$	$3 + \frac{7}{12} = \frac{7}{12} + 3$	$\frac{1}{2} + \frac{1}{2} < 2 - \frac{1}{2}$
VI	$\frac{2}{3} + \frac{3}{4} < \frac{2}{4} + \frac{3}{3}$	$\frac{1}{2} + \frac{1}{3} > \frac{1}{5} + \frac{3}{5}$	$\frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}$	$\frac{2}{3} + \frac{1}{3} > \frac{1}{3} + \frac{1}{3}$

123 Exploring intuitively the greater than, less than, and equality relations among fractions and among sums of fractions.

Tasks: *Pupils record in the blue boxes the correct relation sign to complete each sentence; if they wish, they may mark the unmarked rectangles with fractions.*

Purpose: To extend understanding of the equality and inequality relations between fractions and between whole numbers and fractions.

Unifying Ideas: Structure; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Yardstick; masking tape.

Materials for Pupils: Tracing paper to copy the subdivided rectangle.

The Lesson:

A yardstick that is blank on one side is a handy aid in developing the idea involved on this page. (A piece of masking tape covering one side of a two-sided yardstick is an easy improvisation.) Thirty-six inches can easily be divided into halves, thirds, fourths, sixths, ninths, and twelfths.

With the blank side toward the class, say: Show me a point that would divide the stick into 2 parts of equal length.

How many inches in a yard? What numbers would you expect to find on the other side? If I put the yardstick on my finger, will it balance like a seesaw? How many feet in a yard? How many in half a yard? How many inches in a foot? How many inches in a foot and a half?

How can we divide the stick into 3 parts of equal length? Can you guess where the marks would be? Do these marks look right to everyone? How can we tell if we are reasonably close? (Cut a piece of paper that is as long as one part and compare it with the other parts. Put two marks on the board, and compare each part.) What numbers should we find on the other side opposite our marks? How many inches between the marks? How many feet? How many yards? (Same development with fourths and sixths.) Make a table:

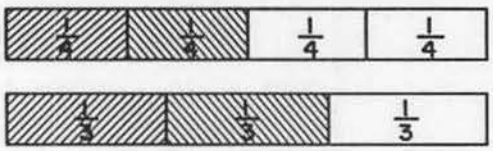
Yard	Inches
1/2	18
1/3	12
1/4	9
1/6	6
2/3	24
2/4	18
3/4	27

Which is larger . . . 1/2 of a yard or 1/3 of a yard? . . . 1/3 or 1/4? . . . 1/3 or 1/6? . . . 1/6 or 1/4? Which is larger . . . 1/2 or 2/4? . . . 1/2 or 3/6? . . . 1/3 or 2/6? Which is smaller . . . 1/2 + 1/2 or 1/4?

Write examples such as these on the board and explain, if necessary, the use of the signs of inequality:

$$\frac{1}{2}+\frac{1}{2}>\frac{1}{4}$$
$$\frac{1}{2}-\frac{1}{2}<\frac{1}{4}$$
$$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$
$$\frac{1}{4}+\frac{1}{4}<\frac{1}{3}+\frac{1}{3}$$

Illustrate the last example—and others like it:



Headline for the Day: Consider the first completed sentence in any column (A-D) on page 123 as a headline for a story. Tell a story to fit that headline.

Bill had some marbles. He had less than 40. He decided to keep three out of every four marbles and give the rest to his brother, Jim, who had no marbles.

He arranged them in groups of four and had none left over.

Could Bill have ended up with 51 marbles? No

Could Jim get as many as 20 from Bill? No

Who would have more? BILL

Could Bill and Jim end up with the same number of marbles? No

How many did Bill have in the beginning? How many after the division? How many did Jim have after the division?

We can't answer these last questions but there are only a certain number of possibilities. Let's make a chart to see how many these are. Start with the smallest number Bill could have had in the beginning.

Number Bill had after the division:	3	6	9	12	15	18	21	24	27	30
Number Bill had in the beginning:	4	8	12	16	20	24	28	32	36	40

Number Bill gave Jim:	1	2	3	4	5	6	7	8	9	10

A group of five boys and girls spent Saturday afternoon wrapping bandages for the Red Cross. While walking home, they talked about how fast they worked.

Alice said that she wrapped three bandages every four minutes.

"Well," said Ben, "it took me only three minutes to wrap two bandages."

Kathryn said that it took her six minutes to wrap five bandages. Paul reported seven bandages in nine minutes.

"I wrapped nine in twelve minutes. Isn't that the best record?" said Jerry. No

Who had the best record? KATHRYN

Number of bandages Alice wrapped	3	6	9	12	15	18	21	24	27	30	33	36
Time in minutes	4	8	12	16	20	24	28	32	36	40	44	48

Number of bandages Ben wrapped	2	4	6	8	10	12	14	16	18	20	22	24
Time in minutes	3	6	9	12	15	18	21	24	27	30	33	36

Number of bandages Kathryn wrapped	5	10	15	20	25	30	35	40	45	50	55	60
Time in minutes	6	12	18	24	30	36	42	48	54	60	66	72

Number of bandages Paul wrapped	7	14	21	28	35	42	49	56	63	70	77	84
Time in minutes	9	18	27	36	45	54	63	72	81	90	99	108

Number of bandages Jerry wrapped	9	18	27	36	45	54	63	72	81	90	99	108
Time in minutes	12	24	36	48	60	72	84	96	108	120	132	144

124-125 Exploring rates.

Tasks: *Pupils use the information in the stories to complete each table.*

Purpose: To further explore relations among fractions via applications.

Unifying Ideas: Structure; functions and relations; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

If you have been following our suggestions on previous pages, this page and pupil page 125 will be very easy for your pupils.

The questions concerning the best record is interesting. It can be answered by comparing the outputs for a common time. In 36 minutes, Alice wrapped 27 bandages, Ben wrapped 24, Kathryn wrapped 30, Paul wrapped 28, and Jerry wrapped 27. Hence, Kathryn had the best record.

Drill

Here is a suggestion for a drill sheet.

	3	10	13	26
x 5	15		65	
x 10		100		
x 15				

	7	10	17	27
x 3		30		
x 20				540
x 23				

	5	20	25	45
x 4				
x 30				
x 34				

	8	20	28	56
x 7				
x 10				
x 17				

		30		67
x 9	63			
x 20			740	
x 29				

		40		23
x 8			368	
x 50	300			
x 58				

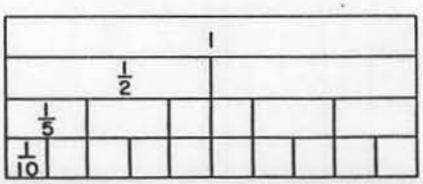
	9	20	29	
x 6				
x 40				
x 46				

x 2	16	60	76	38
x 60				
x 62				

13 x 16 25 x 14 36 x 21

15 x 15 53 x 12 79 x 14

Headline for the Day: Tell stories about this picture.



Frank is a clerk in a candy store. Different kinds of candy are sold at different prices. No pennies are allowed in this store.

Chocolates are five cents each or one for five cents. Licorice sells at three pieces for five cents. Peppermints are sold two for five cents; and gum drops are four for five cents. There are five pieces of butterscotch in each five-cent purchase. Pieces of butterscotch sold, therefore, at five pieces for five cents.

Frank decided to make a chart so he would not make mistakes.

Number of chocolates	1	2	3	4	5	6	7	8	9	10	11
Price in cents	5	10	15	20	25	30	35	40	45	50	55
Number of peppermints	2	4	6	8	10	12	14	16	18	20	22
Price in cents	5	10	15	20	25	30	35	40	45	50	55
Number of licorice sticks	3	6	9	12	15	18	21	24	27	30	33
Price in cents	5	10	15	20	25	30	35	40	45	50	55
Number of gum drops	4	8	12	16	20	24	28	32	36	40	44
Price in cents	5	10	15	20	25	30	35	40	45	50	55
Number of pieces of butterscotch	5	10	15	20	25	30	35	40	45	50	55
Price in cents	5	10	15	20	25	30	35	40	45	50	55

a. Marcia bought eight pieces of candy and spent less than 15 cents. What did she buy?
GUMDROPS

b. Stewart bought four pieces of candy for 10 cents and his sister bought 12 pieces for 20 cents. Did they buy the same kind?
No

c. Jack spent a quarter and got 15 pieces of candy. Would he have had more pieces if he had bought peppermints? No

d. Henrietta bought 27 pieces of one kind. What did she buy?
LICORICE

Mary had saved cola bottle caps. One day she stacked them in piles of six. She decided to give them all to her younger brothers — one out of every two to Phil, one out of every three to Bob, and the rest to Dick.

Make a chart showing how the division might go.

	Possible totals	Phil	Bob	Dick	B + D	P + D	P + B
e.	6	3	2	1	3	4	5
f.	12	6	4	2	6	8	10
g.	18	9	6	3	9	12	15
h.	24	12	8	4	12	16	20
i.	30	15	10	5	15	20	25
j.	36	18	12	6	18	24	30

Tasks: Pupils use the information in the stories to complete each table.

Purpose: To further explore relations among fractions via applications.

Unifying Ideas: Structure; functions and relations; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

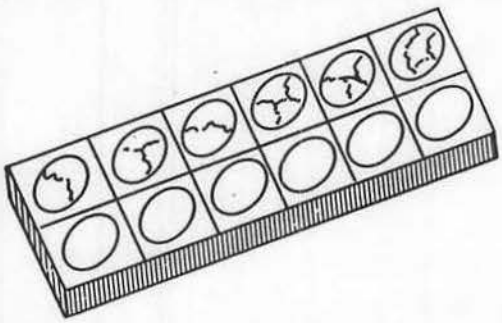
The Lesson:

You may wish to introduce the work with the three stories that follow:

Story 1

Bill said that 6 eggs in a carton of a dozen were cracked. His sister said: But it looks to me as if 3 in each half-dozen are broken. His little brother said: In each group of 2 eggs, 1 is broken. When his mother looked at the carton, she said: Why half of them are broken.

Here is a sketch of the situation. Who was right?



Write on the board:

Bill: 6 eggs are cracked
Sister: 3 in each half-dozen are broken.
Little brother: 1 out of every 2 is broken.
Mother: Half are broken.

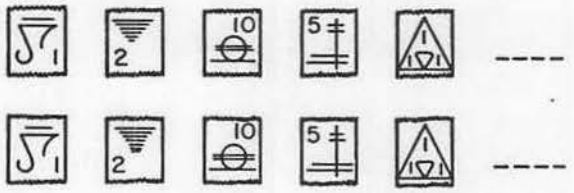
Are they all saying much the same thing? In mathematics, we say that numbers have many names. We say, for example, that:

$$\underline{1/2} = 2/4 = \underline{3/6} = 4/8 = 5/10 = \underline{6/12}$$

The underlined names for 1/2 suggest something quite similar to the remarks about the broken eggs.

Story 2

Alice saved stamps. Her best friend, Maxine, also saved stamps. Alice decided to give Maxine one out of each pair of duplicates. So, she arranged them in pairs across her desk, something like this:



How many did she give Maxine? Do you have enough information? Which of the following sentences need to be completed so that you will have enough information?

A. Alice had ___ stamps arranged in pairs on her desk.
B. The stamps were worth $___.
C. She had ___ pairs arranged on her desk.
D. Alice gave Maxine ___ out of every pair of stamps.
E. Alice had ___ stamps of which she had no duplicates.
F. Of those so arranged, she kept ___ stamps.

Completions for any one of the sentences A, C, and F will provide enough information to answer the question conclusively.

Without completing A, C, or F, have pupils draw a chart of some of the possibilities, if they know how many stamps Alice gave Maxine.

Headline for the Day: Complete each sentence. Tell what is alike about all of the sentences.

$3\overline{)6}$, $5\overline{)10}$, $15\overline{)30}$, $\_\overline{)40}$, $50\overline{)\_}$, $28 \div 14 = \square$, $\square \times 8 = 16$, $9 \times \square = 18$, $2 \times 35 = \square$, $80 \div \square = 40$

Number to Maxine	1	5	50		
No. of stamps arranged in pairs	2	10	100	etc.	

Story 3

In grocery stores, you will find cans, cartons, bottles, and packages marked this way:

3/15	2/11	4/25
5/20	5/22	5/13

What do you think those marks mean? (3 for 15¢; 2 for 11¢; 4 for 25¢; etc.)

If a can is marked 3/15, how much do you think you would pay for 1 can? This may lead to a discussion. Someone will argue that if 3 cans cost 15¢ then 1 can would cost 5¢ because $15 \div 3 = 5$. Someone else may contend that such is not the case, because then the cans would be marked 1/5. If the price is meant to indicate a bargain for purchasing 3 cans then the price for 1 can must be 6¢. By purchasing 3 cans for 15¢, the customer saves 1¢ per can.

On the board draw a chart for each case:

No. of cans	1	2	3	4	5	6	7	8	9	10
Price in cents	5	10	15			30				

No. of cans	1	2	3	4	5	6	7	8	9	10
Price in cents	6	12	15	21	27	30				

We agree that the price is 3 cans for 15¢ in either case. How much would 6 cans cost? How much would 9 cans cost? The same in both records?

Now, how about 1 can? We have agreed that 1 can might cost 5¢ or 6¢. In each case, how much would 2 cans cost? (10¢ in the first case; 12¢ in the second.)

But someone might argue that if 1 can costs 6¢ and 3 cans cost 15¢ then you probably get a small bargain for buying 2 cans at a time. So, while 1 can costs 6¢, 2 cans may cost only 11¢. That would look like this:

		Savings
1 can	6¢	0
2 cans	11¢	1¢
3 cans	15¢	3¢

So, we need another chart:

No. of cans	1	2	3	4	5	6	7
Price in cents	6	11	15			30	

Now introduce the first situation given on the pupil page, without referring to the page. After getting the chart started, direct the class to the page.

How much of the page can you complete without any further help? If possible, have pupils work in small groups. Otherwise, continue with a class discussion of each example.

On the pupil page we are using shorthand notations within a special context. These notations are similar to 3/28, which, in a similar situation, would be read "3 cans for 28¢." Avoid using the word *fraction.* To emphasize the danger in using this word, consider: If Ralph bought 1 chocolate for 5¢ and 3 sticks of licorice for 5¢, he would have bought 4 pieces of candy and paid 10¢. If we were to think of the shorthand expressions 1/5 and 3/5 as fractions, it might lead us to assert mathematically that:

$$\frac{1}{5} + \frac{3}{5} = \frac{4}{5}$$

and this would lead to the false conclusion that Ralph bought 4 pieces of candy for 5¢. Or, we might be tempted to write:

$$\frac{1}{5} + \frac{3}{5} = \frac{4}{10.}$$

This is certainly a false statement, if we are relating this to fractions in arithmetic.

Since a notation can have a variety of uses, it is extremely important to shy away from rules and to keep our wits about us so that we are always able to understand what the notation refers to in a specific instance.

If we could buy 3 pieces of something for 5¢ then we would assume that 6 pieces would cost 10¢. If we should buy 3 pieces or 6 pieces, the cost of each piece would be the same. Now it is possible to express that condition quite sensibly. We are tempted to say that:

$$\frac{3}{5} = \frac{6}{10} \text{ and } \frac{3}{5} = \frac{9}{15,} \text{ etc.}$$

because this suggests the idea well and we feel that it is a true statement in arithmetic.

Question *a* on pupil page 125 asks what kind of candy Marcia selected if she bought 8 pieces for less than 15¢. She couldn't have bought chocolates (8

for 40¢), peppermints (8 for 20¢), butterscotch, or licorice. In the last two items there is no sale of 8 pieces in either case. So she must have bought 8 gumdrops for 10¢.

However, someone might argue that 8 pieces of licorice would cost less than 15¢, because 9 cost 15¢. Also, that butterscotch seems to be 1¢ per piece—and 8 would cost 8¢. Both are certainly sensible observations. But, remember, you can't use pennies in the store.

Question *b* also may set off a discussion. If Stewart bought 4 pieces of the same kind of candy and his sister selected all 12 from the same kind then the answer to the question is: No, Stewart bought peppermints and his sister bought licorice.

Perhaps Stewart paid 5¢ for 1 chocolate and 5¢ for 3 pieces of licorice—4 pieces for 10¢. (How many other ways could he pay 10¢ for 4 pieces?) His sister might have spent 10¢ for 2 chocolates and 10¢ for 10 pieces of butterscotch. If so, the answer is still: No, they did not buy the same kind. But is there a combination purchase that would lead to answering Yes to the question? If you prefer to avoid this kind of discussion, simply assume that the purchase, in each case, was limited to one kind of candy.

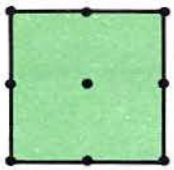
Questions that have several answers help keep children awake to the context in which the problem arises. They will become less apt to take the numerical expressions out of context and misapply some well-remembered rule. They will keep their main attention focused on the real problem, using a knowledge of arithmetic as a tool when needed. The arithmetic involved is usually of the most elementary kind. A firm grasp on the context of the situation is an essential prerequisite in knowing how to proceed.

Drill

Here are some stories for headline practice:

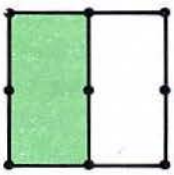
1. Charlie spent 35 cents on candy bars. Each candy bar cost 7 cents.
2. The class dictionary has about 1000 pages. Each page has about 40 words on it.
3. Joe found 16 golf balls. He gave 1/8 of the balls to Clyde.

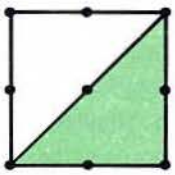
$1/8 \text{ of } 16 = 2$ $\qquad$ $16 \div 8 = 2$

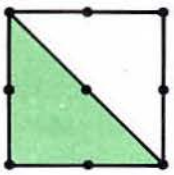
IF we agree that  suggests 1

THEN the shaded area in each picture below suggests →

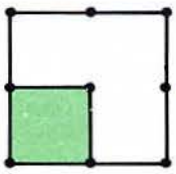
ACCEPT OTHER CORRECT FRACTIONS.

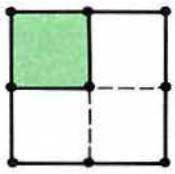
a  → $\frac{1}{2}$

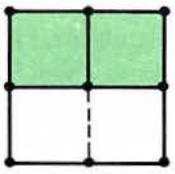
b  → $\frac{1}{2}$

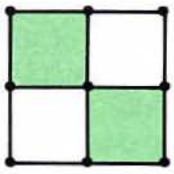
c  → $\frac{1}{2}$

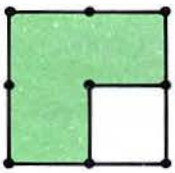
d → $\frac{1}{2}$

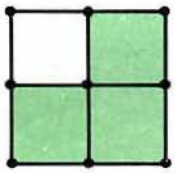
e  → $\frac{1}{4}$

f  → $\frac{1}{4}$

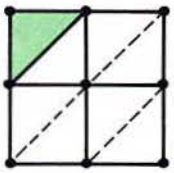
g  → $\frac{1}{2}$

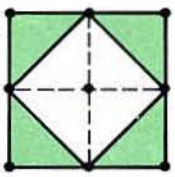
h  → $\frac{1}{2}$

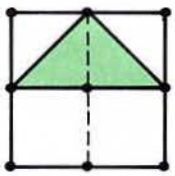
i  → $\frac{3}{4}$

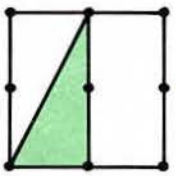
j  → $\frac{3}{4}$

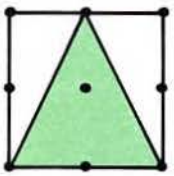
k → $\frac{1}{8}$

l  → $\frac{1}{8}$

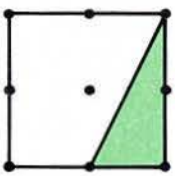
m  → $\frac{1}{2}$

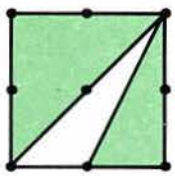
n  → $\frac{1}{4}$

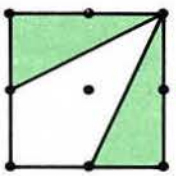
o  → $\frac{1}{4}$

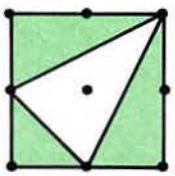
p  → $\frac{1}{2}$

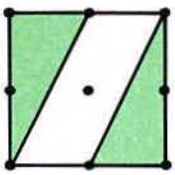
Hard work below!

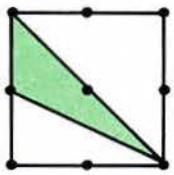
q  → $\frac{1}{4}$

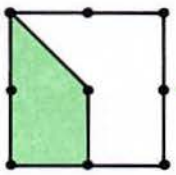
r  → $\frac{3}{4}$

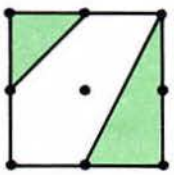
s  → $\frac{1}{2}$

t  → $\frac{5}{8}$

u  → $\frac{1}{2}$

v  → $\frac{1}{4}$

w  → $\frac{3}{8}$

x  → $\frac{3}{8}$

The unshaded areas suggest →

f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{5}{8}$

126-128 Exploring ratios.

Tasks: Pupils use the horizontal line adjacent to each representation as a fraction bar in recording the fraction of the unit square tinted green in each representation. They perform a similar task at the bottom of the page by recording the fraction of the unit square unshaded in each representation.

Purpose: To explore fraction concepts via areas of various geometric figures.

Unifying Ideas: Geometry; measurement.

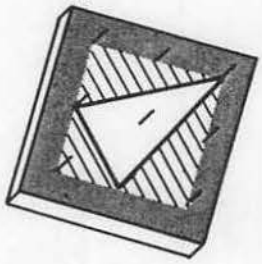
Materials for Teacher: Counting Board; rubber bands; graph board.

Materials for Pupils: Geo-boards; rubber bands; graph paper.

The Lesson:

You will be well rewarded for the few minutes it takes to drive 9 thin nails into a piece of wood. Arrange them in a 3 × 3 array, from 2 to 4 inches apart, simulating the pattern of black dots used on pupil page 126. With this board and some rubber bands, you can create each situation presented on this page except exercise *k*. To make the model more helpful, paint the region inside or outside the array some color so that the two regions are distinguishable.

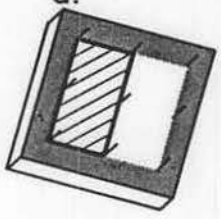
t.



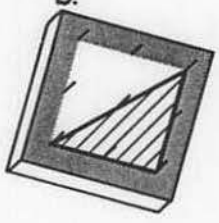
You may wish to use several rubber bands in an example such as *t*, though it can be done (as shown above) with a single rubber band.

If we agree that the size of the region outlined by the nails suggests 1 then how much does the region inside the rubber band(s) suggest? The following diagrams illustrate examples *a*, *b*, *c*, and *d*:

a.

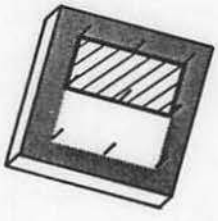


b.



c.

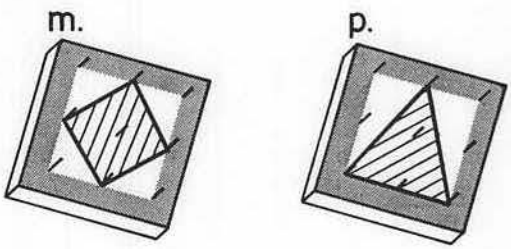
d.



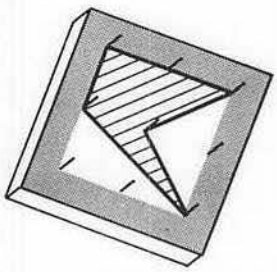
more

Headline for the Day: Label the unlabeled points.

Are there any other ways to arrange a rubber band so that 1/2 is inside the band and the other 1/2 is outside? Many ways. Two of the clearest are shown in examples *m* and *p*:



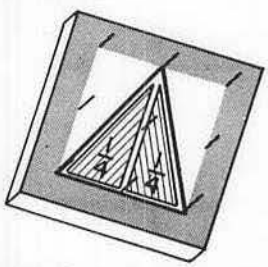
Other ways are shown in examples *h*, *s*, and *u* on pupil page 126. Yet another way that is not at all obvious is:



(If you can make several such boards, or have pupils make them, they will be in demand as game boards.)

Another helpful device is to mark out an array of dots on a piece of paper or thin cardboard. Space the dots the same distance apart as the nails. After you have outlined a region, cut out a piece of paper the same shape and size, and mark on it the size of the paper in relationship to the whole region that suggests 1. Do this particularly with exercises *a*, *b*, *e*, *l*, *n*, and *o*, and perhaps *u*, marked 1/2, 1/2, 1/4, 1/8, 1/4, 1/4, and 1/2.

In each example, a combination of pieces of paper can be used to cover the region under consideration. Exercise *p*—a most important one—is probably not obvious until this is done:



With these simply constructed materials, the pupils will have a most intriguing time discovering some very basic ideas of geometry and mensuration. A few may even suspect that the area of a triangle can be found by multiplying half its base by its altitude.

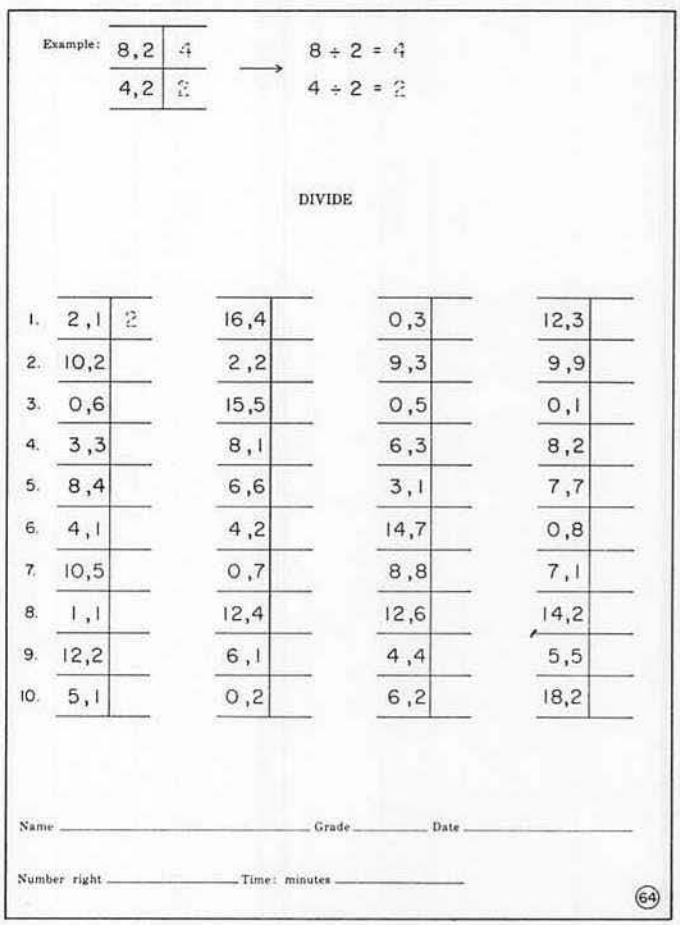
Once familiar with the model you have made, some pupils might construct boards to illustrate the problems on pupil page 126.

Drill

Here are some stories for headline practice:

1. Neil went to the candy store. He noticed on one counter that 75 lollipops had been bundled so that each bundle was made up of 5 lollipops.
2. Chauncey's mother sold a used lamp for $1.89 and bought a new one for $5.00.

Consider using as a drill sheet or as a test on division page 64 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C:



Example: 8,2 | 4 → 8 ÷ 2 = 4
4,2 | 2 → 4 ÷ 2 = 2

DIVIDE

1.	2,1	2	16,4		0,3		12,3	
2.	10,2		2,2		9,3		9,9	
3.	0,6		15,5		0,5		0,1	
4.	3,3		8,1		6,3		8,2	
5.	8,4		6,6		3,1		7,7	
6.	4,1		4,2		14,7		0,8	
7.	10,5		0,7		8,8		7,1	
8.	1,1		12,4		12,6		14,2	
9.	12,2		6,1		4,4		5,5	
10.	5,1		0,2		6,2		18,2	

Name _____ Grade _____ Date _____

Number right _____ Time: minutes _____

64

Here are suggestions for two more drill sheets.

6 × 3	5 × 4	7 × 7	3 × 9
12 × 15	25 × 8	35 × 14	15 × 18
24 × 75	125 × 16	175 × 28	75 × 36

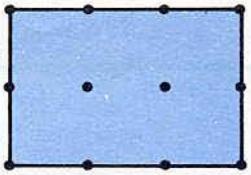
1 × 1 = ___	11 × 11 = ___	21 × 21 = ___	31 × 31 = ___
2 × 2 = ___	12 × 12 = ___	22 × 22 = ___	32 × 32 = ___
3 × 3 = ___	13 × 13 = ___	23 × 23 = ___	33 × 33 = ___
4 × 4 = ___	14 × 4 = ___	24 × 24 = ___	34 × 34 = ___
5 × 5 = ___	15 × 15 = ___	25 × 25 = ___	35 × 35 = ___
6 × 6 = ___	16 × 16 = ___	26 × 26 = ___	36 × 36 = ___
7 × 7 = ___	17 × 17 = ___	27 × 27 = ___	37 × 37 = ___
8 × 8 = ___	18 × 18 = ___	28 × 28 = ___	38 × 38 = ___
9 × 9 = ___	19 × 19 = ___	29 × 29 = ___	39 × 39 = ___
10 × 10 = ___	20 × 20 = ___	30 × 30 = ___	40 × 40 = ___

...÷3	30	9	39	30	21	51	60	24	84	$3\overline{)36}$	$3\overline{)72}$
...÷6	36	42	78	60			120	30	150	$6\overline{)162}$	$6\overline{)300}$
				10	3	13					
...÷8				80	16	96	240	32	272	$8\overline{)184}$	$8\overline{)200}$
	7	9	16								
...÷2	200	16	216	90	14	104	300	80	380	$2\overline{)72}$	$2\overline{)500}$
...÷9			117	81					243	$9\overline{)108}$	$9\overline{)135}$
	10	3			4	13	20	7			
...÷4	100	4		80	16	96	120	16	104	$4\overline{)500}$	$4\overline{)140}$
			24								
...÷7	70	35	105	210	56	266				$7\overline{)168}$	$7\overline{)693}$
							10	9	19		

70 , 63 , ___ , 49 , ___ , ___ , ___ , ___ , ___ , ___ , ___

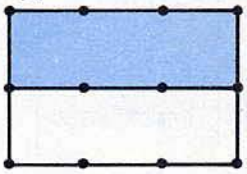
80 , 72 , ___ , ___ , ___ , 40 , ___ , ___ , ___ , ___ , ___

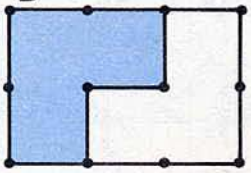
150 , 135 , 120 , ___ , ___ , ___ , ___ , ___ , ___ , ___ , ___

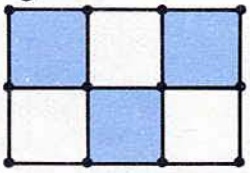
IF we agree that  suggests 1

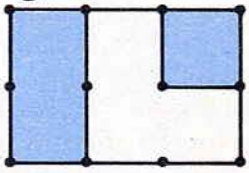
THEN the shaded area in each picture below suggests

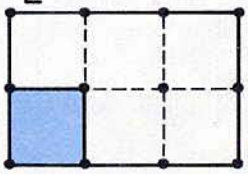
ACCEPT OTHER CORRECT FRACTIONS.

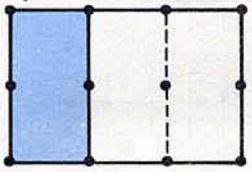
A  $\frac{1}{2}$

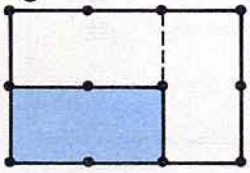
B  $\frac{1}{2}$

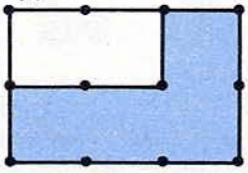
C  $\frac{1}{2}$

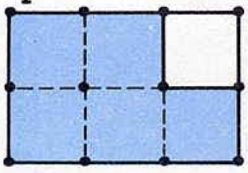
D  $\frac{1}{2}$

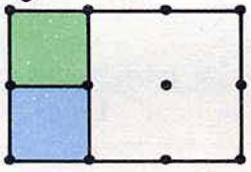
E  $\frac{1}{6}$

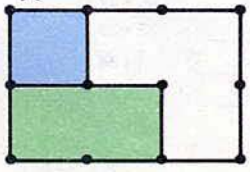
F  $\frac{1}{3}$

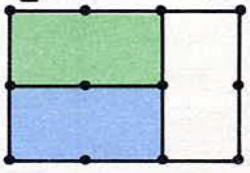
G  $\frac{1}{3}$

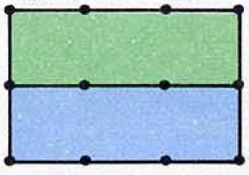
H  $\frac{2}{3}$

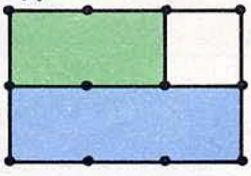
I  $\frac{5}{6}$

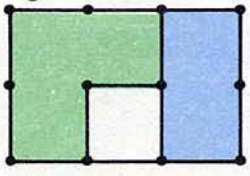
J  $\frac{1}{3}$

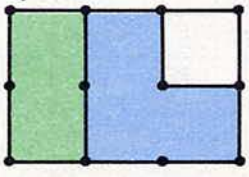
K  $\frac{1}{2}$

L  $\frac{2}{3}$

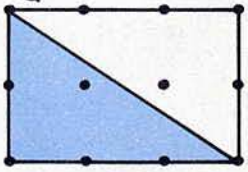
M  $\frac{2}{2}$ or 1

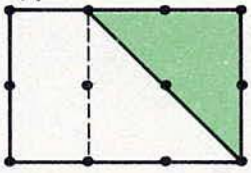
N  $\frac{5}{6}$

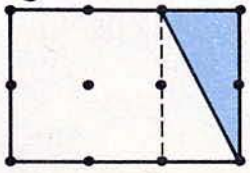
O  $\frac{5}{6}$

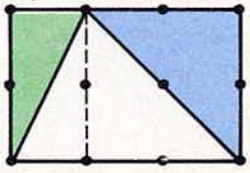
P  $\frac{5}{6}$

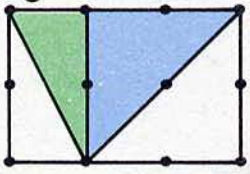
Hard work below!

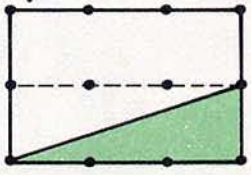
Q  $\frac{1}{2}$

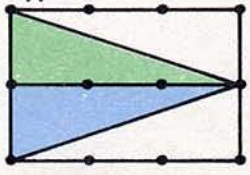
R  $\frac{1}{3}$

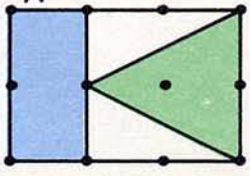
S  $\frac{1}{6}$

T  $\frac{1}{2}$

U  $\frac{1}{2}$

V  $\frac{1}{4}$

W  $\frac{1}{2}$

X  $\frac{2}{3}$

The unshaded areas suggest

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	T	V	X
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{0}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{3}$

Notes on Pupil Page 127

Tasks: *Pupils use the horizontal line adjacent to each representation as a fraction bar in recording the fraction of the unit rectangle tinted blue or green in each representation. They perform a similar task at the bottom of the page by recording the fraction of the unit rectangle unshaded in each representation.*

Purpose: To continue exploring fraction concepts via areas of various geometric figures.

Unifying Ideas: Geometry; measurement.

Materials for Teacher: Counting Board; graph board.

Materials for Pupils: Geo-boards; rubber bands; graph paper.

The Lesson:

Pupil page 127 is similar to pupil page 126. On pupil page 127, however, a region of different size and shape is used to suggest 1. It is obvious that another model—woods, nails, and rubber bands—would be most helpful, and several models would help liven the activity. However, you may wish to take a somewhat different approach.

Let's consider that this is the floor plan of a rectangular room. The man who sweeps it says that he can do it in an hour. Sometimes he sweeps it a part at a time. But, if he adds the times together, it always is 1 hour.

How long would it take him to sweep the blue region in exercise A? in exercise B? in exercise C? in exercise D? (A half-hour, or 30 minutes, for each.)

When part of the floor plan is colored blue and part green, this indicates that he swept the blue part first and then the green part. The uncolored portion indicates the part of the floor still to be swept.

In how many other examples on pupil page 127 does the diagram indicate that half of the floor has not been swept? (K, Q, T, U, and W)

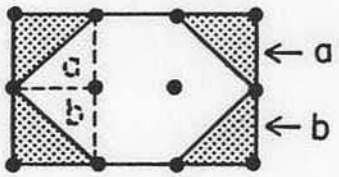
How much of the floor has been swept in exercise E? How much time did it take? (1/6 swept in 10 minutes.) How much is left . . . and how long will it take? (5/6 to sweep in 50 minutes.)

How much has been swept in exercise F? And how long did it take? (1/3 in 20 minutes.)

In how many other exercises has 1/3 of the floor been swept and 2/3 not swept at all? (G, J, and R)

In how many exercises will it take 20 minutes to finish sweeping the floor? (H, L, and X)

Suppose that only one part in each corner were swept. A diagram would look like this:



How much is swept? How long did it take?

You may need to use paper cutouts. Four pieces—1 to fit over each corner—represent the part swept. Rearrange pieces labeled *a* and *b* so that the 4 together clearly cover 1/3 of the diagram.

How much is still to be swept? How long will it take? (2/3 left to sweep in 40 minutes.)

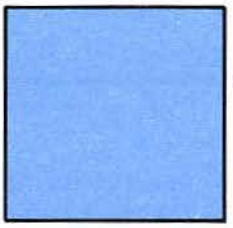
How much was swept in each of the 4 corners? How long did it take to sweep each corner? (1/12 swept in 5 minutes.)

Drill

Here are some stories for headline practice:

1. There are 63 kindergarten pupils, 70 first-grade pupils, 57 second-grade pupils and 52 third-grade pupils attending Eastside Elementary School.
2. The pear trees in a nearby orchard are arranged in 10 rows of 100 trees each.
3. Roy's dog ate 1/6 of a bag of dog food each day. The bag of dog food weighed 12 pounds when full.

Headline for the Day: Label the unlabeled points.

IF we agree that  suggests 1

THEN the shaded area in each picture below suggests

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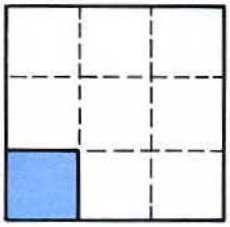
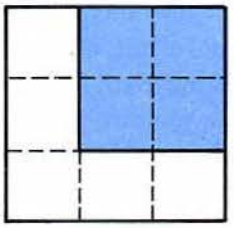
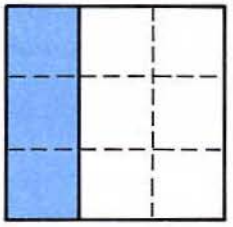
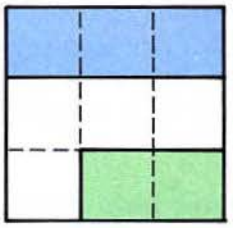
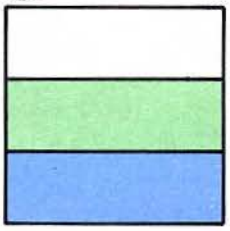
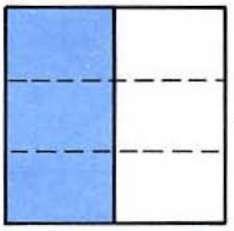
1.  $\frac{1}{9}$

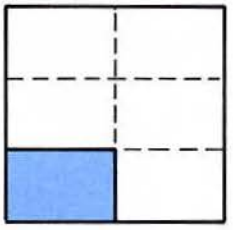
2.  $\frac{4}{9}$

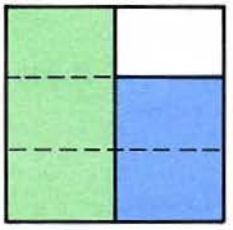
3.  $\frac{1}{3}$

4.  $\frac{5}{9}$

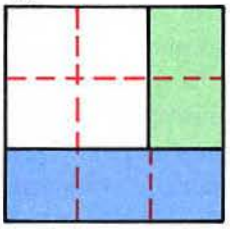
5.  $\frac{2}{3}$

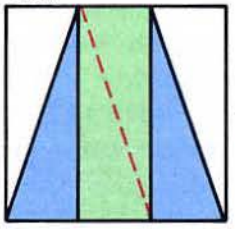
6.  $\frac{1}{2}$

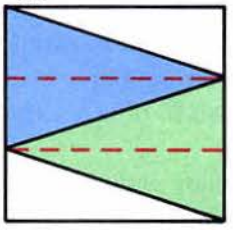
7.  $\frac{1}{6}$

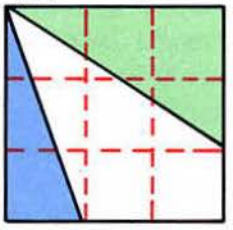
8.  $\frac{5}{6}$

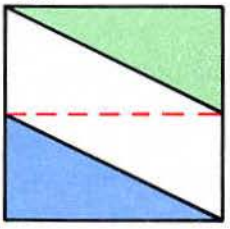
Hard work below!

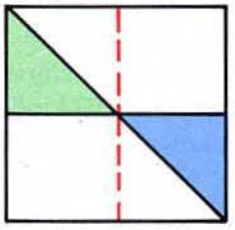
9.  $\frac{5}{9}$

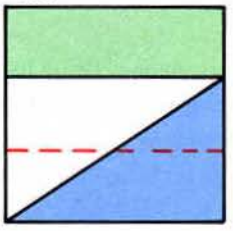
10.  $\frac{2}{3}$

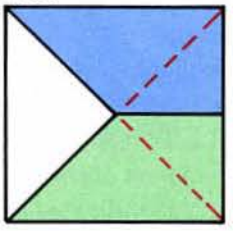
11.  $\frac{2}{3}$

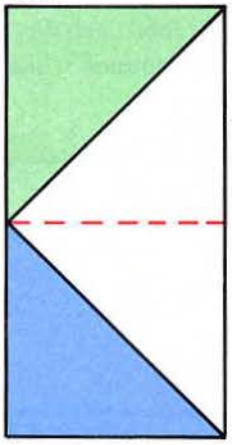
12.  $\frac{1}{2}$

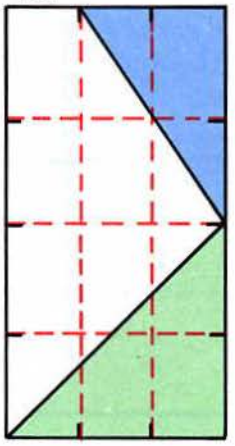
13.  $\frac{1}{2}$

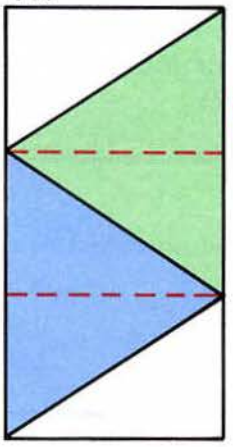
14.  $\frac{1}{4}$

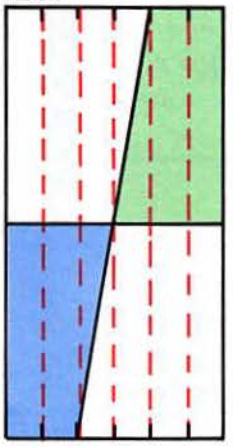
15.  $\frac{2}{3}$

16.  $\frac{3}{4}$

17.  $\frac{2}{2}$

18.  $\frac{5}{6}$

19.  $\frac{1}{3}$

20.  $\frac{5}{6}$

Notes on Pupil Page 128

Tasks: *Pupils use the horizontal line adjacent to each representation as a fraction bar in recording the fraction of the unit square tinted blue or green.*

Purpose: To continue exploring fraction concepts via areas of geometric figures.

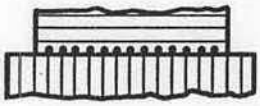
Unifying Ideas: Geometry; measurement.

Materials for Teacher: Counting Board; graph board.

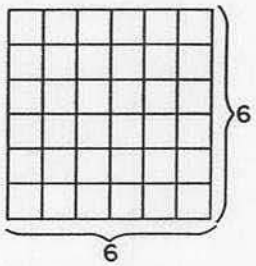
Materials for Pupils: Geo-board; rubber bands; ruler; lined paper; crayons.

The Lesson:

See that all children have lined paper and a ruler or straightedge. Then give them the following directions: Turn a piece of lined paper sideways and use it to measure off spaces one line apart along the top edge of another piece of lined paper. Do the same at the bottom edge of the paper. Now, connect the dots at the top and bottom of the paper with a ruler and pencil so that the whole page is divided into many small squares.



See how many large squares you can cut out that are 6 spaces on each side. How many small squares will there be in each large square? (36) Draw a diagram on the board.



6 × 6 = 36

Most lined paper will provide at least 8 such squares on one page. After discussion, agree on the minimum number of large squares each child ought to have when he is done.

Turn to pupil page 128. Imagine that your large squares are the same size as the solid blue square at the top of the page. With a blue and a green crayon, color in parts of your large squares so they look like the first 8 examples.

Those who finish the work first can continue making large pieces to be colored to fit examples 9 through 16.

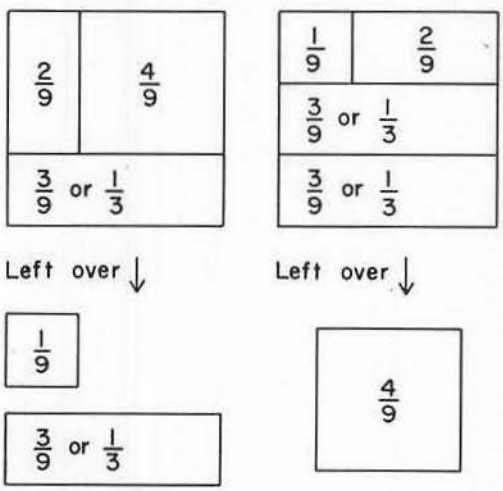
When you finish coloring a large piece, label it with the number of the example.

After the class has finished the first 8 examples,

Headline for the Day: Mabel had 100 marbles. She gave 1/2 of them to Pandora, 1/4 of them to Gunther, 1/5 of them to Mason. How many marbles did each of her friends receive? How many marbles did Mabel keep for herself?

discuss the work on the page. Let's agree that the size of the blue region will suggest 1 on this page. If that is true, how much of the large square is colored blue in exercise 1? (1/9) How much in exercise 2? (4/9) Exercises 3 and 4? (3/9 or 1/3, and 5/9). Have the children mark these results in the colored sections of their 6 × 6 squares.

Cut out several pieces of paper the same size as the large squares you have cut out and colored. Then cut out the 5 parts of the squares you colored in exercises 1, 2, 3, and 4. Can you arrange some of the colored pieces so that they will fit neatly on top of one of the large squares? Can you arrange them so that you have 1 colored piece left over? Can you arrange to have 2 colored pieces left over? The answer to both questions is Yes:



Paste down the pattern with 2 left over. Save the extra pieces.

Have the children approach exercises 5, 6, and 7 similarly. This time they will need 2 large squares to cover neatly . . . and, using the pieces left over from the other activity, they will have pieces marked 1/9, 1/3, 2/3, 1/2, 1/6, and 5/9.

The results are graphic representations of sums of fractions that equal 1.

Drill

Here is a suggestion for a drill sheet:

$3\overline{)27}$	$3\overline{)60}$	$3\overline{)87}$	$3\overline{)147}$	$3\overline{)2400}$
$6\overline{)20}$	$6\overline{)6}$	$6\overline{)156}$	$6\overline{)52}$	$6\overline{)78}$
$9\overline{)81}$	$9\overline{)900}$	$9\overline{)981}$	$9\overline{)105}$	$9\overline{)927}$
$5\overline{)100}$	$5\overline{)100}$	$5\overline{)40}$	$5\overline{)128}$	$\overline{)256}$
$8\overline{)56}$	$8\overline{)20}$	$8\overline{)27}$	$8\overline{)54}$	$8\overline{)108}$
$4\overline{)36}$	$4\overline{)28}$	$4\overline{)64}$	$4\overline{)23}$	$4\overline{)39}$
$7\overline{)1400}$	$7\overline{)350}$	$7\overline{)9}$	$7\overline{)1813}$	$7\overline{)4242}$
$6\overline{)48}$	$6\overline{)30}$	$6\overline{)3000}$	$6\overline{)3228}$	$6\overline{)269}$
$9\overline{)54}$	$9\overline{)270}$	$9\overline{)4500}$	$9\overline{)4824}$	$9\overline{)2412}$
$8\overline{)800}$	$8\overline{)160}$	$8\overline{)5}$	$8\overline{)125}$	$8\overline{)3000}$